Motivic cohomology 陳相上明講

§0 Introduction

Let X be a smooth separated scheme/k. Hodge–Atiyah and Lichtenbaum conjectured around 1882–1987 that there should exist certain complexes $\mathbb{Z}(n)$ $(n\geq 0)$ of sheaves in Zariski topology on Sm/k which have the following properties:

1. $\mathbb{Z}(0)$ is the constant sheaf $\mathbb{Z}$.
2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$ $\quad \mathcal{O}^*(X)$
3. For every field F/k, $H^n_{Zar}(F,\mathbb{Z}(n)) = K^M_n(F)$, where $K^M_n(F)$ is the n-th Milnor K-group of F.
4. $H^n_{Zar}(X,\mathbb{Z}(n)) = CH^n(X)$, where $CH^n(X) = \mathbb{Z}\{\text{cycles of codim } n\}/\text{rational equiv.}$ is the n-th Chow-group.
5. There is a natural spectral sequence with $E_2^{p,q} = H^p_{Zar}(X,\mathbb{Z}(q)) \Rightarrow K_{2q-p}(X)$, where $K_n(X) = \pi_n(\mathbb{Z}\times BGL(\infty)^+)$ is the Quillen K-theory of X. It is very nice with $\mathbb{Q}$. This spectral sequence degenerates and one has $H^n_{Zar}(X,\mathbb{Z}(n))_{\mathbb{Q}} = gr^n_\gamma K_{2n-p}(X)$ where $gr^\bullet_\gamma$ are quotients of the γ-filtration.

We denote by $H^{p,q}(X,\mathbb{Z}) = H^p_{Zar}(X,\mathbb{Z}(q))$ the motivic cohomology of X.

It turns out that the $H^{p,q}$ remain unchanged if we replace Zariski topology by Nisnevich topology if one uses étale topology, the corresponding cohomologies are denoted by $H^{p,q}_{ét}(X,\mathbb{Z})$, the Lichtenbaum motivic cohomology. It admits the comparison

$$H^{p,q}_{ét}(X,\mathbb{Z}/n) = H^p_{ét}(X,\mu_n^{\otimes q})$$

if $\mathrm{char}\,k\nmid n$. The Bloch–Kato–Suslin conjecture, proved by Voevodsky in 2011, states that

$$H^{p,q}(X,\mathbb{Z}/n) \to H^{p,q}_{ét}(X,\mathbb{Z}/n)$$

is an isomorphism if $p=q$ and monomorphism if $p=q+1$. So one obtains

$$H^{q,q}(X,\mathbb{Z}/n) = H^q_{ét}(X,\mu_n^{\otimes q})$$

if $p\leq q$. If we take $X=\mathrm{Spec}\,k$, this is the Milnor conjecture, proved by Voevodsky in 1996:

$$K^M_q(k)/n = H^{q,q}(k,\mathbb{Z}/n) = H^q_{ét}(k,\mu_n^{\otimes q}).$$

Unlike finite coefficients, the $H^{p,q}(X,\mathbb{Z})$ is quite hard to compute for p small. The open question is the Beilinson–Soulé conjecture: $H^{p,q}(X,\mathbb{Z}) = 0$ if $p<0$. (vanishing)

If $\mathrm{char}\,k=0$, this is known for number field, function fields of genus 0 curves over number fields and their infinite limits.
If $\mathrm{char}\,k>0$, this is known for finite field and global field.

The motivic cohomologies could be realized in a triangulated category $DM(k)$, s.t.

$$H^{p,q}(X,\mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(X),\mathbb{Z}(q)[p]),$$

where $\mathbb{Z}(X)$ is the motive of X. Moreover, we can define motivic homology $H_{p,q}(X,\mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(q)[p],\mathbb{Z}(X))$.
From the aspect of motive, we can derive theorem of all cohomology which can be represented in DM. The main devices are the following:

