Thm 2.28 There is a unique sheafification functor $a: PSh(S) \to Sh(S)$ s.t. the following diagram commutes:

$$\begin{array}{ccc} F \in PSh(S) & \xrightarrow{a} & Sh(S) \ni F_1, F_2 \\ \downarrow & & \downarrow \\ PSh(Sm/S) & \longrightarrow & Sh(Sm/S) \end{array}$$

Proof: Suppose $F_1|_{Sm/S} = F_2|_{Sm/S} = (F|_{Sm/S})^+$, $s \in F_1(Y) = F_2(Y)$ and $T \in Cor_S(X,Y)$. We want to prove $F_1(T) = F_2(T)$. There is a Nis covering $p: U \to Y$ s.t. $s|_U = t^+$, where $t \in F(U)$. Consider the cartesian square
$$\begin{array}{ccc} T_U & \to & X \times U \\ \downarrow & & \downarrow \\ T & \to & X \times Y \end{array}$$
, the T is a disjoint union of spec of Henselians ($X = \operatorname{Spec} A$, A Henselian). So the map $T_U \to T$ has a section $s: T \to T_U$. Denote by $D = \operatorname{Im}(s) \in Cor_S(X,U)$. Then $p \circ D = T$. Then we see that the diagram

$$\begin{array}{ccc} F_1(X) & = & F_2(X) \\ \uparrow F_1(D) & & \uparrow F_2(D) \\ F_1(U) & = & F_2(U) \\ \uparrow F_1(p) & & \uparrow F_2(p) \\ F_1(Y) & = & F_2(Y) \end{array}$$

$F_1(T)$, $F_2(T)$, only for s

commutes. So $F_1(T) = F_2(T)$.

Now we want to make $(F|_{Sm/S})^+$ a sheaf with transfers. Suppose $y \in (F|_{Sm/S})^+(Y)$ and $y|_U = z^+$, where $p: U \to Y$ is a Nis covering and $z \in F(U)$. By shrinking U we may suppose that the z is sent to zero in $F(U \times_Y U)$. We have a commutative diagram

$$\cdots\ \mathbb{Z}(U \times_Y U) \to \mathbb{Z}(U) \to \mathbb{Z}(Y) \to 0 \quad \text{exact (by 2.27)}$$

$$\begin{array}{ccccccc} 0 \to \operatorname{Hom}(\mathbb{Z}(Y), (F|_{Sm/S})^+) & \to & \operatorname{Hom}(\mathbb{Z}(U), (F|_{Sm/S})^+) & \to & \operatorname{Hom}(\mathbb{Z}(U \times_Y U), (F|_{Sm/S})^+) \\ [y] \longmapsto & & z & \longmapsto & 0 \\ & & \uparrow & & \uparrow \\ & & F(U) \ni z & \longmapsto 0 \in & F(U \times_Y U) \end{array}$$

So there is an element $[y]: \mathbb{Z}(Y) \to (F|_{Sm/S})^+$ s.t. $[y]|_U = z$. Then for every $f \in Cor_S(X,Y)$, define the transfer of y w.r.t. f by the composite

$$\mathbb{Z}(X) \xrightarrow{f} \mathbb{Z}(Y) \xrightarrow{[y]} (F|_{Sm/S})^+ \rightsquigarrow F(f)(y) \in (F|_{Sm/S})^+(X). \ \square$$

Prop 2.29 Suppose $X \in Sm/S$ and $\{U_1, U_2\}$ is a Zariski covering of X. We have an exact sequence:

$$\begin{array}{c} (x,y) \longmapsto x+y \\ 0 \to \mathbb{Z}(U_1 \cap U_2) \to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \xrightarrow{+} \mathbb{Z}(X) \to 0. \quad (\text{MV-sequence}) \\ a \longmapsto (a, -a) \end{array}$$

Proof: The $U_1 \amalg U_2$ is a Nisnevich covering of X. Applying Thm 2.27, we obtain an exact sequence

$$\mathbb{Z}(U_1) \oplus \mathbb{Z}(U_1 \cap U_2)^{\oplus 2} \oplus \mathbb{Z}(U_2) \xrightarrow{d} \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \xrightarrow{+} \mathbb{Z}(X) \to 0,$$

where $d(x,y,a,b) = (a-y, y-a)$. So $\operatorname{Im}(d) = \operatorname{Im}(\mathbb{Z}(U_1 \cap U_2) \to \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2))$. $\square$

Def 2.30 Define Sim as a category whose objects are $[n] = \{0, \cdots, n\}$, $n \in \mathbb{N}$, $\operatorname{Hom}([n],[m]) = \{\text{non-decreasing map } [n] \to [m]\}$. For any category $\mathcal{C}$, a (co)simplicial object in $\mathcal{C}$ is a functor $Sim^{op} \to \mathcal{C}$.
$(Sim \to \mathcal{C})$

