Lecture 5   28 March 2022

Remark:

Recurrent $\Longrightarrow$ All bounded harmonic function are const.

$\not\Longleftarrow$ not true in general

(Pf: ($\Rightarrow$) u bounded harmonic $\Rightarrow$ $\tilde{u}(x) := u(x) - \inf_{v \in V} u \geq 0$

$\tilde{u}$ harmonic $\Rightarrow$ superharmonic

$\Longrightarrow$ $\tilde{u}$ constant $\Rightarrow$ u constant)

Transient $\Longleftarrow$ Exists non-constant bounded harmonic function.

$\not\Longrightarrow$

Transient $\nearrow$ exists non-constant bdd harmonic $\nearrow$ finite energy + harmonic + bdd

$\searrow$ no non-const finite energy

Transient $\searrow$ all bdd harmonic are const.

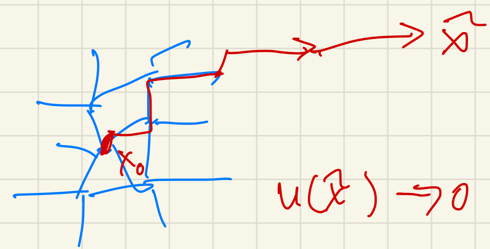
Message: By considering different class of (super-)harmonic functions, we get some feature of the graph (Γ, r).

Recall: (Smooth Theory) On $\mathbb{R}^2$, all bounded harmonic function are constant.

Corollary (Γ, r) finite network. Every superharmonic function is constant.

Corollary (Γ, r) infinite network. If u harmonic on (Γ, r) and $\underbrace{u(x) \to 0 \quad \text{as} \quad x \to \infty}$, then $u \equiv 0$

(We say $\lim u(x) = 0$ as $x \to \infty$ if for every $\varepsilon > 0$ $\{x \in V \mid |u(x)| > \varepsilon\}$ is finite)



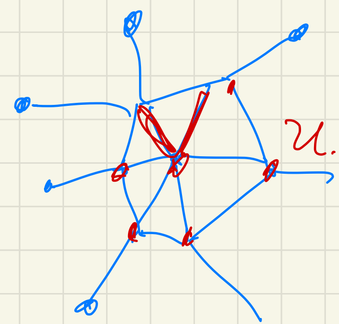
$u(\tilde{x}) \to 0$ as $d(\tilde{x}, x_0) \to \infty$

pf: Suppose $\delta := u(x) \neq 0$ for some $x \in V$.

($\lim u(x) = 0$ as $x \to \infty$) $\Rightarrow \mathcal{U} := \{x \in V \mid |u(x)| > \frac{|\delta|}{2}\}$ is finite

$\Rightarrow$ $u \leq \frac{|\delta|}{2}$ over $V - \mathcal{U}$.

Maximum principle $\Rightarrow$ $u \leq \frac{|\delta|}{2}$ over V.



$\Rightarrow$ Contradiction

$\Rightarrow$ $|\delta| = 0$.

$\Rightarrow$ $u \equiv 0$.

(Baby case for Dirichlet problem:
Prescribe $u \equiv 0$ along "boundary at infinity" $\Rightarrow$ $u \equiv 0$ over interior Γ)

($\Rightarrow$ uniqueness for general case).

Non-negative superharmonic

- transient.

- examples: Green's function $G(\cdot, y)$ for each fixed y.

Q: Every non-negative superharmonic function is a linear combination of Green's function?

Ans: Not in general.

Riesz decomposition Theorem

(Γ, r) transient. Let $u : V \to \mathbb{R}$ non-negative superharmonic

Then there exists unique $f : V \to \mathbb{R}$ and v harmonic

$$u = G(f) + v$$

where ~~and~~ v is the greatest harmonic minorant of u $(*)$.

and v, f are non-negative.

$(*)$. If $\tilde{v}$ is harmonic s.t. $0 \le \tilde{v} \le u$

$$\Rightarrow 0 \le \tilde{v} \le v \le u$$

Pf: $(\mathrm{Id} - P)u \geq 0$

$\Rightarrow \quad u(x) \geq (Pu)(x) \geq (P^2u)(x) \geq \dots \geq 0$

① $v(x) := \lim_{n\to\infty} (P^n u)(x)$ is finite.

② v non-negative. $\left(v := \lim_{n\to\infty} P^n u \right)$.

③ v harmonic. $(Pv)(x) = v(x)$

(dominated - convergence Thm).

