Rost Complex: $C^p(X, K_n^M) = \bigoplus_{x \in X^{(p)}} K_{n-p}^M(k(x))$

$d_X : C^p(X, K_n^M) \to C^{p+1}(X, K_n^M)$ $\quad x \in X^{(p)}$, $y \in X^{(p+1)}$

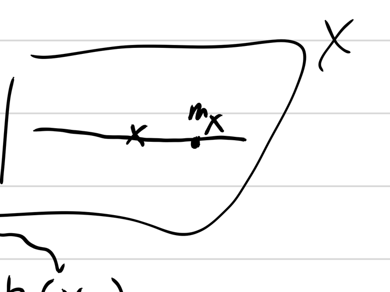
Rmk: $\underline{H^n(C^*(X, K_n^M)) = CH^n(X)}$.

Thm 3.18 For X finite type$/k$, we have $d_X \circ d_X = 0$.

Proof: It suffices to prove that for any integral local X/k, $X = \operatorname{Spec} \overset{\text{local ring of dim 2}}{R}$, we have

$$\sum_{x \in X^{(1)}} \partial^x_{m_X} \circ \partial^{\xi_X}_x = 0.$$

$m_X \to x_0$ close point



1) We may suppose that $k(\underset{\text{close point}}{x_0}) = k$. $\quad k \to R \to k(x_0)$

We choose a lift of a transcendental basis of $k(x_0)/k$ to R. So we find a field $k \subseteq \underset{k(\alpha)}{K} \subseteq R$ s.t. $k(x_0)/K$ is finite $\{\alpha\}$ extension. Choose $u \in X \times_k k(x_0) = X' \xrightarrow{p} x_0 \in X$

Then $\underline{k(u) = k(x_0)}$. Moreover, $d_X \circ d_X = 0 \iff (p^* \circ d_X \circ d_X)_u = 0$

$\iff (d_{X'} \circ d_{X'} \circ p^*)_u = 0$.

2) Reduce the question from X to $\mathbb{A}^2_{(0,0)}$.

Suppose X is a localization of $Y \subseteq \mathbb{P}^n$ at y (rational point) where Y is closed and $\dim Y = 2$. We define

$$H = \underbrace{Gr(1, n-2, n+1)}_{\{(x,V) \mid x \in V\}} \subseteq \mathbb{P}^n \times Gr(n-2, n+1)$$

x: line, V: $Gr(n-2, n+1)$

Then we have a map $(\mathbb{P}^n \times Gr(n-2, n+1)) \setminus H \to \mathbb{P}^2 \times Gr(n-2, n+1)$

$(x, V) \mapsto (P_V(x), V)$, $x \notin V$

where $P_V(x)$ is the projection of x centered at V. Let $F = K(Gr(n-2, n+1))$.

We could show that $\underline{(H \times_{Gr(n-2,n+1)} F) \cap Y_F = \emptyset}$.

$\dim Y = 2$, $\forall V \in Gr(n-2, n+1)$ codim $V = 3$

Find a $V \in Gr(n-2, n+1)$ s.t. $Y \cap V = \emptyset$.

$\underset{\dim = 4n-9}{Gr(1, n-2, n+1)} \xrightarrow{\psi} Gr(n-2, n+1) \sim \dim = 3n-6$

$\downarrow \varphi \to$ rel. dim $3n-9$

$Y \subseteq \mathbb{P}^n$

$\dim \varphi^{-1}(Y) = 3n-7$ so $\overline{\psi(\varphi^{-1}(Y))} \subsetneq Gr(n-2, n+1)$

Hence there is a map $\pi : \mathbb{P}^n_F \setminus H_F \to \mathbb{P}^2_F$ which induces a map $p : Y_F \hookrightarrow \mathbb{P}^n_F \setminus H_F \xrightarrow{\pi} \mathbb{P}^2_F$. This map satisfies $p^{-1}(p(y)) \cap Y_{\{0\}} = \{y\}$.