Def 2.31 For any $n \in \mathbb{N}$, define $\Delta^n = \operatorname{Spec} k[x_0, \cdots, x_n] / (\sum_{i=0}^n x_n) - 1 = \mathbb{A}^n$.

$$\| \qquad \{(x_0, \cdots, x_n) \in \mathbb{A}^{n+1} \mid \sum x_i = 1\}$$

The $\{\Delta^n\} = \Delta^\cdot$ is a cosimplicial object in Sm/k. $f: [n] \to [m]$

$$\operatorname{Hom}(f): \Delta^n \to \Delta^m$$
$$(x_i) \mapsto (y_j)$$
$$y_j = \sum_{f(i)=j} x_i$$

Def 2.32 (Suslin complex) For any $F \in PSh(S)$, define a simplicial object in $PSh(S)$

$$(C_\cdot F)_n = F^{\Delta^n} \qquad F^X(Y) = F(X \times_S Y).$$

It associates to a complex

$$C_* F: \ \to F^{\Delta^n} \xrightarrow{d_n} F^{\Delta^{n-1}} \to \cdots \to F^{\Delta^1} \to F \to 0$$

with $d_n = \sum (-1)^{i-1} \partial_i$, $\partial_i: \Delta^{n-1} \to \Delta^n$ is the i-th face map.

In Thm 2.26, we showed that the cohomology dimension of Nis topology on X is just $\dim(X)$. So for every bounded above complex $C \in C^-(Sh(S))$, we could find an $i: C \xrightarrow{qis} I^\cdot$, where $H^n(X, I^m) = 0$ if $n > 0$, $\forall m$. Then we define $\mathbb{H}^n(X, C) = H^n(I^\cdot(X))$ as the n-th hypercohomology of C w.r.t. X. It is a standard argument to show that $\mathbb{H}^n(X, C)$ is independent of the choice of $I^\cdot$.

Def 2.33 For every $n \in \mathbb{N}$, define the motivic complex

$$\mathbb{Z}(q) = C_* \mathbb{Z}(\mathbb{G}_m^{\wedge q})[-q]$$

where $\mathbb{Z}(\mathbb{G}_m^{\wedge q}) = (\mathbb{Z}(\mathbb{G}_m), 1)^{\wedge q}$ and $\mathbb{Z}(q)^i = C_{q-i} \mathbb{Z}(\mathbb{G}_m^{\wedge q}) = \mathbb{Z}(\mathbb{G}_m^{\wedge q})^{\Delta^{q-i}}$ $(\mathbb{Z}(q) := 0, q < 0)$.
For any group A, we write $\mathbb{Z}(q) \otimes_{\mathbb{Z}} A$ as $A(q)$. $\qquad \mathbb{Z}(q)^i = 0$ if $i > q$

For example, $\mathbb{Z}(0)$ is the constant sheaf $\mathbb{Z} \xrightarrow{qis} C_* \mathbb{Z}$.

Def 2.34 For every $X \in Sm/k$, define

$$H^{p,q}(X, A) = \mathbb{H}^p(X, A(q))$$

to be the $\boxed{\text{motivic cohomology}}$ with coefficients in A.

Prop 2.35 For any $X \in Sm/k$, we have

$$H^{p,q}(X, A) = 0$$

if $p > \dim X + q$.
Proof: Using Lem 1.20, we obtain a spectral sequence

$$H^s(X, \boxed{H^t(A(q))}) \Rightarrow \mathbb{H}^{s+t}(X, A(q)) = H^{s+t,q}(X, A).$$

If $\underset{s+t}{\overset{\|}{p}} > \dim X + q$, either $t > q$ or $s > \dim X$. So $H^s(X, H^t(A(q))) = 0$ in any case. $\square$

§3 Milnor K-theory.

$$\text{K-Theory} \overset{AHSS}{\Leftarrow} \text{Motivic cohomology}$$
$$\cup$$
$$(n,n)\text{-motivic cohomology} \begin{cases} = \text{Milnor K-thy over fields} \\ \text{For general } X, \text{ Gersten complex} \end{cases}$$

$$\mathbb{Z} \oplus F^\times \oplus F^{\times \otimes 2} \oplus \cdots$$
$$\|$$

Def 3.1 For any field F, define $K_*^M(F) = T(F^\times) / \langle x \otimes (1-x) \rangle_{(x \in F^\times \setminus 1)}$ to be the Milnor K-theory of F, which is a graded associative algebra. Denote by $[x]$ the class of $x \in F^\times$ in $K_1^M(F)$.

