Prop A: Let K be a complete discrete valuation field and let K' be a finite normal extension of K of prime degree p. Let k (resp. k') be the residue field of K (resp. K'). Then the following diagram commutes:

$$\begin{tikzcd} K_n^M(K') \arrow[r, "\partial_{K'}"] \arrow[d, "N_{K'/K}"] & K_{n-1}^M(k') \arrow[d, "N_{k'/k}"] \\ K_n^M(K) \arrow[r, "\partial_K"] & K_{n-1}^M(k) \end{tikzcd}$$

Proof: We show that the map $\delta_{K'/K} = \partial_K \circ N_{K'/K} - N_{k'/k} \circ \partial_{K'}$ is equal to zero.
We show that at first $p\delta_{K'/K} = 0$.
Suppose that K'/K is unramified, i.e. $e_{K'/K} = 1$. If K'/K is separable, then k'/k is normal by [Serre, Chp I, §7, Prop 20]. If, in addition, k'/k is separable, then $Gal(K'/K) = Gal(k'/k)$ and we have

$$\begin{aligned} K_*^M(k'/k) \circ \delta_{K'/K} &= K_*^M(k'/k)(\partial_K \circ N_{K'/K} - N_{k'/k} \circ \partial_{K'}) \\ &= \partial_{K'} \circ K_*^M(K'/K) \circ N_{K'/K} - K_*^M(k'/k) \circ N_{k'/k} \circ \partial_{K'} \quad \text{by } e_{K'/K} = 1 \\ &= \sum_{\sigma \in Gal(K'/K)} \partial_{K'} \circ \sigma - \sum_{\bar\sigma \in Gal(k'/k)} \bar\sigma \circ \partial_{K'} \quad \text{by Prop 3.11} \\ &= 0 \quad (\sigma \text{ doesn't change valuation}) \end{aligned}$$

If, instead, k'/k is purely inseparable, we find as above that

$$K_*^M(k'/k) \circ \delta_{K'/K} = \sum_{\sigma \in Gal(K'/K)} \partial_{K'} \circ \sigma - p\,\partial_{K'} \quad (\text{Prop } 3.11)$$

But $\sigma \in Gal(K'/K)$ induces identity map of k' since k'/k is purely inseparable, so the expression above is zero.

If K'/K is purely inseparable, then k'/k is also purely inseparable ([Serre, Chp I, §6, Prop 16]), so the same arguments as above shows

$$K_*^M(k'/k) \circ \delta_{K'/K} = p\partial_{K'} - p\partial_{K'} = 0.$$

Finally since $N_{k'/k} \circ K_*^M(k'/k) = p \cdot Id$, we've proved that $p \cdot \delta_{K'/K} = 0$.

If K'/K is totally ramified, i.e. $e_{K'/K} = p$. If K'/K is Galois, then

$$\begin{aligned} p \cdot \delta_{K'/K} &= p\partial_K \circ N_{K'/K} - p\partial_{K'} \quad (k' = k) \\ &= \partial_{K'} \circ K_*^M(K'/K) \circ N_{K'/K} - p \cdot \partial_{K'} \quad (\text{by } e_{K'/K} = p) \\ &= \sum_{\sigma \in Gal(K'/K)} \partial_{K'} \circ \sigma - p \cdot \partial_{K'} \quad (\text{by Prop } 3.11) \\ &= 0 \quad (\text{by } k' = k). \end{aligned}$$

If K'/K is purely inseparable, then

$$\begin{aligned} p \cdot \delta_{K'/K} &= \partial_{K'} \circ K_*^M(K'/K) \circ N_{K'/K} - p \cdot \partial_{K'} \\ &= p \cdot \partial_{K'} - p\partial_{K'} = 0. \quad (\text{by Prop } 3.11) \end{aligned}$$

So we have proved that $p\delta_{K'/K} = 0$.

