Harmonic functions on infinite (planar) graphs

Lecture 1 (14 March, 2022)

Motivation

Smooth

Riemann mapping

Conformal structures for simply-connected domain

$\mathbb{C}$

unit disk $\mathbb{D}$

Discrete

Circle packing

Given topological triangulation of $\mathbb{D}$, circle packing P_K

fills up $\mathbb{C}$, parabolic circle packing

fills up $\mathbb{D}$, hyperbolic circle packing.

Combinatorics.

Graph: (V, E)

vertices + edges

$c: E \to \mathbb{R}_{>0}$ edge weights

Given infinite graph and initial vertex $x \in V$, what is the expected number of visits to $y \in V$?

infinite $\Downarrow$ recurrent

finite $\Downarrow$ transient.

Harmonic functions

Conformal deformation of D,

$\Updownarrow$

g holo,

$\Downarrow$

$u := \operatorname{Re} g'$ harmonic

$\Updownarrow$

$u|_{\partial D}$ boundary value

Deformations of circle packing

$\Updownarrow$

parametrized by radii r

$\Updownarrow$

$u := \frac{d}{dt}(\log r)$ infinitesimal change of radii

is a discrete harmonic,

$(\Delta u)_i = \sum_j c_{ij}(u_i - u_j) = 0$

$\Updownarrow$

boundary value on ∂D

(Ongoing research).

Random walk or Electric network,

$c: E \to \mathbb{R}_{>0}$, $c: E \to \mathbb{R}_{>0}$

$\Downarrow$, $\Downarrow$

probability , conductance $= \frac{1}{\text{resistance}}$ (电阻)

$\Downarrow$, $\Downarrow$

Not in the course.

Martingale

$u: V \to \mathbb{R}$

$\Downarrow$

harmonic

Voltage $u: V \to \mathbb{R}$

is harmonic

$\sum c_{ij}(u_i - u_j) = 0$

(Kirchhoff's eq).

$\Updownarrow$

boundary value?

Riemann mapping Thm

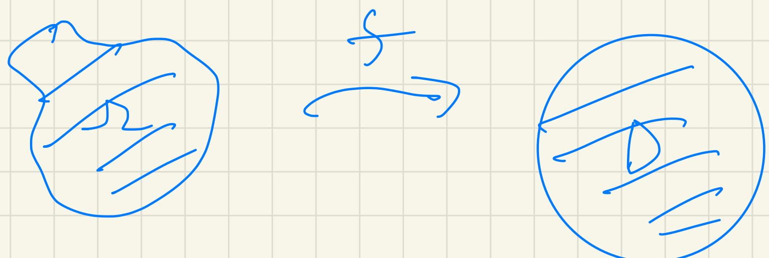
$\Omega \subset \mathbb{C}$ non-empty simply-connected open domain of complex plane.

s.t. $\Omega \subsetneq \mathbb{C}$

Then $\exists$ $f: \Omega \to D :=$ unit disk $= \{z \in \mathbb{C} \mid |z| < 1\}$. biholomorphic

open

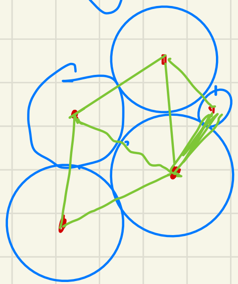
(conformal
$\Downarrow$
angle-preserving)



Here f is unique up to Möbius transformations.

Remark: ① $\mathbb{C}$, D homeomorphic topologically but biholomorphic.

Circle packing (can have different radii)



contact graph: (V, E)

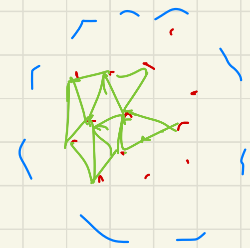
centers $\leftrightarrow$ vertices

two circles touch $\Leftrightarrow$ an edge.

Circle packing Thm

Let $K=(V,E)$ infinite triangulation of open disk.

