Theorem: In a ring category with left duals, the unit object is simple.

proof: Let $X \subset \mathbb{1}$ a simple subobject of $\mathbb{1}$.

$$0 \to X \xrightarrow{i} \mathbb{1} \xrightarrow{!} Y \to 0.$$ exact sequence (where $Y = \mathbb{1}/X$).

$(-)^*$ is exact, so

$$0 \to Y^* \to \mathbb{1} \to X^* \to 0$$ exact too.

Then apply $X \otimes -$ (exact functor by def. of multiry cat).

$$0 \to X \otimes Y^* \to X \to X \otimes X^* \to 0$$

Exo: $X \xrightarrow{p} X'$ epi. with X simple $\Rightarrow X' \simeq 0$ or X.

But $X \otimes X^* \neq 0$ (using $\mathrm{coev}_X \neq 0$)

So by exo. $X \simeq X \otimes X^*$.

Exo: Let $A \xrightarrow{f} B$ a nonzero morphism. Then:

(i) A simple $\Rightarrow f$ monomorphism;

(ii) B simple $\Rightarrow f$ epimorphism.

Nb: $1 \xrightarrow{\mathrm{coev}_X} X \otimes X^*$ is a nonzero morphism.

But $X \otimes X^*$ is simple (because it is iso. to X simple)

So, by previous exo (ii), coev_X must be an epi.

Then: $1 \xrightarrow{\text{epi}} X \otimes X^* \xrightarrow{\sim} X$.

So, we have $1 \twoheadrightarrow X$.

But, we also have $X \subset 1$, so $X \hookrightarrow 1$

The notation $\twoheadrightarrow$ means epimorphism.

$\hookrightarrow$ means monomorphism.

Conclusion: $1 \xrightarrow{\text{epi}} X \xhookrightarrow{\text{mono}} 1$. / So we have a nonzero morphism $1 \to 1$

But: $\mathrm{End}_{\mathcal{C}}(1) = \mathbb{k} \cdot \mathrm{id}_1$

So $f = c \cdot \mathrm{id}_1$ with $c \in \mathbb{k}^*$, so f is iso, then epi and mono must be iso. / So $X \simeq 1$. ∎

Corollary: In a ring category with left duals, the evaluation maps are epimorphisms, and the coevaluation maps are mono.

proof: The evaluation and coevaluation maps are

$$X^* \otimes X \xrightarrow{ev_X} 1, \qquad 1 \xrightarrow{coev_X} X \otimes X^*$$

but 1 is simple (by previous thm), so the result follows by previous cor. ∎

Corollary: Let $\mathcal{C}$ be a ring category with left duals, and $F: \mathcal{C} \to \mathcal{D}$ an exact k-linear monoidal functor into a multiring cat. $\mathcal{D}$. Then F is faithful (so is a tensor functor, by def)

Recall: F faithful means: $\begin{cases} F(X) = 0 \Rightarrow X = 0 & \text{(or equiv. } X \neq 0 \Rightarrow F(X) \neq 0\text{)} \\ F(f) = 0 \Rightarrow f = 0. \end{cases}$

Proof: Let $X \in \mathcal{C}$, $X \neq 0$

Consider the monomorphism $1 \xrightarrow{\text{coev}_X} X \otimes X^*$ (mono. by previous Corollary)

Exo By exactness of F, $1 \to F(X) \otimes F(X^*)$ is also mono.

So, $F(X) \neq 0$.

Let f be a morphism of $\mathcal{C}$ such that $F(f) = 0$

Exo: Again, by exactness, $F(\mathrm{Im}(f)) = \mathrm{Im}(F(f)) = 0$

But we already proved that F is faithful on the object, so $\mathrm{Im}(f) = 0$

So, $f = 0$ ▨ (Exo: $\mathrm{Im}(f) = 0 \Rightarrow f = 0$)

Exercices: Let $\mathcal{C}$ be a multiring category with left duals, and let $X \in \mathcal{C}_{ij}$ and $Y \in \mathcal{C}_{jk}$ be nonzero objects.

(1) Show that $X \otimes Y \neq 0$

(this if $\mathcal{C}$ is ring cat., $(i=j=k)$
$\forall X, Y \in \mathcal{C}$ $X, Y \neq 0 \Rightarrow X \otimes Y \neq 0$)

(2) Deduce that $\text{length}(X \otimes Y) \geq \text{length}(X) \cdot \text{length}(Y)$.

(Jordan-Hölder...)

(there is no "zero divisor" in some sense)

(3) Show that if $\mathcal{C}$ is a ring cat. with left duals, then an invertible object in $\mathcal{C}$ is simple

Def: X is invertible if $X^* \otimes X \xrightarrow{ev_X} 1$ and $1 \xrightarrow{coev_X} X \otimes X^*$ are iso.

