Grothendieck rings and Frobenius-Perron dimensions.

<u>Recall</u>: Let $\mathcal{C}$ be an abelian category.

An object X has finite length if there is a sequence (called filtration)

subobject

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

such that X_i / X_{i-1} is simple

(Jordan-Hölder): The sequence of simple objects $(S_i) = \left(X_i / X_{i-1}\right)$ are independent of the filtration (which always exists), up to isomorphism and permutation. (and every sequence of subobject can be complete as a Jordan-Hölder sequence)

Let $[X : S]$ be the multiplicity of S in (S_i).

Exo: $X \simeq X' \Rightarrow [X:S] = [X':S]$ for any simple objects

Def: Let $\mathcal{C}$ be an abelian $\mathbb{k}$-linear cat., where the objects has finite length. Let $Gr(\mathcal{C})$ be the "Grothendieck group", i.e., the abelian group freely generated by the simple objects $(X_i)_{i\in I}$ (up to iso).

$\forall X \in \mathcal{C}$, define $[X] \in Gr(\mathcal{C})$ as

$$[X] := \sum_{i\in I} [X:X_i] \cdot X_i .$$

Exo: If $0 \to X \to Y \to Z \to 0$ exact, then $[Y] = [X] + [Z]$

In particular: $[X \oplus Z] = [X] + [Z]$

Multiplication of $Gr(\mathcal{C})$:

Let $\mathcal{C}$ be a multiring category (over k).
Let $(X_i)_{i\in I}$ be the simple objects of $\mathcal{C}$ (up to iso)

The tensor product $\otimes$ induces a multiplication of $Gr(\mathcal{C})$:

$$X_i X_j := [X_i \otimes X_j] = \sum_{k\in I} [X_i \otimes X_j : X_k] X_k \quad , (i,j \in I)$$

Rk: Such multiplication is called "fusion rules"; this notation comes from physics. There is a relation between fusion of "particle" in physics and fusion rules. In particle physics, an elementary particle is encoded as an irreducible rep. of a group, and the collision of two particles decompose into other particles -- this is given by the tensor product of irr. rep....

You can still need fusion rings not coming from Rep(G) in physics, in condensed matter physics → look for the notion of anyon.

Rk: The fusion rules do not completely characterize the category (even in the semisimple case) → see the categorification problem.

We can extend the multiplication on every object linearly as

$$[X]\cdot[Y] := \left(\sum_{i\in I}[X:X_i]\,X_i\right)\left(\sum_{j\in I}[Y:X_j]\,X_j\right) = \sum_{i,j\in I}[X:X_i][Y:X_j]X_iX_j$$

Exo: $[X][Y] = [X\otimes Y]$ (hint: use exactness of tensor product)

Lemma: Above multiplication is associative.

proof: By linearity, we can reduce to the simple case.

$$(X_i X_j) X_p = \left(\sum_{h \in I} [X_i \otimes X_j : X_h] X_h \right) X_p$$

$$= \sum_{h \in I} [X_i \otimes X_j : X_h] (X_h X_p)$$

$$= \sum_{h \in I} [X_i \otimes X_j : X_h] \sum_{\ell \in I} [X_h \otimes X_p : X_\ell] X_\ell$$

$$= \sum_{h \in I} \sum_{\ell \in I} \cdots = \sum_{\ell \in I} \left(\sum_{h \in I} \cdots \right) X_\ell .$$

<u>So</u>: $$[(X_i X_j) \cdot X_p : X_\ell] = \sum_{h \in I} [X_i \otimes X_j : X_h][X_h \otimes X_p : X_\ell]$$

(Exo) $$= [(X_i \otimes X_j) \otimes X_p : X_\ell]$$

Idem: $X_i(X_j X_p) = \cdots$

You get: $[X_i(X_j X_p) : X_\ell] = [X_i \otimes (X_j \otimes X_p) : X_\ell]$

The result follows by the associativity of $\otimes$

$$(X_i \otimes X_j) \otimes X_p \simeq X_i \otimes (X_j \otimes X_p). \quad \blacksquare$$

(see def below)

Then: $Gr(\mathcal{C})$ is a $\mathbb{Z}_+$-ring with unit $[\mathbb{1}]$

It is called the Grothendieck ring of $\mathcal{C}$.

Recall (from Chap 3 of the book):

First $\mathbb{Z}_+ := \mathbb{Z}_{\geq 0} = \{ n \in \mathbb{Z} \text{ such that } n \geq 0 \}$.

__Def:__

(i) A $\mathbb{Z}_+$-basis of a ring is a basis $B = \{b_i\}_{i\in I}$ such that

$$b_i b_j = \sum_{k\in I} c_{ij}^k b_k \quad \text{with } c_{ij}^k \in \mathbb{Z}_+$$

(ii) A $\mathbb{Z}_+$-ring is a ring with a __fixed__ $\mathbb{Z}_+$-basis B and with identity $1 = \sum c_i b_i$, where $c_i \geq 0$.