1. If $E\to X$ is an $\mathbb{A}^n$-bundle, we have $\mathbb{Z}(E)\simeq\mathbb{Z}(X)$ in DM.
2. If $U\cup V = X$ is an open covering, we have the MV-sequence $\mathbb{Z}(U\cap V)\to\mathbb{Z}(U)\oplus\mathbb{Z}(V)\to\mathbb{Z}(X)\to\cdots[1]$ in DM.
3. If $Y\subseteq X$ is a closed embedding of codim c in Sm/k, we have a Gysin triangle
$$\mathbb{Z}(X\setminus Y)\to\mathbb{Z}(X)\to\mathbb{Z}(Y)(c)[2c]\to\cdots[1].$$
4. For any vector bundle E of rank n on X, we have the projective bundle formula
$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^{n}\mathbb{Z}(X)(i)[2i],$$
which defines the Chern classes of E.
5. For proper smooth X, the $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(d_X)[2d_X]$ in DM, which implies the Poincaré duality
$$H^{p,q}(X,\mathbb{Z}) = H^{2d_X-p,d_X-q}(X,\mathbb{Z}).$$

This course consists of the following:

1. Intersection theory
2. Milnor K-theory and Rost complex
3. The categories of motives and basic tools.
4. Computation of $H^{p,0}$ and $H^{p,p}$

§1 Intersection Theory

Def 1.1 Let X be a finite type scheme over k. Define
$$Z_c(X) = \bigoplus_{\substack{\text{irr. cl. subvar.}\\ \dim Z = c}} \mathbb{Z}\cdot Z \qquad (Z(X) = \bigoplus_c Z_c(X))$$
$$K_c(X) = \{F \text{ coherent on } X,\ \dim \mathrm{supp}\, F \leq c\}.$$

Def 1.2 Let $X\in Sm/k$, $U,V\subseteq X$ irr. closed and $W\subseteq U\cap V$ be a irr. component. If $\dim_W U + \dim_W V - \dim X = i$, we say that U,V intersect properly at W. (codim W = codim U + codim V)

Thm 1.3 Let $k\subseteq A$ be a noetherian regular local ring, M,N are finite generated A-mod and $\ell(M\otimes_A N)<\infty$. Then
1) $\ell(\mathrm{Tor}_i^A(M,N))<\infty$, $\forall i\geq 0$.
2) $\chi^A(M,N) = \sum_{i=0}^{\dim A}(-1)^i\,\ell(\mathrm{Tor}_i^A(M,N)) \geq 0$.
3) $\dim M + \dim N \leq \dim A$.
4) $\dim M + \dim N < \dim A$ iff $\chi^A(M,N) = 0$.

Proof: See J.P. Serre, Local Algebra.

Rem 1.4 The 3) above shows that for any $W\subseteq U\cap V$, we have codim $W \leq$ codim U + codim V $\Leftrightarrow$ $\dim W \geq \dim U + \dim V - \dim X$. ($X$ is smooth)

Def 1.5: Let $X, W\subseteq U\cap V$ as in Def 1.2. Define the intersection multiplicity $m_W(U,V)$ of U,V at W be
$$\chi^{O_{X,W}}(O_{X,W}/\mathfrak{p}_W,\ O_{X,W}/\mathfrak{p}_V).$$
So $m_W(U,V)\geq 0$ and $m_W(U,V)=0$ iff U,V intersect properly at W.

Def 1.6 Let $X\in Sm/k$, $U\in Z_a(X)$, $V\in Z_b(X)$. If U,V intersect properly at any component, define
$$U\cdot V = \sum_{W\subseteq U\cap V} m_W(U,V)\cdot W \in Z_{a+b-d_X}(X).$$

Ex: $C, D\subseteq\mathbb{P}^2$ curve; $C\to\mathbb{P}^1$, $D\to\mathbb{P}^1$ finite maps f,g; $\deg f = n$, $\deg g = m$.
Bézout thm: $\#(C\cap D) = nm$, for every $x\in C\cap D$, we have $m_x(C,D) = \ell_{O_{\mathbb{P}^2,x}}(F,G)$, Also true for any f,g on surface X.