(4) Let X be an object in a multiring cat. with left duals, such that $X \otimes X^* \simeq 1$. Show that X is invertible.

Example: of a ring category where the unit object is **not** semisimple.

in particular not simple, so in particular, there is no left duals for any object

Consider the category $\mathcal{C}$ of triple (V, W, A), where V, W are finite dim. vector space (over $\Bbbk$ alg closed), and $A: V \to W$ linear map (i.e just a matrix).

Define the tensor product by:

usual tensor product on vector spaces

Kronecker product.

$$(V, W, A) \otimes (V', W', A') := (V \otimes V', W \otimes W', A \otimes A')$$

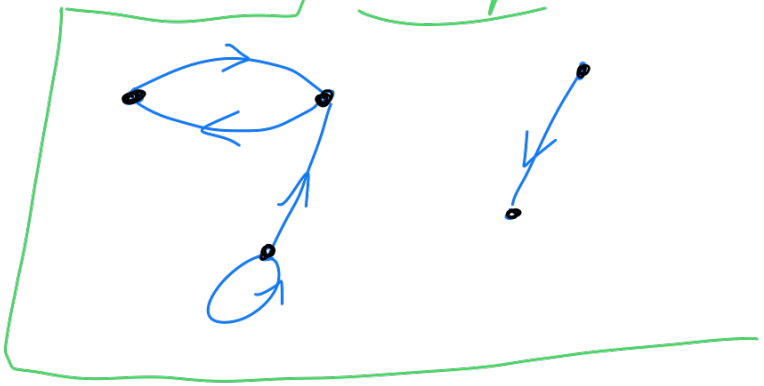
with obvious associativity constraints, and unit $\mathbb{1} = (\Bbbk, \Bbbk, id)$

Exo: Show that $\mathcal{C}$ is a ring cat., and that $\mathbb{1}$ is not semisimple.

Rk: Above category can be defined as the category of finite dim. representations of the quiver of type A_2.

Def: A quiver is a directed graph {vertices, oriented edges}. where loops and multiple arrows between two vertices are allowed.

Example,



Def: A_2 is $\bullet \longrightarrow \bullet$

Def: A representation of a quiver is the assignement of a vector space to each vertix, and a linear map to each oriented arrow (the map goes in the sense of the orientation).

Example: For A_2: $V \xrightarrow{A} W$ it is exactly the def. of triple (V, W, A) as written above.

Absence of self-extension of the unit object:

Def: Let $\mathcal{C}$ be an abelian category. Let $X, Y \in \mathcal{C}$.

Let $\mathrm{Ext}^1(Y,X)$ be the set of exact sequences

$$0 \to X \to Z \to Y \to 0 \qquad \text{(up to iso.)}$$

$\mathrm{Ext}^1(Y,X)$ is called the set of extensions of Y by X.

We will not go into the details here.

$\mathrm{Ext}^1(Y,X)$ has a structure of abelian group. (see Ch. 1 pg 4 of the book)

If $\mathcal{C}$ is the cat. of A-module where A algebra over k (alg. closed)
$\mathrm{Ext}^1(Y,X)$ corresponds to the "outer" derivations $\left\{\begin{matrix}\mathrm{Der} \\ \mathrm{Inner}\end{matrix}\right.$

$\left\{\begin{matrix}\mathrm{Der} = \{D : A \to \mathrm{Hom}_k(Y,X) \mid D(ab) = D(a)b + aD(b)\} \\ \mathrm{Inner} = \{D \in \mathrm{Der} \mid D(a) = [f,a]\}\end{matrix}\right.$

Let k be an alg. closed field of char. 0.

Theorem: Let $\mathcal{C}$ be a finite ring cat. over k, where the unit object is simple. Then $\mathrm{Ext}^1(1,1) = 0$

(in other words, there is no non trivial extension of 1 by itself

i.e, (up to iso) $0 \to 1 \to Z \to 1 \to 0$ exact $\Rightarrow Z \simeq 1 \oplus 1$.)

proof: See the book on p71.

Exo: Deduce that if a finite ring cat. $\mathcal{C}$ over k (alg. closed, char. 0) has a unique simple object (up to iso) 1. Then $\mathcal{C} \simeq \mathrm{Vec}$.

Counter_exo: in char $p>0$: $\mathrm{Rep}(\mathbb{Z}/p\mathbb{Z})$ over $\mathbb{F}_p$.
Show that $\neq \mathrm{Vec}$.

cat. of finite dim k-vector spaces

Grothendieck ring and Frobenius–Perron dimensions.

We will provide the detailed def. tomorrow.

Without details. $\mathcal{C}$ multitensor cat.

$(Gr(\mathcal{C}), +, \times)$ Grothendieck ring.

elements: generated by iso classes of simple objects.

$+$: from $\oplus$

$\times$: from $\otimes$