Proposition: A finite ring category $\mathcal{C}$ with left duals is a tensor category (i.e. it also has right duals).

Idea of the proof: Show that the functor $(-)^*$ is essentially surjective
(i.e. $\forall X \in \mathcal{C} \quad \exists X' \in \mathcal{C}$ s.t. $(X')^* \simeq X$)
(recall that if $A^* = B$ is a left dual of A, then A is a right dual B)

We will start to prove that on the simple objects. | We will first need to show that if X is simple then so is X^*.

FPdim means Frobenius–Perron dimension,
Recall that $\mathrm{FPdim}(X^*) = \mathrm{FPdim}(X)$.

Let $d_1 < d_2 < \dots < d_r$ be the (ordered) list of possible FPdim of simple objects of $\mathcal{C}$. (i.e. $\forall S$ simple $\exists i$ s.t. $\mathrm{FPdim}(S) = d_i$).

Let $\mathcal{O}(\mathcal{C})$ be the simple objects of $\mathcal{C}$ up to isomorphism. (It is a finite set by assumption)

Let $\mathcal{O}_i(\mathcal{C})$ subset of ______ (up to iso) with FPdim $= d_i$, $(i = 1, \dots)$

The proof of the proposition requires to prove the following lemma:

<u>Lemma</u>: Let $L \in \mathcal{O}_i(\mathcal{C})$. Then L^* is simple and $\exists L' \in \mathcal{O}_i(\mathcal{C})$ such that $L \simeq (L')^*$. (in other words, $(-)^*$ induces a bijection of $\mathcal{O}_i(\mathcal{C})$).

(because finite set).

<u>proof</u>: If L^* is not simple, then there exists a nonzero map $K \longrightarrow L^*$ with K simple. (you can make a sequence of strict subobject $L^* \supset \dots$ and by finiteness, it must reach a simple object K)

and $\mathrm{FPdim}(K) < \mathrm{FPdim}(L^*) = \mathrm{FPdim}(L) = d_i$

If $i = 1$, then $\mathrm{FPdim}(K) < d_1$, contradiction (d_1 is the smallest possible FPdim of a simple object).

<u>So</u>: $(-)^*$ is stable on $\mathcal{O}_1(\mathcal{C})$.

<u>Claim</u>: If $(-)^*$ is stable on $\mathcal{O}_i(\mathcal{C})$ for some i, then it is a bijection.

proof of the claim: we are reduced to prove that it is an injection (because $\mathcal{O}_i(\mathcal{C})$ is a finite set).

Let $L_1, L_2 \in \mathcal{O}_i(\mathcal{C})$ such that $L_1^* \simeq L_2^*$. We will show that $L_1 \simeq L_2$.

$L_1^* \simeq L_2^* \Rightarrow \mathrm{Hom}_{\mathcal{C}}(L_1^*, L_2^*)$ is nonzero. But $(-)^*$ is full, so

$\mathrm{Hom}_{\mathcal{C}}(L_2, L_1)$ is also nonzero, but L_1, L_2 are simple, so by Schur's Lemma the nonzero element of $\mathrm{Hom}_{\mathcal{C}}(L_1, L_2)$ must be an iso, i.e. $L_1 \simeq L_2$.

So $(-)^*$ is injective from a finite set to itself, so is bijective.

In particular, $\forall L \in \mathcal{O}_i(\mathcal{C})$, $\exists L' \in \mathcal{O}_i(\mathcal{C})$ s.t. $(L')^* \simeq L$. $\boxed{\checkmark}$ (claim).

We know that $(-)^*$ is stable on $\mathcal{O}_1(\mathcal{C})$. We deduce that Lemma is OK for $i=1$. We will make a proof by induction for general i.

Let us come back to L and $K \overset{\text{simple}}{\subset} L^*$ from the beginning

$\mathrm{FPdim}(K) < d_i$ we can apply the induction on K: $\exists K'$ simple such that $(K')^* \simeq K$.

There is a nonzero map $K \to L^*$, $\mathrm{Hom}_{\mathcal{C}}(K, L^*) \neq 0$.

but (again) the functor $(-)^*$ is full, so $\mathrm{Hom}_{\mathcal{C}}(L, K') \neq 0$.

with L, K' simple. So again by Schur's lemma $L \simeq K'$.

Contradiction with $\mathrm{FPdim}(K) < d_i = \mathrm{FPdim}(L)$.