For example, $K_0^M(F) = \mathbb{Z}$ and $K_1^M(F) = F^\times$.

$$[x] + [y] = [xy]$$

Prop 3.2 1) $[x][y] + [y][x] = 0$ 2) $[x][x] = [x][-1]$
Proof: 1) $[x][-x] = [x]\left[\frac{1-x}{1-x^{-1}}\right] = [x][1-x] + [x^{-1}][1-x^{-1}] = 0$

So $[x][y] + [y][x] = [x][-x] + [x][y] + [y][x] + [y][-y] = [x][-xy] + [y][-xy]$
$$= [xy][-xy] = 0$$

(2) $[x][x] = [x][-1] + [x][-x] = [x][-1]$.

Prop 3.3. Let K be a field and v be a normalized discrete valuation on K. Let $k(v) = \mathcal{O}_v / m_v$ be the residue field. Then $\exists!$ homomorphism

$$\partial_v: K_n^M(K) \to K_{n-1}^M(k(v)) \qquad f \in k[T]$$

s.t. $\forall u_1, \cdots, u_{n-1} \in \mathcal{O}_v^*$ and $x \in K^*$

$$\partial_v([x][u_1] \cdots [u_{n-1}]) = v(x)[\bar{u}_1] \cdots [\bar{u}_{n-1}] \qquad v(f) = \operatorname{ord}_p(f)$$

where $\bar{u}_i$ is the class of u_i in $k(v)^*$.
Proof: The uniqueness is clear. For the existence, choosing a uniformizer π, define a graded ring morphism

$$\theta_\pi: K_*^M(K) \to K_*^M(k(v))[\underset{\deg = 1}{\varepsilon}] / (\varepsilon^2 - \varepsilon[-1])$$
$$[\pi^i u] \mapsto [\bar{u}] + i \cdot \varepsilon$$
$$u \in \mathcal{O}_v^*$$

Then define by equation $\theta_\pi(z) = s_\pi(z) + \partial_v(z) \cdot \varepsilon$. Hence

$$\theta_\pi([\pi^i u][u_1] \cdots [u_{n-1}]) = ([\bar{u}] + i \cdot \varepsilon)[\bar{u}_1] \cdots [\bar{u}_{n-1}] = [\bar{u}][\bar{u}_1] \cdots [\bar{u}_{n-1}] + i \cdot [\bar{u}_1] \cdots [\bar{u}_{n-1}] \varepsilon.$$
$\square$

Thm 3.4 (Homotopy invariance) There is a split exact sequence

$$0 \to K_*^M(k) \xrightarrow{i} K_*^M(k(T)) \xrightarrow{(\partial_p)} \bigoplus_{P \text{ irr monic}} K_{*-1}^M(k[T]/p) \to 0$$

Proof: We have $s_\pi \circ i = \mathrm{id}$. Now we want to construct an isomorphism

$$\bigoplus_{P \text{ irr. monic}} K_n^M(k[T]/p) \xrightarrow{\tau_{n,p}} K_{n+1}^M(k(T)) / K_{n+1}^M(k)_r$$

with inverse (∂_p). We define $\tau_{n,p}$ inductively on the degree of P. Suppose $P = T - \lambda$. We define $\tau_{n,p}$ to be the composite

$$K_n^M(k[T]/p) \xrightarrow{ev_\lambda} K_n^M(k) \xrightarrow{\cdot [p]} K_{n+1}^M(k(T)) / K_{n+1}^M(k)$$

which maps $[\bar{f}_1] \cdots [\bar{f}_n]$ to $[p][f_1(\lambda)] \cdots [f_n(\lambda)]$, $f_1, \cdots, f_n \in k[T]$.
Moreover, $\partial_q \circ \tau_{n,p} = \begin{cases} \mathrm{id} & \text{if } q = p \\ 0 & q \neq p \end{cases}$

For general P and $f_1, \cdots, f_n \in k[T]$ with $\deg(f_i) < \deg P$, we define

$$\tau_{n,p}([\bar{f}_1] \cdots [\bar{f}_n]) = [P][f_1] \cdots [f_n] - \sum_{\deg q < \deg P} \tau_{n,q}(\partial_q([P][f_1] \cdots [f_n])).$$

One checks that it is well-defined and prove that it satisfies

1) $\partial_q \circ \tau_{n,p} = \begin{cases} \mathrm{id} & q = p \\ 0 & q \neq p \end{cases}$ $\qquad$ 2) $\sum_{\deg q < \deg P} \tau_{n,q}(\partial_q(x)) = x$ if $x = \sum [f_{i1}] \cdots [f_{in}]$, $\deg f_{ij} < \deg P$ $\square$