$y \in Y$, $\downarrow q$

So for any $s \in K^M_*(K(Y))$, we have

$$\sum_{\substack{u \in (\mathbb{P}^2_F)^{(1)} \\ p(y) \in \bar{u}}} \partial^u_{p(y)} \circ \partial^{\xi_{\mathbb{P}^2_F}}_u (p_* q^*(s)) = (\underline{d_{\mathbb{P}^2_F} \circ d_{\mathbb{P}^2_F}} (p_* q^*(s)))_{p(y)}$$

$$= (p_* q^*(d_Y \circ d_Y(s)))_{p(y)}$$

$$= K^M_*(F/k) \circ (\underline{d_Y \circ d_Y}(s))_y .$$

But $K^M_*(F/k)$ is injective since F/k is purely transcendental. $\quad K^M_*(k) \hookrightarrow K^M_*(k(t))$

3). The statement of $\overset{X}{\mathbb{A}^2_{(0,0)}} = (s,t)$.

We have an exact sequence by Thm 3.4

$$0 \to \underline{K^M_*(k(s)) \xrightarrow{i} K^M_*(k(s,t))} \xrightarrow{(\tau_x)} \bigoplus_{x \in \mathbb{A}^1_{k(s)}} K^M_{*-1}(k(x)) \to 0$$

The τ_x could be obtained by

$$\tau_x(a) = N_{k(x)(t)/k(s,t)}\big([t - t(x)] \cdot K^M_{*-1}(k(x)(t)/k(x))(a) \big).$$

$a \in K^M_{*-1}(k(x))$; $t(x)$ is the generater of $k(x)/k(s)$

One checks that $\underset{y \in \mathbb{A}^1_{k(s)}}{\partial_y} \circ \tau_x = \begin{cases} \text{id} & \text{if } y = x \\ 0 & \text{else} \end{cases}$, since $[t - t(x)] K^M_*(k(x)(t)/k(x))(a)$

has nonzero valuation only at $t - t(x)$, which lies over the valuation of x in $k(s)$.

Suppose $b \in K^M_*(k(s,t))$, $y = (s) \in X = \mathbb{A}^2$. If $b \in \operatorname{Im}(i)$, then $d_X \circ d_X(b) = 0$ by the naturality of pull-back along $p : \mathbb{A}^2 \to \mathbb{A}^1$ $\quad p^* : k(s) \to k(s,t)$).

$(s,t) \mapsto (s)$ $\quad i = K^M_*(p^*)$

$(\dim \mathbb{A}^1 = 1)$

So assume $b = \tau_x(a)$. Then it suffices to prove

$$\boxed{\partial^y_0 \circ \partial^{\xi_{\mathbb{A}^2}}_y \circ \tau_x = -\partial^x_{(0,0)}} \iff \underline{\sum_{0 \in y} \partial^y_0 \circ \partial^{\xi_{\mathbb{A}^2}}_y = 0}$$

here $x \in \mathbb{A}^1_{k(s)}$ corresponds to a divisor $\bar{x}$ in $\mathbb{A}^2$ different from $\{s = 0\}$.

Details are left as HW.

divisors in $\mathbb{A}^2 = \begin{cases} \{s = 0\} \\ \text{a point } x \in \mathbb{A}^1_{k(s)} \end{cases}$ $\quad \square$

§4. Comparison Theorems, I.

In this section we are going to compute $H^{n,n}(\operatorname{Spec} F, \mathbb{Z})$, F a field.

Prop 4.1 We have $H^{p,q}(\operatorname{Spec} F, \mathbb{Z}) = H_{q-p}(C_* \mathbb{Z}(\mathbb{G}_m^{\wedge q})(\operatorname{Spec} F))$ for all p, q.

