Lecture 3 (21 March 2022)

$$
\begin{tikzcd}
\text{1-chain} \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\text{pairing}"'] & \text{0-chain} \arrow[r, "\partial"] \arrow[d, leftrightarrow] & \mathbb{R} \\
\text{1-cochain} & \text{0-cochain} \arrow[l, "\partial^*"'] &
\end{tikzcd}
$$

Ex ① $(\partial i)_x = \sum_{y \in U(x)} i(x,y)$

② $(\partial^* u)_{xy} = u_y - u_x$

③ $\partial^* u = 0 \iff u$ is constant

Proposition 1-cochain ξ is coboundary i.e. $\exists u$ s.t. $\partial^* u = \xi$

$\iff$ $\langle \xi, z \rangle = 0$ for all finit cycle 1-chain z (i.e. $\partial z = 0$).

Pf ($\Rightarrow$) Assume ξ is coboundary, i.e. $\exists$ u s.t. $\partial^* u = \xi$

For any cycle z,

$$\langle \xi, z \rangle = \langle \partial^* u, z \rangle = \langle u, \partial z \rangle = \langle u, 0 \rangle = 0$$

($\Leftarrow$) Ex.

Anoter way to say : co-exact 1-form $\perp$ closed 1-form

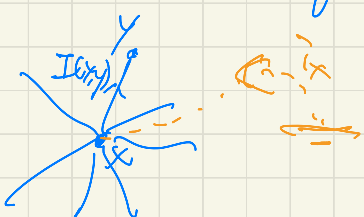
Def. $R: \{1\text{-chain}\} \longrightarrow \{1\text{-cochain}\}$,

$\quad\quad i \longmapsto r\, i$

$r(x,y) = r(y,x)$ is the resistance

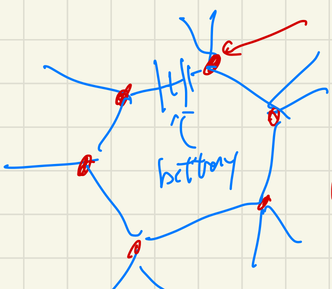
(currents $\longleftrightarrow$ difference of voltage).

Kirchoff's equations: Let I 1-chain.



Conservation of charges

$\partial I + i = 0$ where i 0-chain represent external current.



well-defined voltage at nodes

$u: V \to \mathbb{R}$

$u_y - u_x$ = voltage difference

$= r(x,y)\, I(x,y) + F$ where F is 1-cochain

$\Longleftrightarrow \langle R(I) + F, Z \rangle = 0$ for all finite cycles Z.

$$(*)\quad \begin{cases} \partial I + i = 0 \\ \langle R(I) + F, z \rangle = 0 \end{cases}$$

Define $\hat{I} := R^{-1}(R(I) + F) \iff \left(I = \hat{I} - R^{-1}(F) \right)$

$$(*) \iff \begin{cases} \partial \hat{I} + \left(i - \partial R^{-1}(F) \right) = 0 \\ \langle R(\hat{I}), z \rangle = 0 \end{cases}$$

$$\iff (**) \begin{cases} \partial \hat{I} + \hat{i} = 0 \quad \text{—}\ (\mathrm{I}) \quad \text{where } \hat{i} = i - \partial R^{-1}(F). \\ \langle R(\hat{I}), z \rangle = 0 \quad \text{—}\ (\mathrm{II}) \end{cases}$$

Express $(**)$ in terms of voltage

$(\mathrm{II}) \Rightarrow \exists\ u : V \to \mathbb{R}$ s.t. $u_y - u_x = r(x,y)\, I(x,y)$

Sub $I(x,y) = \frac{1}{r(x,y)}(u_y - u_x)$ into ①

$$\boxed{\begin{aligned}\sum_{y \in V(x)} \frac{1}{r(x,y)}(u_y - u_x) &= -i_x \\ &\parallel \\ \sum_{y \in V(x)} c(x,y)(u_y - u_x)\end{aligned}}$$

for $x \in V$.

$\left(c(x,y) = \frac{1}{r(x,y)} \right)$

discrete Poisson's eq.

<u>Def</u>, u is potential of current I.

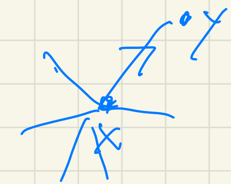
Associated Markov chain

$\Gamma = (V, E)$, c conductance.

V state space

$X_1, X_2, \ldots X_n \ldots$ random variables (n discrete time)

$$P(X_{n+1} = y \mid X_n = x) = p(x,y) = \frac{c(x,y)}{\sum_{y \in V(x)} c(x,y)}$$



Remark: ① $p(x,y) = \dfrac{c(x,y)}{\sum_{y \in V(x)} c(x,y)} \neq \dfrac{c(y,x)}{\sum_{x \in V(y)} c(y,x)} = p(y,x)$

(Although $c(x,y) = c(y,x)$)

② P $V \times V$ matrix

$$\begin{array}{cc} & y\text{-th column.} \\ x\text{-th} & \begin{pmatrix} & & \\ & p(x,y) & \\ & & \end{pmatrix} \end{array}$$

Stochastic matrix.

$$p^n(x,y) := (P^n)(x,y) = \sum_{x_1, x_2 \ldots x_{n-1} \in V} p(x,x_1)\, p(x_1,x_2) \ldots p(x_{n-1},y)$$

$$\neq (p(x,y))^n$$

③. $d(x,y) :=$ min # edges connecting $x, y \in V$.

$=$ combinatorial distance.

$$p^n(x,y) > 0 \iff n \geq d(x,y).$$

$$p^n(x,y) = P(X_{m+n} = y \mid X_m = x)$$

(4).

