Recall: Cor_S: $\mathrm{Obj} = Sm/S$, $S \in Sm/k$

$\mathrm{Cor}_S(X,Y) = \mathbb{Z}\{C \subseteq X \times_S Y\}$, $C \to X$ finite surj.

$X \xrightarrow{f} Y \xrightarrow{g} Z$ $f, g \in \mathrm{Cor}_S$

$$\begin{array}{ccc} & & X \times Z \\ & & \uparrow p_{13} \\ & & X \times_S Y \times_S Z \xrightarrow{p_{23}} Y \times_S Z \\ & & \downarrow p_{12} \\ & & X \times_S Y \end{array}$$

$$g \circ f = p_{13*}\left(p_{23}^*(g) \cdot p_{12}^*(f)\right)$$

Prop 2.7 The composition law is associative.

Proof. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ are morphisms in Cor. We have Cartesian squares

$$\begin{array}{ccc} XYZW & \longrightarrow & XZW \\ \downarrow & & \downarrow p^{XZW}_{XZ} \\ XYZ & \xrightarrow{p^{XYZ}_{XZ}} & XZ \end{array} \qquad \begin{array}{ccc} XYZW & \longrightarrow & XYW \\ \downarrow & & \downarrow \\ YZW & \longrightarrow & YW \end{array}$$

Now
$$h\circ(g\circ f) = p^{XZW}_{XW*}\left(p^{XZW*}_{ZW}(h) \cdot p^{XZW*}_{XZ} p^{XYZ}_{XZ*}\left(p^{XYZ*}_{YZ}(g) \cdot p^{XYZ*}_{XY}(f)\right)\right)$$
$$= p^{XZW}_{XW*}\left(p^{XZW*}_{ZW}(h) \cdot p^{XYZW}_{XZW*} p^{XYZW*}_{XYZ}\left(p^{XYZ*}_{YZ}(g) \cdot p^{XYZ*}_{XY}(f)\right)\right)$$
$$= p^{XZW}_{XW*}\left(p^{XZW*}_{ZW}(h) \cdot p^{XYZW}_{XZW*}\left(p^{XYZW*}_{YZ}(g) \cdot p^{XYZW*}_{XY}(f)\right)\right)$$
$$= p^{XZW}_{XW*}\, p^{XYZW}_{XZW*}\left(p^{XYZW*}_{ZW}(h) \cdot p^{XYZW*}_{YZ}(g) \cdot p^{XYZW*}_{XY}(f)\right)$$
$$= p^{XYW}_{XW*}\, p^{XYZW}_{XYW*}\left(\cdots \right)$$
$$= p^{XYW}_{XW*}\left(p^{XYZW}_{XYW*}\left(p^{XYZW*}_{ZW}(h) \cdot p^{XYZW*}_{YZ}(g)\right) \cdot p^{XYW*}_{XY}(f)\right)$$
$$p^{XYZW}_{XYW*}\, p^{XYZW*}_{YZW}\left(p^{YZW*}_{ZW}(h) \cdot p^{YZW}_{YZ}(g)\right)$$
$$= p^{XYW}_{XW*}\left(p^{XYW*}_{YW}\, p^{YZW}_{YW*}\left(\cdots\right) \cdots f\right)$$
$$= (h\circ g)\circ f \qquad \square$$

Thm 2.8 $\mathcal{O}: X \mapsto \mathcal{O}(X)$, $\mathcal{O}^*: X \mapsto \mathcal{O}^*(X)$. We have $\mathcal{O}(g\circ f) = \mathcal{O}(f)\circ\mathcal{O}(g)$, $\mathcal{O}^*(g\circ f) = \mathcal{O}^*(f)\circ\mathcal{O}^*(g)$

$C \subseteq X \times Y \to Y$, $C \to X$ finite surj. $\mathcal{O}(C): \mathcal{O}(Y) \to \mathcal{O}(X)$, $\mathcal{O}(Y) \to \mathcal{O}(C) \xrightarrow{Tr} \mathcal{O}(X)$

Sketch of the proof: (for $\mathcal{O}$) For every $a \in \mathbb{N}$, $X \in Sm/k$, define $\boxed{\mathcal{M}_a(X)} = \bigoplus_{\dim V = a} K(V)$. So we have a pairing $\mathcal{O}(X) \times Z_a(X) \to \mathcal{M}_a(X)$.
$(s, V) \mapsto s|_V$

For $f: X \to Y$, $C \subseteq X$ finite, $s \in K(C)$, we define $f_*(s) = Tr_{K(C)/K(f(C))}(s)$, $f_*(s) \in \mathcal{M}_a(Y)$.