(iii) A unital $\mathbb{Z}_+$-ring is a $\mathbb{Z}_+$-ring where $1 \in B$.

Note that a $\mathbb{Z}_+$-ring is a free $\mathbb{Z}$-module, i.e. of the form $\mathbb{Z}^r$ (as $\mathbb{Z}$-module), where $r = |B|$ called the rank

__Notation:__ $[b_i b_j : b_k] = c_{ij}^k$

Def: A $\mathbb{Z}_+$-ring (with $\mathbb{Z}_+$-basis B) is called <u>transitive</u> if $\forall X, Z \in B$, $\exists Y_1, Y_2 \in B$ such that

$$[XY_1 : Z] \neq 0 \quad \text{and} \quad [Y_2 X : Z] \neq 0.$$

<u>Proposition:</u> If $\mathcal{C}$ is a ring category with left duals, then $Gr(\mathcal{C})$ is a transitive unital $\mathbb{Z}_+$-ring.

<u>proof:</u> We already proved that in a ring category with left duals, the unit object is simple. So $Gr(\mathcal{C})$ is unital.

We also proved that $\mathbb{1} \xrightarrow{coev_X} X \otimes X^*$ is a monomorphism, then so $\mathbb{1}$ is a simple subobject of $X \otimes X^*$, and by Jordan-Hölder we can complete $\mathbb{1} \subset X \otimes X^*$ as a Jordan-Hölder filtration

<u>Then</u>: $[X \otimes X^* : 1] > 0$.

<u>Now</u>: let Z be a simple object.

$Z \otimes 1 \simeq Z$, so $[Z \otimes 1 : Z] = 1$

<u>Then</u>; $[(X \otimes X^*) \otimes Z ; Z] > 0$

By associativity: $[(X \otimes X^*) \otimes Z : Z] = [X \otimes (X^* \otimes Z) ; Z]$

<u>So</u>: there is a simple object Y_1 such that:

$[X^* \otimes Z : Y_1] > 0$ and $[X \otimes Y_1 : Z] > 0$.

<u>Idem</u>: $[Z \otimes (X^* \otimes X) : Z] > 0$, so ..., $\exists Y_2$ s.t. $[Y_2 \otimes X : Z] > 0$.

⚠ $[X^* \otimes X : 1] \neq 0$ because $X^* \otimes X \xrightarrow{ev_X} 1$ (epi), so $0 \to Y \hookrightarrow X^* \otimes X \twoheadrightarrow 1 \to 0$ with $X^* \otimes X / Y = 1$

Definition (Frobenius–Perron dimension):

Let A be a transitive unital $\mathbb{Z}_+$-ring of finite rank

Let $X \in B$, define M_X to be the fusion matrix, i.e. given by left multiplication:

$$M_X = \left([XY : Z] \right)_{Y,Z \in B}$$

$$\mathrm{FPdim}(X) := \| M_X \|$$ (matrix norm).

Theorem: Above $\mathrm{FPdim}(X)$ is a positive eigenvalue of M_X.

(in fact, the maximal one).

proof: Frobenius–Perron theorem (see book Section 3.2)

→ well known: (Perron, 1907), (Frobenius, 1912).

Theorem: FPdim is a ring homomorphism:

$$\mathrm{FPdim}: A \longrightarrow \mathbb{C},$$

(as a linear extension from the elements of B).

($\to$ see the book for the proof.)

According to previous proposition, we can talk about FPdim for a finite ring category $\mathcal{C}$ with left duals.

Example/Exo: Let $\mathcal{C} = \mathrm{Rep}(G)$ be the category of finite dim. rep. of a finite group G over $\mathbb{C}$. Then $\mathrm{FPdim}(X) = \dim_{\mathbb{C}}(X)$

Hint:

- $\mathbb{C}G \simeq \bigoplus_{X\in B} X^{\oplus \dim X}$, so $[\mathbb{C}G] = \sum_{X\in B} \dim X\,[X]$. (well-known)
- $\forall X, Y$, $\mathrm{Hom}_{\mathcal{C}}(X \otimes \mathbb{C}G, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}G, {}^*X \otimes Y)$ (proved before)
- $\dim_{\mathbb{C}}(\mathrm{Hom}(\mathbb{C}G, Z)) = \dim_{\mathbb{C}}(Z)$. (Exo)

Then: $X \otimes \mathbb{C}G = (\dim_{\mathbb{C}} X)\,\mathbb{C}G$

+ use Prop 3.3.6 in the book.