Then, the initial assumption "L^* is not simple" is false,

so L^* is simple, so $(-)^*$ is stable on $\mathcal{O}_i(\mathcal{C})$, $\forall i$!

Thus, by the previous claim, it is a bijection. $\boxtimes$ ← of the lemma.

Let us come back to the proof of the proposition.

By the previous lemma: the contravariant functor $(-)^*$ induces a bijection / permutation on $\mathcal{O}_i(\mathcal{C})$, so on $\mathcal{O}(\mathcal{C})$ (finite set also by finiteness).

So, the covariant functor $G = (-)^{**}$ is also a permutation of $\mathcal{O}(\mathcal{C})$

But a permutation of a finite set is an element of the group S_n (with n = ord. of the set). But S_n is a finite group, and every element g of a finite group has finite order (i.e. smallest r such that $g^r = e$).

Then, $\exists\, r$ such that $F := \underbrace{G \circ \cdots \circ G}_{r \text{ times}}$ denoted G^r satisfies

$F(L) \simeq L \quad \forall L \in \mathcal{O}(\mathcal{C})$.

Exo: Show that F is exact and fully faithful.

Let $L \in \mathcal{O}(\mathcal{C})$, consider its projective cover $P(L)$.

It exists by finite assumption.

Exo: $F(P(L))$ is projective.

$$P(L) \xrightarrow{p} L \quad \text{epi.}$$

$$F \downarrow$$

$$F(P(L)) \xrightarrow{F(p)} F(L) \simeq L \quad \text{epi.}$$

What about $F(P(L)) \xrightarrow{p'} L'$ epi ? $\quad p'$ epi. L' simple. ??

F is full: $\quad F \uparrow$

$$P(L) \to L' \quad \text{epi}$$

Exo: $P(L) \to L'$ epi with L' simple $\Rightarrow L' \simeq L$.

So: $L' \simeq L$ (because F is full), and $F(P(L))$ has a unique simple quotient $F(L) \simeq L$.

Then, by the universality of the projective cover:

$$\begin{tikzcd} F(P(L)) \arrow[d, dashed] \arrow[dr] & \\ P(L) \arrow[r] & L \end{tikzcd}$$

$\exists$ epi: $F(P(L)) \to P(L)$

Exo: F keeps the length of the object.

Then, $F(P(L))$ and $P(L)$ have the same length.

Exo: Deduce that the epi $F(P(L)) \to P(L)$ is an iso.

Exo: Deduce that F is surjective on iso. classes of projective objects.

Exo (look for a ref): Any object of $\mathscr{C}$ is iso. to some $\mathrm{Coker}(m)$ with $P_1 \xrightarrow{m} P_2$ for some projective P_1, P_2.

Exo: Deduce from previous exo and the fact that F is fully faithful that F is surjective on iso. classes of objects (i.e. essentially surj).

Fact: fully faithful + essentially surjective $\iff$ equivalence

So F is an equivalence.

In particular, $(-)^*$ must be essentially surjective, i.e. $\forall X \in \mathscr{C}$ $\exists X'$ such that $(X')^* \cong X$. [Recall that if $A^* = B$ then A is a right dual of B]

So X' is a right dual of X so right dual always exist, so it's rigid, so tensor

Proposition: Let P be a projective object in a multiring category. If $X \in \mathcal{C}$ has a left dual (resp. right) then the object $P \otimes X$ (resp. $X \otimes P$) is projective.

proof: We will prove the left case (the right is similar).

Recall: An object Q is projective iff $\mathrm{Hom}_{\mathcal{C}}(Q, -)$ is exact.

Here we need to show that $\mathrm{Hom}_{\mathcal{C}}(P \otimes X, -)$ is exact.

But X has a left dual, so (by the natural adjunction iso) the above functor is exact iff $\mathrm{Hom}_{\mathcal{C}}(P, - \otimes X^*)$ is exact, but this last is exact as the composition of two exact functors

- $- \mapsto - \otimes X^*$ (exact by def of multiring cat)
- $- \mapsto \mathrm{Hom}_{\mathcal{C}}(P, -)$ (exact because P is projective)

<u>Corollary</u>: If $\mathcal{C}$ is a multiring cat. with left duals (for example a multitensor category), then $1 \in \mathcal{C}$ is projective object iff $\mathcal{C}$ is semi simple.

See proof tomorrow.