$\exists$ unique embedded circle packing P_K s.t.



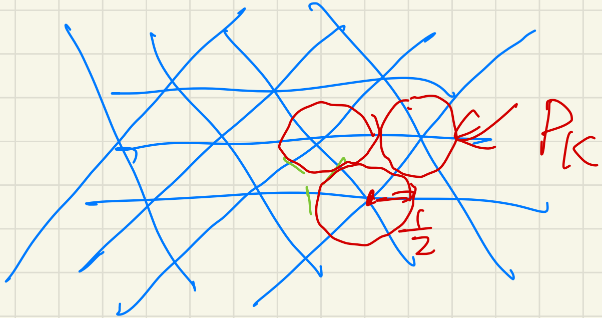
the contact graph of P_K is K

and the circle packing fills up either $\mathbb{C}$ or unit disk $\mathbb{D}$.

(Only one case can happen).

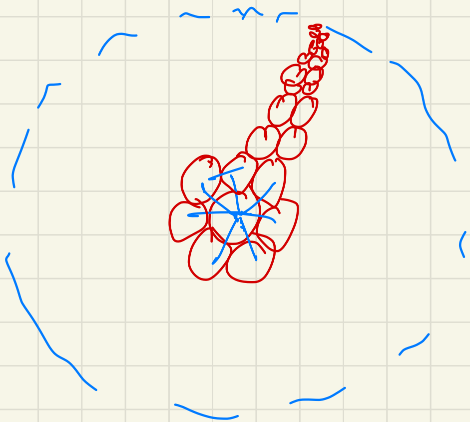
Rmk: Combinatorics K $\rightarrow$ P_K fills up $\mathbb{C}$ or $\mathbb{D}$.

E.g.



triangle lattice

P_K fills up $\mathbb{C}$.



P_K fills up $\mathbb{D}$.

Combinatorics.

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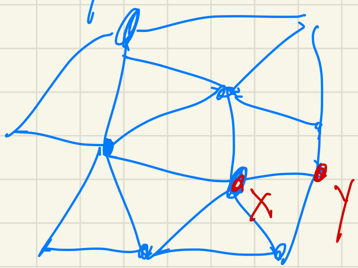


$c : E \to \mathbb{R}_{>0}$

$c_{ij} = c_{ji}$

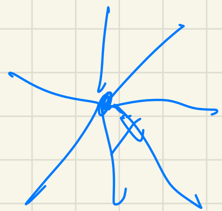
Usually assume $c \equiv 1$,

Random walk: Particle at x can jump to a neighboring vertex y.

probability $p(x,y) := \dfrac{c_{xy}}{\sum_{y \sim x} c_{xy}}$

Known cases for hyperbolic circle packings:

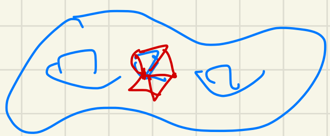
① If we know all vertices have degree > 6.



$\deg(x) = 7$

② Finite Triangulation of genus-g surface, $g > 1$

↓



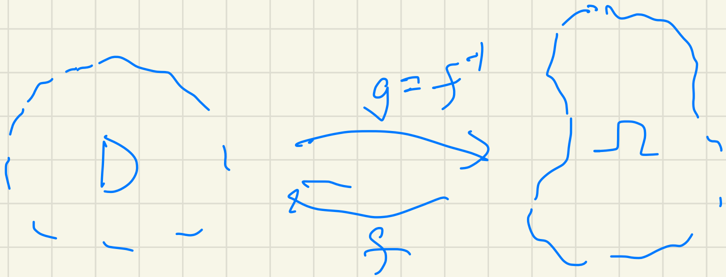
lift it to universal cover

↓

infinite triangulation of $\mathbb{R}^2$

Classical harmonic functions

~ conformal deformation.



$g : D \to \mathbb{C}$ holomorphic.