It suffices to show that, for every $z \in K_n^M(K')$, there exists an m prime to p s.t.

$$\left.\begin{aligned} m\,\delta_{K'/K}(z) &= 0 \\ p\,\delta_{K'/K}(z) &= 0 \end{aligned}\right\} \Rightarrow \delta_{K'/K}(z) = 0$$

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr] & \\ L \arrow[dr] & & K' \arrow[dl] \\ & L' & \end{tikzcd} \qquad L \otimes_K K' = L'$$

Suppose that L is an extension of K of degree prime to p and $L' = [L, K']$ be the field generated by L, K' in $\bar K$. So $[L' : L] = p$.
By Prop 3.9, the following diagram commutes

$$\begin{tikzcd} K_n^M(K') \arrow[r] \arrow[d, "N_{K'/K}"] & K_n^M(L') \arrow[d, "N_{L'/L}"] \\ K_n^M(K) \arrow[r] & K_n^M(L) \end{tikzcd}$$

Moreover,

$$\left\{\begin{aligned} & e_{L'/L} \cdot [k(\mathcal{O}_{L'}) : k(\mathcal{O}_L)] = p \\ & e_{K'/K} \cdot [k(\mathcal{O}_{K'}) : k(\mathcal{O}_K)] = p \\ & e_{L'/L} \cdot e_{L/K} = e_{L'/K} = e_{K'/K} \cdot e_{L'/K'} \end{aligned}\right. \Rightarrow \left\{\begin{aligned} & e_{L'/L} = e_{K'/K} \\ & k(\mathcal{O}_L) \otimes_k k' = k(\mathcal{O}_{L'}). \end{aligned}\right.$$

Hence we have a commutative diagram $\boxed{\text{Prop } 3.9}$

$$\begin{tikzcd} K_n^M(k') \arrow[r] \arrow[d, "N_{k'/k}"] & K_n^M(k(\mathcal{O}_{L'})) \arrow[d, "N_{k(\mathcal{O}_{L'})/k(\mathcal{O}_L)}"] \\ K_n^M(k) \arrow[r] & K_n^M(k(\mathcal{O}_L)) \end{tikzcd}$$

We now fix an element $z \in K_n^M(K')$. Then we have

$$\begin{aligned} e_{L/K} \cdot K_*^M(k(\mathcal{O}_L)/k) \circ \delta_{K'/K} &= e_{L/K} \cdot K_*^M(k(\mathcal{O}_L)/k) \circ (\partial_K \circ N_{K'/K} - N_{k'/k} \circ \partial_{K'}) \\ &= \partial_L \circ K_*^M(L/K) \circ N_{K'/K} - e_{L/K} N_{k(\mathcal{O}_{L'})/k(\mathcal{O}_L)} \circ K_*^M(k(\mathcal{O}_{L'})/k') \circ \partial_{K'} \\ &= \partial_L \circ K_*^M(L/K) \circ N_{K'/K} - N_{k(\mathcal{O}_{L'})/k(\mathcal{O}_L)} \circ \partial_{L'} \circ K_*^M(L'/K') \\ &= \partial_L \circ N_{L'/L} \circ K_*^M(L'/K') - N_{k(\mathcal{O}_{L'})/k(\mathcal{O}_L)} \circ \partial_{L'} \circ K_*^M(L'/K') \\ &= \delta_{L'/L} \circ K_*^M(L'/K') \quad (*) \end{aligned}$$

$N_{k(\mathcal{O}_L)/k}$

We claim that for a given $z \in K_n^M(K')$, there exist L/K s.t. the $(*)(z) = 0$. Then by applying $N_{k(\mathcal{O}_L)/k}$, we obtain $[L : K] \cdot \delta_{K'/K}(z) = 0$, hence we are done.

It remains to prove the claim. Suppose $\bar L$ is the algebraic extension of K obtained in Prop 3.8 w.r.t. p. Then $K' \otimes_K \bar L$ is also a field. Then $K_*^M(K' \otimes_K \bar L / K')(z)$ can be written as $\sum [x][y_2] \cdots [y_n]$ where $x \in K' \otimes_K \bar L$, $y_i \in \bar L$, by similar statements in Prop 3.10.
So there is a $K \subseteq L \subseteq \bar L$, $p \nmid [L : K]$, s.t. $K_*^M(L'/K')(z) = \sum [x][y_2] \cdots [y_n]$, $L' = K' \otimes_K L$, $x \in L'$, $y_i \in L$.
Hence we may assume that

$$z = \sum [x][y_2] \cdots [y_n], \ x \in K' \text{ and } y_i \in K.$$

One discusses the cases when K'/K is $\left\{\begin{array}{l}\text{totally ramified} \\ \text{unramified}\end{array}\right.$. Details are left as HW.

