② Unit axiom: the functor of left and right mult. by 1:

$$L_1 : X \mapsto 1 \otimes X \quad , \quad R_1 : X \mapsto X \otimes 1$$

are autoequivalence on $\mathcal{C}$.

<u>Lemma</u>: There are natural isomorphisms

$$l : L_1 \to \mathrm{Id}_{\mathcal{C}} \quad , \quad r : R_1 \to \mathrm{Id}_{\mathcal{C}} .$$

<u>proof</u>: We will prove that for l (r is similar).

We will make such an l such that:

$\iota : 1\otimes 1 \to 1$

$$\begin{tikzcd}
1\otimes(1\otimes X) \arrow[rr, "\bar{a}^{-1}_{1,1,X}"] \arrow[dr, "L_1(l_X)"'] & & (1\otimes 1)\otimes X \arrow[dl, "\iota\otimes id_X"] \\
& 1\otimes X &
\end{tikzcd}$$

We want $L_1(l_X) = (\iota\otimes id_X)\circ \bar{a}^{-1}_{1,1,X}$.

But, as we proved yesterday, because L_1 is an equivalence of category, it induces a bijection on the hom-set, in particular, $l_X = L_1^{-1}\left[(\iota\otimes id_X)\circ \bar{a}^{-1}_{1,1,X}\right]$.

inverse in hom-set

$\to$ l_X is isom. OK.

Such an l defined by above l_X is a natural transformation if by def: $\forall\, X \xrightarrow{f} Y$

$$\begin{tikzcd} L_1(X) \arrow[r, "l_X"] \arrow[d, "L_1(f)"'] & Id_{\mathcal{C}}(X) \arrow[d, "Id_{\mathcal{C}}(f)"] \\ L_1(Y) \arrow[r, "l_Y"] & Id_{\mathcal{C}}(Y) \end{tikzcd}$$

commutes.

$$\iff \begin{tikzcd} L_1 \circ L_1(X) \arrow[r, "L_1(l_X)"] \arrow[d, "L_1 \circ L_1(f)"'] & L_1 \circ Id_{\mathcal{C}}(X) \arrow[d, "L_1 \circ Id_{\mathcal{C}}(f)"] \\ L_1 \circ L_1(Y) \arrow[r, "L_1(l_Y)"'] & L_1 \circ Id_{\mathcal{C}}(Y) \end{tikzcd}$$

commutes

the last diagram reformulates and decomposes as follows:

$$\begin{tikzcd}
1\otimes(1\otimes X) \arrow[r, "a^{-1}_{1,1,X}"] \arrow[d, "Id_1\otimes(Id_1\otimes f)"'] & (1\otimes 1)\otimes X \arrow[r, "\iota\otimes id_X"] \arrow[d, "(id_1\otimes id_1)\otimes f"] & 1\otimes X \arrow[d, "Id_1\otimes f"] \\
1\otimes(1\otimes Y) \arrow[r, "a^{-1}_{1,1,Y}"'] & (1\otimes 1)\otimes Y \arrow[r, "\iota\otimes id_Y"] & 1\otimes Y
\end{tikzcd}$$

But the left and right squares commute because

left: $a^{-1}_{1,1,\_}$ is a natural transformation OK.

right: $(id_1\otimes f)\circ(\iota\otimes id_X) = \iota\otimes f$.

$(\iota\otimes id_Y)\circ(\underbrace{id_1\otimes id_1}_{id_1})\otimes f = \iota\otimes f$ $\Bigg\}$ The result follows. ∎

$\boxed{\begin{array}{l}\text{Exo:}\\ Id_{X\otimes Y} = Id_X\otimes Id_Y\end{array}}$

Proposition: $\forall X \in \mathcal{C}$

$$l_{1\otimes X} = id_1 \otimes l_X \quad , \quad r_{X\otimes 1} = r_X \otimes id_1 .$$

⚠ $l_{X\otimes Y} \neq l_X \otimes l_Y$ in general.

Proof: We know that l is natural transformation ($l : L_1 \to Id_{\mathcal{C}}$), i.e.

$$\begin{tikzcd}
L_1(X) \arrow[r, "l_X"] \arrow[d, "f"'] & Id_{\mathcal{C}}(X) \arrow[d, "Id_{\mathcal{C}}(f)"] \\
L_1(Y) \arrow[r, "l_Y"'] & Id_{\mathcal{C}}(Y)
\end{tikzcd}$$

commutes.

Apply it to $f = l_Z$ ($X = 1\otimes Z$, $Y = Z$)

$$\begin{tikzcd}
1\otimes(1\otimes Z) \arrow[r, "l_{1\otimes Z}"] \arrow[d, "id_1\otimes l_Z"'] & 1\otimes Z \arrow[d, "l_Z"] \\
1\otimes Z \arrow[r, "l_Z"'] & Z
\end{tikzcd}$$

commutes

$\Longleftrightarrow$ $l_Z \circ l_{1\otimes Z} = l_Z \circ (id_1 \otimes l_Z)$

but l_Z is iso (you can apply l_Z^{-1})

so the expected equality holds $\boxtimes$

Proposition (triangle diagram): $\forall X, Y \in \mathcal{C}$

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X \otimes id_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "id_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

commutes.

