There is an split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_*^M(k) \arrow[r, "i"] \arrow[dr, "0"'] & K_*^M(k(T)) \arrow[r, "(\partial_P)"] \arrow[d, "-\partial_\infty"] & \bigoplus_{P \text{ irr. monic}} K_{*-1}^M(k[T]/P) \arrow[r] \arrow[dl, "\Sigma N_P"] & 0 . \\ & & K_{*-1}^M(k) & & \end{tikzcd}$$

$v_\infty(\frac{1}{T})=1$

$$N_P : K_*^M(k[T]/P) \to K_*^M(k) \qquad (N_\infty := id)$$

Def 3.5 Suppose $k(a)/k$ is a finite simple extension where the minimal polynomial of a is P. Define

$$N_P =: N_{a/k} : K_*^M(k(a)) \to K_*^M(k). \qquad K_1^M(k) = k^\times \qquad K_0^M(k) = \mathbb{Z}$$

Def 3.6 Suppose K/k is a finite extension and $K = k(a_1, \cdots, a_r)$.
Define $\quad N_{a_1,\cdots,a_r/k} = N_{a_1/k} \circ N_{a_2/k(a_1)} \circ \cdots \circ N_{a_r/k(a_1,\cdots a_{r-1})}$.

$$N_{a_1\cdots a_r/k} = N_{K/k} \to \text{of fields.}$$
$\to$ degree

Thm 3.7 The $N_{a_1,\cdots a_r/k}$ is independent of the choice of $a_1 \cdots a_r$. Hence we obtain the norm map

$$N_{K/k} : K_*^M(K) \to K_*^M(k).$$

The goal of the following is to prove Thm 3.7.

Prop 3.8 Let k be a field and p be a prime. There exists an algebraic extension L of k s.t. every finite extension of L has order a power of p and s.t. $K_*^M(k)_{(p)} \to K_*^M(L)_{(p)}$ is injective.

Proof: Define a partially ordered set as following

$$S = \left\{ (\alpha, \{L_\beta \mid \beta \le \alpha\}) \;\middle|\; \begin{array}{ll} \alpha \text{ is an ordinal} & \to \text{finite} \ge n \in \mathbb{N} \\ & \to \text{limit } \lim_{\substack{\longrightarrow \\ n\in\mathbb{N}}} n = \aleph_0 \\ p \nmid [L_\beta : k] < \infty & \\ [L_{\beta+1} : L_\beta] > 1 & \beta \text{ finite} \\ L_w = \bigcup_{\alpha < w} L_\alpha & w \text{ is limit} \end{array} \right\}$$

$(\alpha, \{L_\beta \mid \beta \le \alpha\}) \le (\alpha', \{L'_{\beta'} \mid \beta' \le \alpha'\})$ if $\alpha \le \alpha'$
$L_\beta = L_{\beta'}$ if $\beta \le \alpha$.

α

α'

We have $\operatorname{card} \alpha \le \operatorname{card} \bar{k} = \operatorname{card} k$. So S is a set. Every totally ordered subset of S has a maximal element by taking union, so there is a maximal element $(\alpha, \{L_\beta \mid \beta \le \alpha\})$ in S. The $L = L_\alpha$ does not have an extension with order prime to p, hence every finite extension of L has order p^n.
For every simple extension $K(a)/K$, the composite

$$K_*^M(K) \to K_*^M(K(a)) \xrightarrow{N_{a/K}} K_*^M(K)$$

is the multiplication by $[K(a) : K]$ by direct computation.
Hence for any $\beta \le \alpha$, the composite

$$K_*^M(L_\beta)_{(p)} \hookrightarrow K_*^M(L_\alpha)_{(p)} \xrightarrow{N_{a_1\cdots a_r/L_\beta}} K_*^M(L_\beta)_{(p)}$$

is an isomorphism, so $K_*^M(k)_{(p)} \to K_*^M(L)_{(p)}$ is injective by transfinite induction. $\square$

Rmk: Suppose k'/k is a field extension and v' (resp. v) is a discrete valuation on k' (resp. k.) s.t. $v'|_k = v$. Then the diagram

$$\begin{tikzcd} K_*^M(k) \arrow[r, "\partial_v"] \arrow[d] & K_{*-1}^M(k(v)) \arrow[d] \\ K_*^M(k') \arrow[r, "e \cdot \partial_{v'}"] & K_{*-1}^M(k(v')) \end{tikzcd},$$

commutes where e is the ramification index, $\pi_v = u \cdot \pi_{v'}^e$, $u \in \mathcal{O}_{v'}^*$.