(4). (Idea: $(Id-P)u = \cancel{(Id-P)G(f)} + \cancel{(Id-P)v}$)

Define $f := u - Pu \geq 0$

Consider $Pf = Pu - P^2 u$

$\vdots$

$P^n f = P^n u - P^{n+1} u$

$\Rightarrow$ sum over all eqs:

$\sum_{j=0}^{n} P^{(j)} f = L.H.S. = R.H.S. = u - P^{n+1} u$

$$\lim_{n\to\infty} \sum_{j=0}^{n} P^{(j)} f = u - v$$

$$\parallel$$

$$G(f).$$

$$\Rightarrow \quad u = G(f) + v.$$

Since f is unique, v is also unique.

Check: $u \geq v \geq 0 \Rightarrow$ harmonic minorant

greatest ??

Let h harmonic s.t. $u \geq h \geq 0$.

$$P^n u \geq P^n h = P^{(n-1)} h = \dots = h$$

$$\Rightarrow \quad v(x) = \lim_{n \to \infty} (P^n u)(x) \geq h(x)$$

$$\Rightarrow \quad v \geq h$$

Finite network (keyword: Effective resistance)

(Γ, r) finite,

$(\ast)$, $\displaystyle\sum_{y \in V(x)} \frac{1}{r(x,y)} \left(u(y) - u(x)\right) = -i(x) \quad \forall x \in V.$

Sol exists? Unique?

$\Rightarrow$ Linear Algebra

$$\Delta : \mathbb{R}^V \to \mathbb{R}^V$$

$$(\Delta u)_x := \sum_{y \in V(x)} \frac{1}{r(x,y)} \left(u(y) - u(x)\right).$$

$\Delta u = 0 \iff u$ harmonic

$\iff$ (finite network $\Rightarrow$ u achieves max value at interior vertex)

$\iff u$ constant.

$\Rightarrow \dim(\operatorname{Ker}(\Delta)) = 1$

$\dim(\operatorname{Im}(\Delta)) = |V| - 1$

Smooth: $v, u : D \to \mathbb{R}$ $\begin{cases} u|_{\partial D} = 0 \\ v|_{\partial D} = 0 \end{cases}$

$(u, v) := \iint_D (\nabla v \cdot \nabla u)\, dA$

$= \iint_D -\Delta v \cdot u \, dA$

$= \iint_D v \cdot (-\Delta u)\, dA$

Consider (0-cochain) $i : V \to \mathbb{R}$ s.t. $\forall u \in \mathbb{R}^V$,

$0 = \boxed{\sum_{x \in V} i(x)(\Delta u)(x)} = (i, \Delta u)$

u 0-cochain

$\partial^* u$ 1-cochain

$R^{-1}\partial^* u$ 1-chain

$\Delta u = \partial R^{-1}\partial^* u$ 0-chain

$$= (\ i\ ,\ \partial R^{-1}\partial^* u\ )$$

$$= (\ \partial^* i\ ,\ R^{-1}\partial^* u\ )$$

$$= (\ R^{-1}\partial^* i\ ,\ \partial^* u\ )$$

$$= (\ \partial R^{-1}\partial^* i\ ,\ u\ )$$

$$= (\ \Delta i\ ,\ u\ ),$$

$\Rightarrow \quad \Delta i = 0$

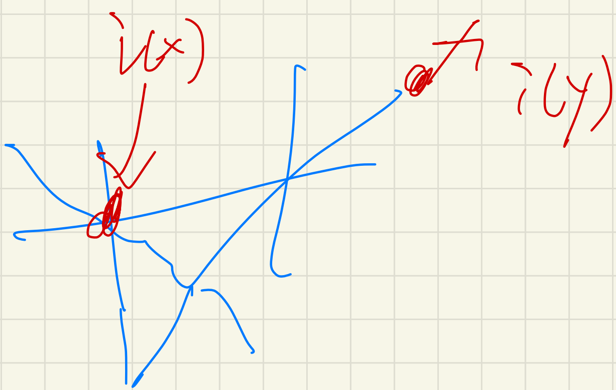
$\Rightarrow \quad i$ constant function.

Hence, $\hat{i} \in$ Image of Δ

$$\Leftrightarrow 0 = \sum_{x \in V} \hat{i}(x) \cdot 1$$

Conclusion: $(*)$ soln exists $\Leftrightarrow$ $\sum_{x \in V} \hat{i}(x) = 0$.

Unique up to constant function.



Thm. (Γ, r) finite. Fix $b \in V$. Then for any $f : (V - \{b\}) \to \mathbb{R}$, there exists a unique u satisfying,

$$(*) \qquad (1-P)\, u(x) = f(x) \qquad \forall x \in V - \{b\},$$

$$u(b) = 0.$$

Plan on Wed: (1). Modify Green's function

(2). $u = G(f)$.

(3) Effective resistance in terms of G.