Rk: The former definition of a monoidal category is Pentagon axiom + triangle axiom. But the definition used in this course is Pent. axiom + Unit axiom (and above prove the triangle axiom)

$$
\begin{tikzcd}
((X\otimes 1)\otimes 1)\otimes Y \arrow[rrrr, "a_{X,1,1}\otimes \mathrm{id}_Y"] \arrow[drr, "(r_X\otimes \mathrm{id}_1)\mathrm{id}_Y"] \arrow[dddd, "a_{X\otimes 1,1,Y}"'] & & {} \arrow[d, phantom, "A"] & & (X\otimes(1\otimes 1))\otimes Y \arrow[dll, "(\mathrm{id}_X\otimes l)\otimes \mathrm{id}_Y"'] \arrow[dddd, "a_{X,1\otimes 1,Y}"] \arrow[r, phantom, "F"] & {} \\
 & & (X\otimes 1)\otimes Y \arrow[d, "a_{X,1,Y}"] & & & \\
{} \arrow[r, phantom, "B"] & {} & X\otimes(1\otimes Y) & {} \arrow[r, phantom, "C"] & {} & \\
(X\otimes 1)\otimes(1\otimes Y) \arrow[urr, "r_X\otimes(\mathrm{id}_1\otimes \mathrm{id}_Y)"] \arrow[drr, "a_{X,1,1\otimes Y}"'] & {} \arrow[r, phantom, "D"] & {} \arrow[r, phantom, "E"] & {} & X\otimes((1\otimes 1)\otimes Y) \arrow[ull, "\mathrm{id}_X\otimes(l\otimes \mathrm{id}_Y)"'] \arrow[dll, "\mathrm{id}_X\otimes a_{1,1,Y}"] & \\
 & & X\otimes(1\otimes(1\otimes Y)) \arrow[uu, "\mathrm{id}_X\otimes l_{1\otimes Y}"'] & & &
\end{tikzcd}
$$

(F): is just the pentagon axiom.

(B)(C): just apply that a is a natural transformation.

(A): Need to have $r_X \otimes id_1 \overset{?}{=} (id_X \otimes l) \circ a_{X,1,1}$:

<u>But</u>: $R_1(r_X) = r_{X\otimes 1} = r_X \otimes id_1$ (by previous proposition)

<u>and</u>: $R_1(r_X) = (id_X \otimes l) \circ a_{X,1,1}$ by definition (ok).

(E): As for (A) but for l. (ok).

So (D) holds, so the result follows because L_1 is an equivalence of cat. ∎

Proposition: $\forall X, Y \in \mathcal{C}$, the following two diagrams commute:

$$
\begin{tikzcd}
(1 \otimes X) \otimes Y \arrow[rr, "a_{1,X,Y}"] \arrow[dr, "l_X \otimes id_Y"'] & & 1 \otimes (X \otimes Y) \arrow[dl, "l_{X \otimes Y}"] \\
& X \otimes Y &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(X \otimes Y) \otimes 1 \arrow[rr, "a_{X,Y,1}"] \arrow[dr, "r_{X \otimes Y}"'] & & X \otimes (Y \otimes 1) \arrow[dl, "id_X \otimes r_Y"] \\
& X \otimes Y &
\end{tikzcd}
$$

Proof: Let us prove the first one (the second is similar).

$$
\begin{tikzcd}
((X\otimes 1)\otimes Y)\otimes Z \arrow[rrrr, "a_{X,1,Y}\otimes id_Z"] \arrow[drr, "(r_X\otimes id_Y)\otimes id_Z"] \arrow[ddd, "a_{X\otimes 1,Y,Z}"'] & & \arrow[d, phantom, "A", near start] & & (X\otimes(1\otimes Y))\otimes Z \arrow[dll, "(id_X\otimes l_Y)\otimes id_Z"'] \arrow[ddd, "a_{X,1\otimes Y,Z}"] & \\
& \arrow[d, phantom, "B"] & (X\otimes Y)\otimes Z \arrow[d, "a_{X,Y,Z}"] & \arrow[d, phantom, "C"] & & \\
& & X\otimes(Y\otimes Z) & & & \mathrm{F} \\
(X\otimes 1)\otimes(Y\otimes Z) \arrow[urr, "r_X\otimes id_{Y\otimes Z}"] \arrow[drr, "a_{X,1,Y\otimes Z}"'] & \arrow[phantom, r, "D"] & & \arrow[phantom, l, "E"] & X\otimes((1\otimes Y)\otimes Z) \arrow[ull, "id_X\otimes(l_Y\otimes id_Z)"'] \arrow[dll, "id_X\otimes a_{1,Y,Z}"] & \\
& & X\otimes(1\otimes(Y\otimes Z)) \arrow[uu, "id_X\otimes l_{Y\otimes Z}"] & & &
\end{tikzcd}
$$

As before, (E): pentagon axiom.

(B), (C): a is a natural transformation (with $id_{Y \otimes Z} = id_Y \otimes id_Z$)

(A), (D): apply previous proposition (triangle diagram).

So E commutes.

But E is what we want after the application of the functor G which is an equivalence of category.

So the result follows. $\blacksquare$