Prop 3.9 Let $k' = k(a)$ be a finite extension of k and let P be the minimal polynomial of a. Let L/k be an extension and suppose

$$P = \prod_i P_i^{e_i}$$

is the prime decomposition of P in $L[t]$. Set $k' \subseteq L_i' = L[t]/P_i$ and $a_i = \bar{t} \in L_i$.
We have a commutative diagram

$$\begin{tikzcd} K_*^M(k') \arrow[r, "(e_i)"] \arrow[d, "N_{a/k}"] & \bigoplus_i K_*^M(L_i') \arrow[d, "\Sigma N_{a_i/L}"] \\ K_*^M(k) \arrow[r] & K_*(L) \end{tikzcd}.$$

Proof: Let $f_1 \cdots f_n \in k[t]$ be prime to P. Then

$$\partial_{P_i}([P][f_1]\cdots[f_n]) = e_i [\bar{f_1}] \cdots [\bar{f_n}]$$

So there is a commutative diagram

$$\begin{tikzcd} K_*^M(k(t)) \arrow[r] \arrow[d, "(\partial_P)"] & K_*^M(L(t)) \arrow[d, "(\partial_Q)"] \\ \bigoplus_P K_{*-1}^M(k[T]/P) \arrow[r, "(\operatorname{ord}_Q(P))"] & \bigoplus_Q K_{*-1}^M(L[T]/Q) \end{tikzcd}$$

$\operatorname{ord}_\infty(\infty) = 1$.

So the statement follows from the definition of map $\bigoplus_P K_*^M(k[T]/P) \xrightarrow{N_P} K_*^M(k)$. $\square$

Prop 3.10 Let k be a field and $\overset{k(a)}{\overset{\|}{k'}}/k$ be prime degree. Then the map

$$N_{a/k} : K_n^M(k') \to K_n^M(k)$$

is independent of the choice of the generator a
Proof: Suppose all finite extension of have order p^n. For f, g being monic with same degree, $f = g + h$, $\deg h < \deg f$. If $h = 0$, $[f][g] = [f][-1]$. Otherwise

$$([h]-[f])([g]-[f]) = [\tfrac{h}{f}][\tfrac{g}{f}] = [\tfrac{h}{f}][1-\tfrac{h}{f}] = 0.$$

So $[f][g] = [h][g] - [h][f] + [f][-1]$. This shows that any element in $K_*^M(k')$ is a sum of $[f_1] \cdots [f_n]$ where f_i are irreducible (or constant) and $[k':f_i] \deg f_i \ge \deg f_i$.
But then $f_1, \cdots f_n$ are constant by condition of k'/k. So for any choice of a, we have $N_{a/k}([f_1] \cdots [f_n]) = [\underset{N_{k'/k}}{\underset{\|}{N_{a/k}}}(f_1)][f_2] \cdots [f_n]$, which is independent of a

For general field k, it suffices to show that the map

$$N_{a/k} : K_*^M(k')_{(p)} \to K_*^M(k)_{(p)}$$

does not depend on a for every prime p. By Prop 3.8, $\exists L/k$ s.t. every finite extension of L has degree a power of p and $K_*^M(k)_{(p)} \to K_*^M(L)_{(p)}$ is injective.

