Lecture 7 — 2 April 2022

(1) Review def of R_{eff}.

(2). $Y \subset V \times V$ s.t. $xy \in Y \Leftrightarrow yx \in Y$.

collection of "oriented" edges

(3). Recall given 1-chain $I(x,y) = -I(y,x)$

$$W(I) = \frac{1}{2} \sum_{xy \in Y} r(x,y) \left(I(x,y) \right)^2$$

Consider $z = k(\delta(a) - \delta(b))$ (In particular $k = 1$)

and I solves the Kirchoff's equation's (1) $\partial I = -z$

(2) $\langle I, Z \rangle = 0$ for all finite circles.

Let

Let u be potential at I $\ni$ $r(x,y)\, I(x,y) = \cancel{u(y) - u(x)}$, $u(x) - u(y)$,

Claim: $W(I) = R_{eff}(a,b)$.

Note: $R_{eff}(a,b) = \dfrac{u(a) - u(b)}{K} = u(a) - u(b)$.

$$W(I) = \frac{1}{2} \sum_{xy \in E} r(x,y) \left(I(x,y)\right)^2 = \frac{1}{2} \sum_{xy \in E} \overset{(u(x)-u(y))}{\left(\cancel{u(y) - u(x)}\right)} I(x,y)$$

$$= \sum_{x \in V} u(x) \underbrace{\left(\sum_{y \in V(x)} I(x,y) \right)}_{} \qquad (\partial I)(x) = \sum_{y \in V(x)} I(x,y)$$

$$= u(a) - u(b) \qquad (\because K = 1).$$

$$= R_{eff}(a,b).$$

Remark: Thomson's principle gives upper bound for $R_{eff}(a,b)$.

Pick any 1-chain $\tilde{I}$ s.t. $\partial \tilde{I} = \delta_a - \delta_b$. (Might fail 2nd Kirchoff's eq.)

Then $W(\tilde{I}) \geq R_{eff}(a,b)$.

<u>Thm</u> (Dirichlet's Principle).

Take $a, b \in V$. We consider

$$u(a) = 1$$
$$u(b) = 0$$

u harmonic on $V \setminus \{a,b\}$.

Then,

$$D(u) := \frac{1}{2}\sum_{[x,y]\in E} c(x,y)\big(u(x)-u(y)\big)^2 \le D(\hat{u})$$

where $\hat{u} : V \to \mathbb{R}$ s.t. $\hat{u}(a) = 1$, $\hat{u}(b) = 0$.

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pf: Consider $d : V \to \mathbb{R}$ which $d = \hat{u} - u$. $\Rightarrow \begin{cases} d(a) = 0 \\ d(b) = 0 \end{cases}$

$$\frac{1}{2}\sum_{[x,y]\in E} c(x,y)\big(u(x)-u(y)\big)\cdot\big(d(x)-d(y)\big) = \sum_{\substack{x=a \\ \text{and } x=b}} d(x) \sum_{y\in V(x)} c(x,y)\big(u(x)-u(y)\big)$$

$$= 0.$$

$$D(\hat{u}) = D(u+d) = D(u) + D(d) + \cancelto{0}{\frac{1}{2}\sum c(x,y)\big(u(x)-u(y)\big)\big(d(x)-d(y)\big)}$$

$$\Rightarrow \quad D(\hat{u}) \ge D(u)$$

and equality holds iff d is constant $\Leftrightarrow d \equiv 0$.

$\Leftrightarrow \tilde{u} \equiv u$

In this case, $u(a) = 1$, $u(b) = 0$ $\Rightarrow$ voltage.

$(V = RI)$ $\Rightarrow$ $1 = u(a) - u(b) = I_{tot} R_{eff}(a,b)$.

$\Rightarrow$ $R_{eff}(a,b) = \dfrac{1}{I_{tot}}$

Let I 1-chain be current in Γ associated to u.

$$\Rightarrow \quad \partial I(x) = \sum_{y \in V(x)} c(x,y)\big(u(x) - u(y)\big) = \begin{cases} 0 & \forall x \in V - \{a,b\}, \\ \kappa & x = a \\ -\kappa & x = b. \end{cases}$$

$$\Rightarrow \quad \cancel{I_{tot}} = \kappa(\delta_a - \delta_b).$$

Claim: $D(u) = \dfrac{1}{R_{eff}(a,b)}$

$$D(u) = \frac{1}{2} \sum c(x,y) \big(u(x)-u(y)\big)^2$$

$$= \frac{1}{2} \sum \big(u(x)-u(y)\big) I(x,y)$$

$$= \sum_{x \in V} u(x) \underbrace{\left(\sum_{y \sim x} I(x,y) \right)}_{= \partial I(x)}$$

$$= \chi \big(u(a) - u(b)\big)$$

$$= \chi = \frac{1}{R_{eff}(a,b)}$$

Pick any $\tilde{u} : V \to \mathbb{R}$ s.t. $\tilde{u}(a) = 1$, $\tilde{u}(b) = 0$,

$$D(\tilde{u}) \geq D(u) = \frac{1}{R_{eff}(a,b)},$$

$$\Rightarrow \quad R_{eff}(a,b) \geq \frac{1}{D(\tilde{u})}$$

Dirichlet's principle gives lower bound of $R_{eff}(a,b)$.

Currents and potentials with finite energy

Def (Γ, r) $\quad I$ 1-chain

$$W(I) = \frac{1}{2} \sum_{xy \in Y} r(xy) \left(I(xy)\right)^2$$

$$\| I \|_1 = \left(W(I)\right)^{\frac{1}{2}}$$

H_1 linear space of 1-chains I with finite energy $W(I) < \infty$.