Def 1.7 Let Y be finite type over k, $F\in K_a(X)$. Define
$$Z_a(F) = \sum_{\dim Z = a} l_{O_{Z,\eta}}(F_\eta)\cdot\overline{\eta}\in Z_a(X).$$

Def 1.8 By GrTm32, Chap3, Ex 6.8, every coherent F on $X\in Sm/k$ has a resolution $0\to E_N\to E_{N-1}\to\cdots\to E_0\to F\to 0$, where E_i's are locally free of finite rank. So for any coherent G, we define
$$\mathrm{Tor}_i(F,G) = H_i(E\otimes G).$$

Prop 1.9. Let $X\in Sm/k$, $F\in K_a(X)$, $G\in K_b(X)$ s.t. F,G intersect properly. Then we have
$$Z_a(F)\cdot Z_b(G) = \sum_{i=0}^{d_X}(-1)^i Z_{a+b-d_X}(\mathrm{Tor}_i(F,G)).$$

Proof: We may assume X is affine and count the coefficients of $\overline{\eta}$ where $\dim\overline{\eta} = a+b-d_X$. We have to show:
$$\chi(F,G) = \sum_{\eta\in\overline{\eta}\cap\overline{\eta}} ((F)_\eta)((G)_\eta)\, m_{\overline{\eta}}(\overline{\eta},\overline{\eta})$$
(Anti-d. Analysis)
$$0\to F_N\to F_{N-1}\to F_{N-2}\to\cdots$$
$$\chi(F,G) = \chi(F_N,G) - \chi(F_{N-1},G) + \chi(F_{N-2},G) - \cdots$$

Both F and G admits a filtration $0=M_0\subseteq\cdots\subseteq M_s$ s.t. $M_i/M_{i-1} = O_X/\mathfrak{p}_i$, $\mathfrak{p}_i$'s a prime ideal. By additivity of both sides, we may assume $F = M_j/M_{j-1}$ and $G = O_X/\mathfrak{q}$ ($\mathfrak{q}$ a prime ideal). So LHS $= m_{\overline{\eta}}(\overline{\eta},\overline{\eta})$. (since $\dim\overline{\eta} = a+b$)

If $\dim\overline{\eta}<a$ or $\dim\overline{\eta}<b$, the equality follows by $((F)_\eta) = ((G)_\eta) = 0$.

If $\dim\overline{\eta}\geq a$ or $\dim\overline{\eta}\geq b$, the $\overline{\eta}$ do not intersect properly at $\overline{\eta}$, so LHS $= 0 = $ RHS. □

Prop 1.10 The intersection product is (commutative.)

Prop 1.11 The intersection product is associative.

Proof: Suppose $F\in K_a(X)$, $G\in K_b(X)$, $H\in K_c(X)$ and they intersect properly.
$$\begin{array}{ccc} F & & \\ \uparrow & & \\ L & \to & M \\ & & \uparrow \\ & & G \end{array} \qquad \begin{array}{ccc} H & & \\ \uparrow & & \\ N & \to & P \\ & & \uparrow \\ & & Q \end{array}$$
Define a double complex $M_{ij} = L_i\otimes_A N_j$. Then
$$E^{1,2}_{p,q} = \mathrm{Tor}_p(F,\mathrm{Tor}_q(G,H)),\quad {}^{2,1}E_{p,q} = \mathrm{Tor}_p(\mathrm{Tor}_q(F,G),H)$$
Fact: $0\to C_0\to C_1\to\cdots\to C_n\to 0$, an additive function ℓ on C_i. Then $\sum(-1)^i\ell(C_i) = \sum(-1)^i\ell(H_i(C))$. (complex)

Suppose L. We have
$$\sum_{p,q}(-1)^{p+q}\ell(({}^{1,2}E)^q_p) = \sum_q(-1)^q\ell(H_q(\mathrm{Tot}(M_{ij}))) = \sum_{p,q}(-1)^{p+q}\ell(({}^{2,1}E)^q_p) \quad (1)$$
$$\|$$
$$Z_a(F)\cdot(Z_b(G)\cdot Z_c(H)) \qquad \|$$
$$(Z_a(F)\cdot Z_b(G))\cdot Z_c(H)$$

Def 1.12 Suppose $X_1, X_2\in Sm/k$, $F_1\in K_{a_1}(X_1)$, $F_2\in K_{a_2}(X_2)$. Define
$$Z_{a_1}(F_1)\times Z_{a_2}(F_2) = Z_{a_1+a_2}(p_1^*F_1)\cdot Z_{a_1+a_2}(p_2^*F_2)$$
where $p_i: X_1\times X_2\to X_i$ is the projection. (Check that it is well-defined!)