$\frac{\partial g}{\partial z}$ holomorphic

$u := \mathrm{Re}\left(\frac{\partial g}{\partial z}\right) : D \to \mathbb{R}$ real-valued

u is harmonic, i.e. $\Delta u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0.$

$v := \mathrm{Im}\left(\frac{\partial g}{\partial z}\right)$ is also harmonic

$$\begin{pmatrix} \text{conformal deformation} \\ \text{of} \\ \text{disk} \end{pmatrix} \longleftrightarrow g \text{ holomorphic} \longleftrightarrow u \text{ harmonic}$$

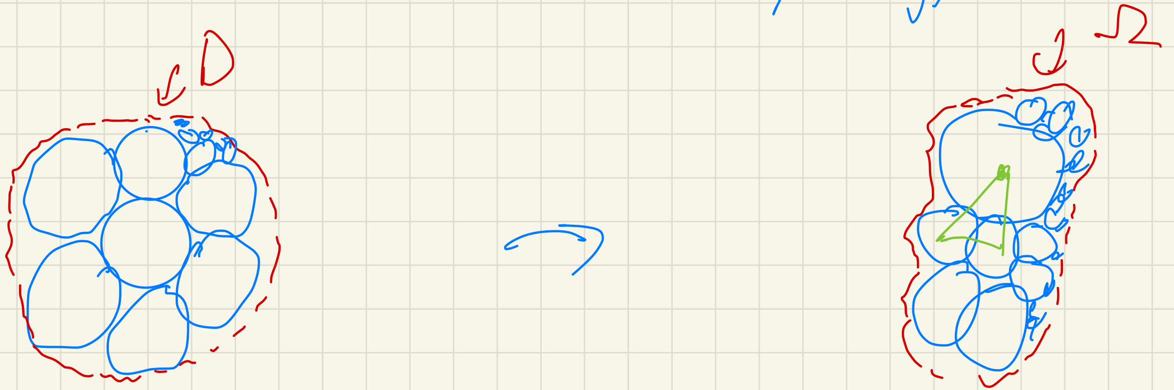
Claim: Harmonic functions are parametrized by boundary values (Dirichlet boundary problem).

$$(*)\ \begin{cases} \Delta u = 0 & \text{over open disk } D. \\ u = h & \text{over } \partial D \end{cases}$$

For some given continuous function $h : \partial D \to \mathbb{R}$.

$(*)$ has a unique solution

Harmonic functions for circle packing,



Deformation of circle packings,

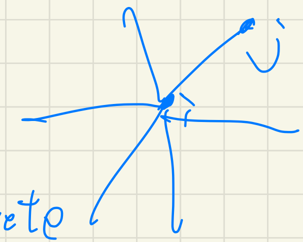
~ parametrized by Euclidean radii $r : V \to \mathbb{R}_{>0}$

contrained by some nonlinear equation.

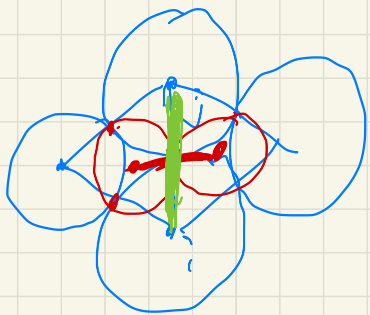
~ $r_t : V \to \mathbb{R}_{>0}$ gives 1-parameter family of circle packings

$$u := \frac{d}{dt} \log r_t \Big|_{t=0} : V \to \mathbb{R}. \quad \text{(change of radii)}$$

$$\Rightarrow \quad (\Delta u)_i := \sum_{j \sim i} c_{ij} (u_j - u_i) = 0.$$



$(\Delta u : V \to \mathbb{R})$ i.e. u is discrete harmonic function.



$$c_{ij} := \frac{\text{length of dual edge}}{\text{length of primal edge}}$$

Remark: — $G=(V,\bar{E})$ not necessary to be planar for the class

— no circle packing after today