Introduction:

The notion of fusion category is a "categorification" of the notion of fusion ring $\mathcal{F}$ (which is elementary, combinatorial). It's given by a finite set $B = \{b_1, \dots, b_r\}$ together with fusion rules:

$$b_i b_j = \sum_{k=1}^{r} n_{ij}^{k} b_k \quad , \text{with } n_{ij}^{k} \in \mathbb{N}_{\geq 0}.$$

+ 4 axioms we will write after.

You should think this notion as a generalization of the multiplication on a finite group ...

of the tensor product on the representations of a finite group.

Here are the axioms:

- Associativity: $b_i (b_j b_k) = (b_i b_j) b_k$
- Neutral: $b_1 b_i = b_i b_1 = b_i$
- Inverse/Adjoint/Dual: $\forall i \; \exists! \; i^*$ with $n_{ik}^1 = n_{ki}^1 = \delta_{i^* k}$.
- Frobenius reciprocity: $n_{ij}^k = n_{i^* k}^j = n_{kj^*}^i$

In general, we can understand that as a representation ring of a "virtual" group (or quantum symmetries).

$\exists$ ring $*$-homomorphism $\dim : \mathcal{F} \to \mathbb{R}_{\geq 0}$

$\dim(b_i)$ can be non-integral.

<u>Golden example:</u> Yang-Lee fusion ring.

$B = \{b_1, b_2\}$, $b_2^2 = b_1 + b_2$ $(\dim(b_1) = 1)$.

Apply dim: $\dim(b_2)^2 = 1 + \dim(b_2)$

$x^2 = 1 + x$. $\Rightarrow x = \dfrac{1+\sqrt{5}}{2} = \phi$ <u>Golden ratio</u>

We will see that from any fusion category, we can associate a fusion ring called its "Grothendieck ring".

But, not every fusion ring is Grothendieck, if it is, it's called "categorifiable".

Ex: $B = \{b_1, b_2\}$, $b_2^2 = b_1 + n b_2$

Thm: it's categorifiable iff $n = 0, 1$

One of the main problem of this theory is to know whether a given fusion ring is categorifiable, which means that its Pentagon Equation (PE) admits a solution, but in general, the P.E is a too big polynomial system to be solved directly...

But if a solution exists, we know that there are only finitely many ones (up to equivalence) → "Ocneanu rigidity"

We will follow the book "Tensor Categories", it is available online at the following link (see WeChat).
↳ we will start it from Chap 2. (Monoidal categories).

Before that, we will (very briefly) recall what is a category, a functor, a natural transformation, ... (not in above book).

Category $\mathcal{C}$:

$X \xrightarrow{f} Y$

- collection of objects $X \in \mathcal{C}$
- " of morphisms $f \in \hom_{C}(X,Y) = C(X,Y)$

We will always assume $\hom_{\mathcal{C}}(X,Y)$ to be a set (locally small category)

- composition: $f \in \hom_{\mathcal{C}}(X,Y)$, $g \in \hom_{\mathcal{C}}(Y,Z)$
  $\Rightarrow g \circ f \in \hom_{\mathcal{C}}(X,Z)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

- identity: $\exists\, id_X \in \hom_{\mathcal{C}}(X,X)$, such that
  $\forall f \in \hom_{\mathcal{C}}(X,Y)$, $f \circ id_X = f$, $id_Y \circ f = f$.
- associativity: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T \Rightarrow h \circ (g \circ f) = (h \circ g) \circ f$

⚠ $X \xrightarrow{f} Y$ is "just" an arrow (as for a graph)

X is <u>not</u> a set (in general), just an abstract object.

Isomorphism of $\mathcal{C}$: $X \xrightarrow{f} Y$. such f is an isomorphism

if $\exists g: Y \to X$ such that: $g \circ f = Id_X$; $f \circ g = id_Y$.

<u>Functor</u>: $F: \mathcal{C} \to \mathcal{C}'$

$X \mapsto F(X)$.

$$X \xrightarrow{f} Y.$$
$$\Downarrow$$
$$F(X) \xrightarrow{F(f)} F(Y).$$

such that:
$F(id_X) = id_{F(X)}$
and
$F(f \circ g) = F(f) \circ F(g)$

For example, the identity functor $F = Id_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$

$Id_{\mathscr{C}}(X) = X$, $Id_{\mathscr{C}}(f) = f$

Natural transformation ϕ: $F_1, F_2 : \mathscr{C} \to \mathscr{C}'$

$\phi : F_1 \to F_2$. such that $\forall X, Y \in \mathscr{C}$
$\forall X \xrightarrow{f} Y$.

$$\begin{tikzcd}
F_1(X) \arrow[r, "\phi_X"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\
F_1(Y) \arrow[r, "\phi_Y"'] & F_2(Y)
\end{tikzcd}$$

commute:

$F_2(f) \circ \phi_X = \phi_Y \circ F_1(f)$.