Ex! H_1 Hilbert space.

$$\langle I, \tilde{I} \rangle = \frac{1}{2} \sum_{xy \in Y} r(x,y) \, I(x,y) \, \tilde{I}(x,y)$$

Check: complete metric space.

Def : j 0-chain,

$$\|j\|_0^2 = \sum_{x \in V} \frac{1}{c(x)} (j(x))^2$$

H_0 Hilbert space := space of 0-chain j s.t. $\|j\|_0^2 < \infty$.

Prop : $\partial : H_1 \to H_0$ is continous,

(Recall: ∂ linear is continuos $\Leftrightarrow$ ∂ is bounded

$\Leftrightarrow$ $\displaystyle \|\partial\| := \max_{I \in H_1} \frac{\|\partial I\|_0}{\|I\|_1} < \infty$)

Pf : Given $I \in H_1$,

$$\|\partial I\|_0^2 = \sum_{x\in V} \frac{1}{c(x)} \left(\sum_{y\in V(x)} I(x,y) \right)^2$$

$$= \sum_{x\in V} \frac{1}{c(x)} \left(\sum_{y\in V(x)} \sqrt{c(x,y)} \frac{I(x,y)}{\sqrt{c(x,y)}} \right)^2$$

(Cauchy-Schwartz ineq. :
$\{a_i\}$ $\{b_i\}$,
$\left(\sum a_i b_i\right)^2 \le \sum a_i^2 \sum b_i^2$
$\parallel$
$\langle a, b\rangle^2 \le \|a\|^2 \cdot \|b\|^2$)

$$\le \sum_{x\in V} \frac{1}{c(x)} \left(\underbrace{\left[\sum_{y\in V(x)} \left(\sqrt{c(x,y)}\right)^2 \right]}_{\parallel \atop c(x)} \left(\sum_{y\in V(x)} \frac{(I(x,y))^2}{c(x,y)} \right) \right)$$

$$= \sum_{x\in V} \sum_{y\in V(x)} r(x,y) \left(I(x,y) \right)^2$$

$$= 2 \|I\|_1^2$$

$$\Rightarrow \|\partial\| = \max_{I\neq 0} \frac{\|\partial I\|_0}{\|I\|_1} \le \sqrt{2}$$

$\Rightarrow \quad \partial$ is bounded

$\Rightarrow \quad \partial$ is continous.

Remark: Suppose $u: V \to \mathbb{R}$ s.t. $u(x)-u(y) = I(x,y)\, r(x,y)$.

$$W(I) = \frac{1}{2} \sum r(x,y) \left(I(x,y)\right)^2 = \frac{1}{2} \sum \frac{1}{r(x,y)} \left(u(x)-u(y)\right)^2$$

$$= \frac{1}{2} \sum c(x,y) \left(u(x)-u(y)\right)^2$$

$$= D(u).$$

Def: For every $u: V \to \mathbb{R}$, the number $D(u)$ Dirichlet energy of u. If $D(u) < \infty$, we say u Dirichlet finite

Kirchoff's eq.

~~Discret Poisson's eq:~~

Given z 0-chain in H_0, find $I \in H_1$

s.t.

$$\partial I + z = 0$$

$$\langle I, Z \rangle = 0 \quad \text{for all finite cycles } Z,$$

It has at most 1 soln I $\Leftrightarrow$ all harmonic functions with finite energy are constant,

In terms of u, Poisson's eq:

$$(Id - P)u = f \qquad \text{where } f(x) = \frac{-z(x)}{c(x)}.$$

$c f = \eta \in H_0 \iff \infty > \sum_{x \in V} \frac{(\eta(x))^2}{c(x)} = \sum_{x \in V} c(x) (f(x))^2$

Denote l^2 to be functions $f: V \to \mathbb{R}$

s.th. $\sum_{x \in V} c(x) (f(x))^2 < \infty$.

(H_0 isomorphic to l^2)

Prop: $P: l^2 \to l^2$ continous

and $\|P\| \leq 1$

pf: Let $f \in \ell^2$.

$P|f| \in \ell^2$?

Check:
$$\sum_{x \in V} c(x) \left(\sum_{y \in V(x)} p(x,y)\, |f| \right)^2 = \sum_{y,z \in V} |f(z)|\,|f(y)|\, w(y,z)$$

(where $w(y,z) = \sum_{x \in V} c(x)\, p(x,y)\, p(x,z)$ $= w(z,y)$)

$$= \sum \left(|f(z)| \sqrt{w(y,z)} \right) \left(|f(y)| \sqrt{w(y,z)} \right)$$

$$\leq \sqrt{\left(\sum |f(z)|^2 (w(y,z)) \right) \left(\sum |f(y)|^2 w(y,z) \right)}$$

$$= \left(\sum_{y,z \in V} w(y,z)\, |f(y)|^2 \right)$$

$$\overset{?}{=} 1 \cdot \sum_{y} c(y)\, |f(y)|^2 < \infty$$

Consider $\sum_z w(y,z) = \sum_z \left(\sum_x c(x)\, p(x,y)\, p(x,z) \right)$

$$= \sum_x c(x)\, p(x,y)$$

$$= \sum_x c(x,y)$$

$$= \sum_x c(y,x)$$

$$= c(y)$$

$$\sum_x p(y,x)$$

$$p(x,y) = \frac{c(x,y)}{c(x)},$$

But $c(x,y) = c(y,x)$

$\Rightarrow \quad \|P\| \le 1$

<u>Ex:</u> P hermitian, i.e.

$$\langle Pf, \tilde{f} \rangle = \langle f, P\tilde{f} \rangle.$$