$u : V \to \mathbb{R}$ regarded as a column vector

$$u = \begin{pmatrix} \\ \\ \end{pmatrix}$$

$$(Pu)_x = \sum_{y \in V} P(x,y)\, u(y) = \sum_{y \in V(x)} P(x,y)\, u(y)$$

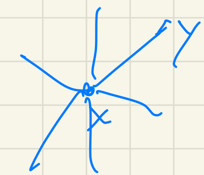
$$\boxed{\sum_{y \in V(x)} c_{xy}(u_y - u_x) = f(x) \iff (Id - P)u = \tilde{f}}$$

$$\tilde{f}_x = -\frac{1}{\sum_{y \in V(x)} c(x,y)} f_x$$

Remark: 5 $\quad \partial R^{-1}(\partial^* u) = -i$

$$\boxed{\text{Smooth: } \quad \Delta = \partial^* \partial}$$

6. $$\sum_{y \in V(x)} p(x,y) = \sum_{y \in V(x)} \left(\frac{c(x,y)}{\sum_{\tilde{y} \in V(x)} c(x,\tilde{y})} \right) = 1$$



But $$\sum_{x} p(x,y) \neq 1$$

Green's function

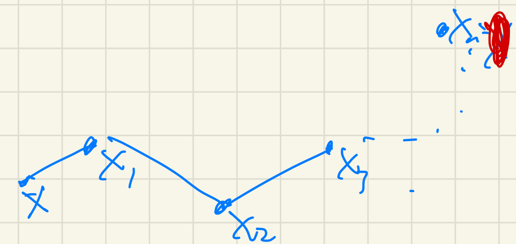
$p^n(x,y)$ = probability for a path of length n starting at x and end at y.

$\shortparallel$

$$\sum_{x_1,x_2,\dots,x_{n-1}\in V} p(x,x_1)\,p(x_1,x_2)\dots p(x_{n-1},y)$$

Expected number of visits for a path of length n starting at x, ~~end at y~~.

$$= \sum_{x_1,\dots,x_{n-1},x_n \in V} \left(\delta_{x_1 y} + \delta_{x_2 y} + \dots + \delta_{x_n y}\right) p(x,x_1)\,p(x_1,x_2)\dots p(x_{n-1},x_n)$$



$$= \sum_{k=1}^{n} \sum_{x_1,\dots,x_n \in V} \delta_{x_k y} \quad p(x,x_1)\,p(x_1,x_2)\cdots p(x_{k-1},x_k)\,p(x_k,x_{k+1})\cdots p(x_{n-1},x_n)$$

$$= \sum_{k=1}^{n} \sum_{x_1,\dots,x_{k-1},\, x_{k+1},\dots,x_n \in V,} p(x,x_1)\cdots\cdots p(x_{k-1},\overset{x_k}{\overset{\shortparallel}{y}})\left(p(\overset{x_k}{\overset{\shortparallel}{y}},x_{k+1})\cdots p(x_{n-1},x_n)\right)$$

$$= \sum_{k=1}^{n} \sum_{x_1,\dots,x_{k-1}} p(x,x_1)\cdots p(x_{k-1},y) \cdot 1$$

$$= \sum_{k=1}^{n} (P^n)(x,y)$$

$$= (Id + P + P^2 + \dots + P^n)(x,y)$$

Green's function

$$G(x,y) := \left(\sum_{k=0}^{\infty} P^k\right)(x,y)$$

= expected number of visits to y for paths starting at x.

$\in [0, \infty]$

$\boxed{\text{Ex. } G(x,y) = \infty \text{ for some } x, y \in V \Leftrightarrow G(\tilde{x},\tilde{y}) = \infty \text{ for any } \tilde{x},\tilde{y}}$

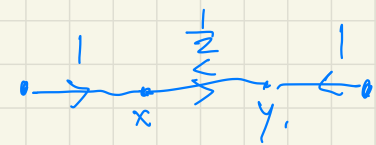
reversible is important.

Definition (Γ, c) is recurrent (parabolic) if $G(x,y)=\infty$ for some vertices x, y

(Γ, c) is (hyperbolic) transient if $G(x,y) < \infty$ for any vertex $x, y \in V$.

Assume (Γ, c) is transient.

$$G = \sum_{n=0}^{\infty} P^n$$



$$PG = GP = \sum_{n=1}^{\infty} P^n = G - Id$$

$$(Id - P)G = G - PG$$
$$= G - (G - Id)$$
$$= Id$$

identity matrix

$$G(Id - P) = G - GP = Id$$

Rmk: $(\Gamma, c) \Leftrightarrow$ reversible Markov chain $p(x,y)$

i.e. $\exists\ c : V \to \mathbb{R}$ s.t.

$$\underset{\overset{\parallel}{c(y,x)}}{c(x,y)} := c(x)\,p(x,y) = \underset{\overset{\parallel}{c(x,y)}}{c(y)\,p(y,x)}.$$

$$\Rightarrow \quad G_1 = (Id - P)^{-1}$$

Be careful : Solve for u in

$$(1-P)\,u = f$$

$$u := G(f)$$

$$\Rightarrow \quad (1-P)\,u = (1-P)\,G(f) = Id(f) = f$$

Need to check : u has finite values.

Assume transient.

Prop: Let $f : V \to \mathbb{R}$ and $G(|f|)(x) < \infty$ for all $x \in V$. Then $u := G(f)$ solves

$$(1-P)u = f.$$

E.g. f is finitly support.

$$|f|(y) = |f(y)|$$

$$0 \leq p(x,y) \leq 1$$

$$0 \leq p^n(x,y) \leq 1$$

p^n is well defined.

If Γ is finite, $\Rightarrow$ $G(x,y)$ is infinite

$\Rightarrow$ recurrent.