So $\forall C \in \mathrm{Cor}_S(X,Y)$, $s \in \mathcal{M}_a(Y)$, $\mathcal{O}(C)(s) = p_{1*}\left(p_2^*(s)|_C\right)$.
$C \subseteq X \times Y$, $C \to X$ finite surj., $X \times Y \xrightarrow{p_2} Y$, $\downarrow p_1$, X

$f: X \to Y$, $C \to X$. Suppose $D \subseteq Y$ (cycle) and C, $f^{-1}(D)$ intersects properly, $s \in \mathcal{O}(X)$,

one shows that $\boxed{f_*(s|_C)|_D = f_*(s|_{C \cdot f^*D})}$. Moreover,

for diagram like
$$\begin{array}{ccc} XYZ & \xrightarrow{p_{23}} & YZ \supseteq C \\ \downarrow p_{12} & & \downarrow p_1 \\ XY & \xrightarrow{q_2} & Y \end{array}$$
one shows that for every $s \in \mathcal{O}(Y\times Z)$, $C \in \mathrm{Cor}(Y,Z)$, we have
$$q_2^*\left(p_{1*}(s|_C)\right) = p_{12*}\left(p_{23}^*(s)|_{p_{23}^*(C)}\right).$$
Finally perform formal calculation. $\square$

Def 2.9. Suppose $F_1, F_2, G \in Psh(S)$. A bilinear function $\varphi: F_1 \times F_2 \to G$ is a collection of bilinear maps
$$\varphi_{X_1,X_2}: F(X_1) \times F(X_2) \to G(X_1 \times_S X_2)$$
for every $X_1, X_2 \in Sm/S$, s.t. $\forall f_i \in \mathrm{Cor}_S(X_i, X_i')$, $i=1,2$, the following diagram commutes

$$\begin{array}{ccc} F_1(X_1') \times F_2(X_2) & \xrightarrow{\varphi_{X_1',X_2}} & G(X_1' \times_S X_2) \\ \downarrow F(f_1)\times Id & & \downarrow G(f_1 \times Id) \\ F_1(X_1) \times F_2(X_2) & \xrightarrow{\varphi_{X_1,X_2}} & G(X_1 \times_S X_2) \end{array}$$
(Same for f_2).

Def 2.10. Define $F_1 \otimes F_2$ to be the presheaf satisfying
$$\mathrm{Hom}(F_1 \otimes F_2, G) = \{\text{Bilinear functions } F_1 \times F_2 \to G\}$$
for every G.

Prop 2.11 The $F_1 \otimes F_2$ exists.

Proof: For every $Z \in Sm/S$, define
$$(F_1 \otimes F_2)(Z) = \bigoplus_{X, Y \in Sm/S} F_1(X) \otimes F_2(Y) \otimes \mathrm{Cor}_S(Z, X \times Y)/\sim,$$
$\varphi \in F_1(X)$, $\psi \in F_2(Y)$

where $\sim$ is the subgroup generated by
$$\varphi \otimes \psi \otimes (f \times Id_Y)\circ h - f^*(\varphi) \otimes \psi \otimes h: \quad f \in \mathrm{Cor}_S(X', X),\ \varphi \in F_1(X),\ \psi \in F_2(Y),\ h \in \mathrm{Cor}_S(Z, X' \times Y)$$
$$\varphi \otimes \psi \otimes (Id_X \times g)\circ h - \varphi \otimes g^*(\psi) \otimes h: \quad g \in \mathrm{Cor}_S(Y', Y),\ \varphi \in F_1(X),\ \psi \in F_2(Y),\ h \in \mathrm{Cor}_S(Z, X \times Y') \quad \square$$

$\bigoplus \mathbb{Z}(Y_j) \to \bigoplus \mathbb{Z}(X_i) \to F \to 0$ $\quad \boxed{\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y)}$

Def 2.12 A pointed presheaf (F, x) is a split injective map $\mathbb{Z} \xrightarrow{x} F$ where $F \in Psh(S)$. For two pointed presheaves $(F_1, x_1), (F_2, x_2)$, define
$$F_1 \wedge F_2 = \frac{F_1 \otimes F_2}{(F_1 \otimes x_2) \oplus (x_1 \otimes F_2)}.$$
$(\mathbb{Z}(\mathbb{G}_m), 1)$ $\quad \mathbb{Z}(\mathbb{G}_m^{\wedge a}) = (\mathbb{Z}(\mathbb{G}_m), 1)^{\wedge a}$

Prop 2.13. $\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y)$, $(F, x) \otimes (G, y) = (F \wedge G, x \otimes y)$. $\square$

Def 2.14. For any $F \in Psh(S)$ and $X \in Sm/S$, define $F^X \in Psh(S)$ by $F^X(Y) = F(X \times_S Y)$. For every $F, G \in Psh(S)$, define $\underline{\mathrm{Hom}}(F, G) \in Psh(S)$ by $\underline{\mathrm{Hom}}(F, G)(X) = \mathrm{Hom}(F, G^X)$.