Suppose k'/k is seperable. Then $L' = L \otimes_k k'$ = a field or products of L. If L' is a field, by Prop 3.9, there is a commutative diagram (resp. $L' = \prod L$)

$$\begin{tikzcd} K_*^M(k') \arrow[r] \arrow[d, "N_{a/k}"] & K_*^M(L') \arrow[d, "N_{L'/L}"] \\ K_*^M(k) \arrow[r] & K_*^M(L) \end{tikzcd} \qquad \text{resp. } \begin{tikzcd} \bigoplus K_*^M(L) \arrow[d, "\Sigma id"] \\ K_*^M(L) \end{tikzcd}$$

hence $N_{a/k}$ is independent of a.

Suppose k'/k is purely inseparable. So $k' = k[t]/(t^p - a)$. If $a^{\frac{1}{p}} \notin L$, then L' is a field. Otherwise $L' = L[t]/(t - a^{\frac{1}{p}})^p$. The Prop 3.9 is applicable to both cases. $\square$.

Prop 3.11 (Assuming Thm 3.7) Let $k \subseteq k' \subseteq K$ be finite extensions of fields.
$k(a)/k$ in particular
1) For any $x \in K_*^M(k')$, $y \in K_*^M(k)$, $N_{k'/k}(x \cdot y) = N_{k'/k}(x) \cdot y$.

2). If K/k is normal and $x \in K_*^M(k')$, we have

$$N_{k'/k}(x)_K = [k':k]_{insep} \sum_{j : k' \to K} j_*(x).$$

3) $N_{k'/k} \circ N_{K/k'} = N_{K/k}$.
Proof: 1) It suffices to assume $k' = k(a)$ by choosing generators of k'/k and Thm 3.7. Then the statement follows from the constructions in Thm 3.4.
3). Clear from Thm 3.7.
2). If k'/k is separable, $k' = k(a)$ Then Prop 3.9 gives a diagram

$$\begin{tikzcd} K_*^M(k') \arrow[r] \arrow[d, "N_{k'/k}"] & \bigoplus_{j : k' \to K} K_*^M(K) \arrow[d, "(j_*)"] \\ K_*^M(k) \arrow[r] & K_*^M(K) \end{tikzcd},$$

which gives the statement.
If k'/k is purely inseparable, it suffices to assume $k' = k(a)$ and proceed inductively. We have $k(a) \otimes_k K = K[t]/(t - a^{\frac{1}{d}})^d$, $d = [k(a):k]$. Again use 3.9.

For general k'/k, denote by k^s the separable closure of k'/k. The $\operatorname{Hom}(k', K) \to \operatorname{Hom}(k^s, K)$ is an isomorphism.
So $N_{k'/k}(x)_K = N_{k^s/k}(N_{k'/k^s}(x))_K = \sum_{j : k^s \to K} i_*(N_{k'/k^s}(x)) = \sum_{j : k' \to K} i_*(N_{k'/k^s}(x)_{k'})$

$$= [k':k]_{insep} \sum_{j : k' \to K} j_*(x). \quad \square.$$

Def: Suppose K is a field with discrete valuation. Fix $a \in (0,1)$, we define $\|x\| = a^{v(x)}$ for every $x \in K$, which is called the absolute value. The completion $\hat{K}$ of K as a metric space is a field with discrete valuation. If $K = \hat{K}$, we say the valuation is complete.

Recall from [Serre, Local Fields, Chap II, §2] that if K is a complete discrete valuation field and if L a finite extension of K, then the discrete valuation on K extends uniquely to a discrete valuation on L and L is also complete w.r.t. this valuation. Moreover, we have

$$[L:K] = e_{L/K} \cdot [k(\mathcal{O}_L) : k(\mathcal{O}_K)],$$

where $e_{L/K}$ is the ramification index.

Prop A. Let K be a complete discrete valuation field, and Let K' be a finite normal extension of K of prime degree P. Let k (resp. k') be the residue field of K (resp K'). The the following diagram commutes:

$$\begin{tikzcd} K_n^M(K') \arrow[r, "\partial_{K'}"] \arrow[d, "N_{K'/K}"] & K_{n-1}^M(k') \arrow[d, "N_{k'/k}"] \\ K_n^M(K) \arrow[r, "\partial_K"] & K_{n-1}^M(k) \end{tikzcd}$$