In particular

$$H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) = \operatorname{coker}\big(\operatorname{Cor}_k(\mathbb{A}^1, \mathbb{G}_m^{\wedge n}) \xrightarrow{\partial_0 - \partial_1} \underline{\operatorname{Cor}_k(\operatorname{Spec} F, \mathbb{G}_m^{\wedge n})} \big).$$

$\quad T_{0,1}$

Proof. $H^{p,q}(\operatorname{Spec} F, \mathbb{Z}) = \underline{H^p(\operatorname{Spec} F, C_* \mathbb{Z}(\mathbb{G}_m^{\wedge q})[-q])} \overset{p_t}{=} H^p((C_* \mathbb{Z}(\mathbb{G}_m^{\wedge q})[-p])(\operatorname{Spec} F))$

$G \mapsto G(\operatorname{Spec} F)$ is an exact functor

$= H_{q-p}(C_* \mathbb{Z}(\mathbb{G}_m^{\wedge q})(\operatorname{Spec} F))$. $\square$

$(\mathbb{G}_m^{\times n})_E \to (\mathbb{G}_m^{\times n})_F$

Suppose E/F is a finite extension. The push forward of cycles give a map $N_{E/F} : H^{n,n}(\operatorname{Spec} E, \mathbb{Z}) \to H^{n,n}(\operatorname{Spec} F, \mathbb{Z})$ by Prop 4.1.

Prop 4.2. Suppose $x \in H^{n,n}(\operatorname{Spec} E, \mathbb{Z})$, $y \in H^{m,m}(\operatorname{Spec} F, \mathbb{Z})$ We have

(1) $N_{E/F}(y_E \cdot x) = y \cdot N_{E/F}(x)$, $\quad N_{E/F}(x \cdot y_E) = N_{E/F}(x) \cdot y$.

(2) If $F \subseteq E \subseteq K$, K is normal over F, we have

$$N_{E/F}(x)_K = [E:F]_{insep} \cdot \sum_{j \in \operatorname{Hom}(E,K)} j(x).$$

(3) If $F \subseteq E' \subseteq E$, then $N_{E/F} = N_{E'/F} \circ N_{E/E'}$.

Proof: The $\underline{\mathbb{Z}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F)}$ is the quotient of free abelian group generated by closed points in $\mathbb{G}_m^{\times n}$ by those points of the form $(x_1, \cdots, 1, \cdots x_n)$.

The exterior product: $\operatorname{Cor}_k(\operatorname{Spec} F, \mathbb{G}_m^{\times n}) \times \operatorname{Cor}_k(\operatorname{Spec} F, \mathbb{G}_m^{\times m}) \to \operatorname{Cor}_k(\operatorname{Spec} F, \mathbb{G}_m^{\times n + m})$

$(c, D) \mapsto c \times D$

gives a ring structure on $\bigoplus_n H^{n,n}(\operatorname{Spec} F, \mathbb{Z})$.

(1) This comes from the projection formula of cycles (Prop 1.22)

(2). We have a Cartesian square

$$\begin{array}{ccc} (\mathbb{G}_m^{\times n})_E & \longleftarrow & (\mathbb{G}_m^{\times n}) \otimes_F (E \otimes_F K) \\ \downarrow & & \downarrow \\ (\mathbb{G}_m^{\times n})_F & \longleftarrow & (\mathbb{G}_m^{\times n})_K \end{array}$$

$$\begin{array}{ccc} K^M_*(E) & \longrightarrow & \bigoplus_{P \in \operatorname{Spec} E \otimes_F K} K^M_*(k(P)) \\ \downarrow N & & \downarrow \sum N_{k(P)/K} \\ K^M_*(F) & \longrightarrow & K^M_*(K) \end{array}$$

We have a similar property like Prop 3.9. Then proceeds as Prop 3.11.

(3). Follow from the transitivity of push-forward of cycles. $\square$

Prop 4.3 Let $p : Z \to \mathbb{A}^1_F$ by a finite surjective morphism and suppose that Z is integral. Let $f_1, \cdots, f_n \in \mathcal{O}^*(Z)$ and

$$p^*(0) = \sum \underset{\mathbb{N}}{n_i^0} \cdot \underset{\text{in } Z}{z_i^0}, \quad p^*(1) = \sum \underset{\mathbb{N}}{n_i^1} \cdot \underset{\text{in } Z}{z_i^1}$$

in $Z_0(Z)$. Define for each $t = 0, 1$

$$\varphi_t = \sum_i n_i^t N_{k(z_i^t)/F}([f_1] \cdots [f_n]|_{z_i^t}) \in K_n^M(F)$$

Then $\varphi_0 = \varphi_1 \in K_n^M(F)$.