Prop 2.15 $\mathrm{Hom}(F \otimes G, H) = \mathrm{Hom}(F, \underline{\mathrm{Hom}}(G, H))$. $\square$

$G \in Psh(S)$. $\mathrm{Hom}(\mathbb{Z}(X) \otimes \mathbb{Z}(Y), G) = \mathrm{Hom}(\mathbb{Z}(X), \underbrace{\underline{\mathrm{Hom}}(\mathbb{Z}(Y), G)}_{= G^Y}) = G(X \times_S Y) = \mathrm{Hom}(\mathbb{Z}(X \times_S Y), G)$

Let us give a brief introduction to Nisnevich topology. (See Étale cohomology. J. Milne, §3, §4).

Def. 2.16 Suppose $f: Y \to X$ is a morphism of finite type.
1) It is called unramified if $\forall y \in Y$, $m_{f(y)} \cdot \mathcal{O}_{Y,y} = m_y$ and $k(y)/k(f(y))$ is a finite separable field extension.
2). It is called étale if it is both flat and unramified.
3). It is called Nisnevich if it is étale and for every $x \in X$, there is a $y \in Y$ s.t. $k(y) = k(x)$.

Ex: Show that if f is Nisnevich, X is locally Noetherian, $\{y \in Y \mid k(y) = k(f(y))\}$ is open in Y.

Def 2.17 A morphism $f: Y \to X$ is called étale (Nisnevich) covering if it is étale (Nisnevich) and surjective.

Def 2.18 Suppose $F \in PSh(S)$. We say that it is a Nisnevich sheaf with transfers if for any $X \in Sm/S$ and Nis covering $\pi: Y \to X$, the sequence
$$0 \to F(X) \xrightarrow{\pi^*} F(Y) \xrightarrow{p_1^* - p_2^*} F(Y \times_X Y)$$
is exact. The category of Nis sheaves with transfers is denoted by $Sh(S)$.

Def 2.19 A local ring A is called Henselian (e.g. complete local rings) if $\forall$ monic $f \in A[T]$ s.t. $\bar{f} = g_0 h_0$ in $k(A)(T)$, where g_0, h_0 are monic and relatively prime, there are monic $g, h \in A[T]$ s.t. $\bar{g} = g_0$, $\bar{h} = h_0$ and $f = gh$.

Thm. 2.20. Let A be a local ring, $X = \mathrm{Spec}\, A$, and $x \in X$ be the closed point. TFAE:
1) A is Henselian
2). Any finite A-algebra B is a direct product of local rings, $B = \prod B_i$. Each B_i is the form B_{m_i}, $m_i \in \mathrm{man}\,\mathrm{Spec}\, B$.
3). If $f: Y \to X$ has finite fibers and separated, $Y = Y_0 \amalg Y_1 \amalg \cdots Y_n$, where $x \notin f(Y_0)$, Y_i $(i \geq 1)$ is finite over X and is the Spec of a local ring.
4). If $f: Y \to X$ is étale and $\exists y \in Y$ s.t. $f(y) = x$ and $k(y) = k(x)$, then f has a section $s: X \to Y$, $f\circ s = id_X$.

Proof: See Milne, §4, Thm 4.2

Rem: $f: Y \to X$, f is Nisnevich covering, there is a $Z_1 \subseteq Z_2 \cdots \subseteq Z_n = X$ (closed), $y \mapsto x$, $k(y) = k(x)$
s.t. $f|_{Z_{i+1} \setminus Z_i}$ has a section.

Now let A be a noetherian ring and $p \in \mathrm{Spec}\, A$. Consider the set $\boxed{I}$ whose elements are pair (B, q), where B is a connected étale A-algebra, $q \in \mathrm{Spec}\, B$, $q \cap A = p$, $k(p) = k(q)$. We say that
$$(B_1, q_1) \leq (B_2, q_2)$$
if there is an A-morphism $f: B_1 \to B_2$ s.t. $f^{-1}(q_2) = q_1$

Let $\mathrm{Spec}\, B \ni q$, $\mathrm{Spec}\, A \ni p$, $k(q) = k(p)$

Prop 2.21 The set I is a direct set and the ring $\varinjlim_{(B,q)} B = A_p^h$ is a Henselian ring which admits the universal property:

For any Henselian A-algebra C s.t. $m_C \cap A = P$, there is a unique $\varphi: A_p^h \to C$ (local homomorphism) s.t. the diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \nearrow \exists! \varphi & \\ A_p^h & & \end{array}$$
$p \leftarrow m_C$, C Hen

Proof. See Milne, §4, Lemma 4.8 and remarks around it. $\square$

Denote by $\mathcal{E}_X$ the small Nisnevich site on X. Obj: $Y \xrightarrow{\text{ét}} X$, $Y_1 \to Y_2$ over X. Suppose X is noetherian and $x \in X$, $F \in PSh(\mathcal{E}_X)$. We write $F_x = F(\mathcal{O}^h_{X,x}) = \varinjlim_{(V,u)} F(V)$ as the stalk of F at x, $V \xrightarrow{\text{ét}} X$, $u \mapsto x$, $k(u) = k(x)$.