$(\delta_{K'/K}(z) = 0)$. $\square$

Prop B: Let k be a field and let k' be a finite normal extension of k of prime degree p. Let $F = k(a)$ be a finite extension and suppose that $F' = k'(a)$ is a field. The following diagram commutes:

$$\begin{tikzcd} K_n^M(F') \arrow[r, "N_{a/k'}"] \arrow[d, "N_{F'/F}"] & K_n^M(k') \arrow[d, "N_{k'/k}"] \\ K_n^M(F) \arrow[r, "N_{a/k}"] & K_n^M(k) \end{tikzcd}$$

Proof: Let v be a discrete valuation on $k(t)/k$ and let $k(t)_v$ be the completion of $k(t)$ at v. Since $k(t)_v/k(t)$ is separable, the minimal polynomial $\pi \in k(t)[x]$ of a generator λ of $k'(t)/k(t)$ decomposes as a product

$$\pi = \prod_{w/v} \pi_{w/v}$$

where $\pi_{w/v} \in k(t)_v[x]$ are distinct monic irr. polynomials and w is any valuation on $k'(t)/k'$ extending v. We have following diagram:

$$\begin{tikzcd} K_{n+1}^M(k'(t)) \arrow[r] \arrow[d, "N_{k'(t)/k(t)}"] \arrow[dr, phantom, "\circlearrowleft 3.9"] & \bigoplus_{w/v} K_{n+1}^M(k'(t)_w) \arrow[r, "\oplus \partial_w"] \arrow[d, "\sum N_{k'(t)_w/k(t)_v}"] \arrow[dr, phantom, "\circlearrowleft \text{ Prop A}"] & \bigoplus_{w/v} K_n^M(k(\mathcal{O}_w)) \arrow[d, "\sum N_{k(\mathcal{O}_w)/k(\mathcal{O}_v)}"] \\ K_{n+1}^M(k(t)) \arrow[r] & K_{n+1}^M(k(t)_v) \arrow[r, "\partial_v"] & K_n^M(k(\mathcal{O}_v)) \end{tikzcd}$$

$y' \quad x'$

$y \quad x$

Now let $\pi \in k[t]$ and $\pi' \in k'[t]$ be minimal polynomials of a over k and k', respectively. Given $x' \in K_n^M(F')$, Thm 3.4 shows that there exists $y' \in K_{n+1}^M(k'(t))$

$F' = k'(a) = k(\mathcal{O}_{\pi'})$

$k' \subseteq k'(t)$

s.t. $\partial_{w_{\pi'}}(y') = x'$ and $\partial_w(y') = 0$ if $w \neq w_{\pi'}, w_\infty$. Then by definition of $N_{a/k'}$,

$$N_{a/k'}(x') = -\partial_{w_\infty}(y').$$

We define $x = N_{F'/F}(x')$ and $y = N_{k'(t)/k(t)}(y')$. The diagram above then shows that $\partial_{v_\pi}(y) = x$, $\partial_v(y) = 0$ if $v \neq v_\pi, v_\infty$, so

$$N_{a/k}(x) = -\partial_{v_\infty}(y).$$

Applying the diagram again for $v = v_\infty$ give

$$\begin{aligned} N_{a/k}(N_{F'/F}(x')) = N_{a/k}(x) = -\partial_{v_\infty}(y) &= -\partial_{v_\infty}(N_{k'(t)/k(t)}(y')) \\ &= -N_{k'/k}(\partial_{w_\infty}(y')) \quad \text{by Prop A.} \\ &= N_{k'/k}(N_{a/k'}(x')). \ \square \end{aligned}$$