A natural isomorphism: $\phi : F_1 \to F_2$ is a natural transformation such that $\forall X \in \mathcal{C}$, ϕ_X is an isomorphism of $\mathcal{C}'$

$(F_1 \xrightarrow{\sim} F_2)$

Category equivalence: $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}$

such that $\exists$ $\phi' : F \circ G \to Id_{\mathcal{C}'}$,
$\phi : G \circ F \to Id_{\mathcal{C}}$ } natural isomorphisms

F is called an equivalence of category, and

$\mathcal{C}$ and $\mathcal{C}'$ are called "equivalent".

If you have a commuting diagram, then its image by a functor is also commuting.

Exercise: Show that the converse is true if the functor is an equivalence of category.

An isomorphism of category, it's as for an equiv. of cat. but with $F \circ G = Id_{\mathcal{C}'}$, $G \circ F = Id_{\mathcal{C}}$.

Rk: isomorphism $\Longrightarrow$ equivalent
$\not\Longleftarrow$

<u>Theorem</u> (assumed) Every category $\mathcal{C}$ is equivalent to a <u>skeletal</u> category $\tilde{\mathcal{C}}$ (axiom of choice ...)

<u>Def</u>: A category is skeletal if every isomorphism class of objects contains one object (i.e. $X \xrightarrow{\simeq} Y \Rightarrow X = Y$)

If $\mathcal{C}$ is <u>not</u> skeletal, then $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are <u>not</u> isomorphic categories.

Lemma 1: An equivalence of category induces a bijection of hom-sets (but not of objects), i.e.

Consider $F: \mathcal{C} \to \mathcal{C}'$ equivalence

then: $\hom_{\mathcal{C}}(X, Y) \longrightarrow \hom_{\mathcal{C}'}(F(X), F(Y))$

$f \longmapsto F(f)$

is a bijection.

proof: We will make f from $F(f)$. By def: $\Phi: F \to G$

$$\begin{tikzcd} G \circ F(X) \arrow[r, "\Phi_X"] \arrow[d, "G \circ F(f)"'] & X \arrow[d, "f"] \\ G \circ F(Y) \arrow[r, "\Phi_Y"'] & Y \end{tikzcd}$$

commutes, i.e. $f \circ \Phi_X = \Phi_Y \circ G(F(f))$, but Φ is a natural iso. so, Φ_X is inversible and so: $f = \Phi_Y \circ G(F(f)) \circ \Phi_X^{-1}$. ☒

There is another property (we will not prove here): of F equiv.

(2) $\forall X' \in \mathscr{C}'$, $\exists X \in \mathscr{C}$ s.t. $F(X) \simeq X'$.

Result: F is an equivalence iff (1) & (2).

Rk: $F(f)$ is an isomorphism iff f is so.

Monoidal category $(\mathcal{C}, \otimes, a, 1, \iota)$

- $\mathcal{C}$ is a category (assumed locally small)
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ bifunctor (tensor product)
- $a : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ natural isomorphism

  where $(- \otimes -) \otimes - : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ trifunctor.

  $(X, Y, Z) \mapsto (X \otimes Y) \otimes Z$.

Idem for $- \otimes (- \otimes -)$.

- $1 \in \mathcal{C}$ an object.
- $\iota : 1 \otimes 1 \xrightarrow{\sim} 1$ isomorphism.

Two axioms:

① Pentagon axiom: $\forall\, W, X, Y, Z \in \mathcal{C}$

$$\begin{tikzcd}
 & ((W\otimes X)\otimes Y)\otimes Z \arrow[dl, "a_{W,X,Y}\otimes id_Z"'] \arrow[dr, "a_{W\otimes X,Y,Z}"] & \\
(W\otimes(X\otimes Y))\otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"'] & & (W\otimes X)\otimes(Y\otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"] \\
W\otimes((X\otimes Y)\otimes Z) \arrow[rr, "id_W\otimes a_{X,Y,Z}"'] & & W\otimes(X\otimes(Y\otimes Z))
\end{tikzcd}$$

commutes

② Unit axiom: the functors of left and right mult. by 1:

$$L_1 : X \mapsto 1 \otimes X \quad , \quad R_1 : X \to X \otimes 1 .$$

are equivalence of $\mathcal{C}$ (autoequivalence);

$(1, \iota)$ is called the "unit" object of $\mathcal{C}$

Next time, we will prove the existence of natural isom.

$$l : L_1 \to \mathrm{Id}_{\mathcal{C}} \quad , \quad r : R_1 \to \mathrm{Id}_{\mathcal{C}} .$$