About Exercise 3 of last course:

$\mathcal{C}$ is a monoidal category, V is an object in $\mathcal{C}$.

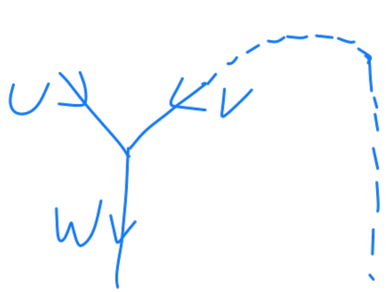
If V has a left dual V^*, show that there is natural iso:

$$\mathrm{Hom}_{\mathcal{C}}(U\otimes V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(U, W\otimes V^*)$$

Hint: consider iso $f \overset{\alpha}{\longmapsto} (f\otimes \mathrm{id}_{V^*})\circ(\mathrm{id}_U \otimes \mathrm{coev}_V)$

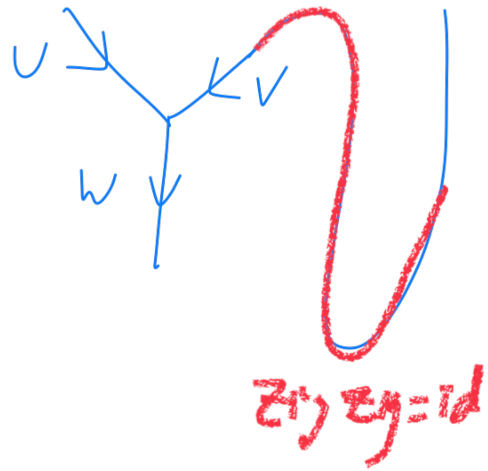
with inverse $g \overset{\beta}{\longmapsto} (\mathrm{id}_W \otimes \mathrm{ev}_V)\circ(g\otimes \mathrm{id}_V)$

Pictorially:



Picture for α.

Picture for β



Picture for $\beta\circ\alpha = \mathrm{id}$

Def: An object in a monoidal category is called rigid if it has left and right duals. A monoidal category is called rigid if every object in it is rigid.

Example 1: The category Vec of finite dim. $\Bbbk$-vector spaces is rigid: the left and right dual of V is its dual space V^* (of linear forms),

where $\mathrm{ev}_V : V^* \otimes V \to \Bbbk$ (contraction)
$f \otimes v \mapsto f(v)$.

$\mathrm{coev}_V : \Bbbk \to V \otimes V^*$
$1 \mapsto \sum_i v_i \otimes v_i^*$

where (v_i) is a basis of V, (v_i^*) its dual basis in V^* (note that this sum is independent of the choice of basis).

Note that there is no coev_V in the infinite dim. case (see the book for details).

Example 2: The category $\mathrm{Rep}(G)$ of finite dimensional rep. of a group G (over $\Bbbk$) is rigid.

Just take V^*, ev_V, coev_V as for Vec.

Note that:

$$\rho_{V^*}(g) := \left(\rho_V(g)^{-1}\right)^*$$

$\in G$

rep.

dual object.

matrix inverse

matrix adjoint

Example 3: The category Vec_M is rigid iff the monoid M is a group, with $\delta_g^* = {}^*\delta_g = \delta_{g^{-1}}$.

$$\mathrm{ev}_{\delta_g} : \underbrace{\delta_{g^{-1}} \otimes \delta_g}_{=\delta_1} \to \delta_1$$

$$\mathrm{coev}_{\delta_g} : \delta_1 \longrightarrow \underbrace{\delta_g \otimes \delta_{g^{-1}}}_{=\delta_1}$$

(take id, for both.)

Vec_G^ω is also rigid, you can take ev and coev as before, up to multiplying by appropriate constants $\omega(-,-,-)$. [EGNO]

Final remarks on this chapter (monoidal cat.):

Let $\mathcal{C}$ be a monoidal category with left (resp. right) duals.

By previous results, there is a contravariant left (resp. right) ← arrows reversing (<> covariant)

duality functor (fully faithful): $X \mapsto X^*$, $f \mapsto f^*$ (resp. $X \mapsto {}^*X$, $f \mapsto {}^*f$)

You get monoidal functors from $C^\vee \to C^{op}$.

dual cat. (arrow reversed) opposite cat. (tensor product reversed)

Then, $X \longmapsto X^{**}$, $X \longmapsto {}^{**}X$ are monoidal functors.

Now for a rigid monoidal cat., the functors $C^\vee \to C^{op}$ above are monoidal equivalent (mutually quasi-inverse)

Then, for rigid monoidal cat. the notion of dual cat $(C^\vee)$ and opposite cat. (C^{op}) are equivalent.

Then: $X \mapsto X^{**}$, $X \mapsto {}^{**}X$ are autoequivalence, but possibly non-trivial (in the sense that V^* and ${}^*V$ could be non-isomorphic) recall that ${}^*(V^*) \simeq ({}^*V)^* \simeq V$.

Exercice: Let $\mathcal{C}$ and $\mathcal{D}$ be rigid monoidal categories. Let $F_i : \mathcal{C} \to \mathcal{D}$ $(i=1,2)$ be monoidal functors. Show that a natural transformation of monoidal functors $\eta : F_1 \to F_2$ must be a natural isomorphism (it's false in the non-rigid case).

We finished the chapter about monoidal categories (Chap 2 in the book)
We follow the recommendation at the end of the intro. of the book.
So now we will teach chapter 4. (tensor category).

Def: Let $\mathbb{k}$ be a field. A multitensor cat. $\mathcal{C}$ (word will be defined just after)
is a locally finite $\mathbb{k}$-linear abelian rigid monoidal cat. $\mathcal{C}$
such that the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on morphisms.
Such $\mathcal{C}$ is called indecomposable if it is not equivalent to a direct sum of nonzero multitensor categories. If in addition,
$\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \simeq \mathbb{k}$ ($\mathbb{1}$ is simple), then $\mathcal{C}$ is called a tensor category.
A (multi)-fusion cat. is a finite semisimple (multi)tensor cat.

In other words: A fusion cat. $\mathcal{C}$ (over $\Bbbk$) is a

1. $\Bbbk$-linear
2. semi simple
3. rigid
4. monoidal category
5. with finitely many (iso. classes of) simple objects
6. with finite dim. hom-space.
7. $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$.

Note that the notion of fusion category is a "categorification" of the notion of fusion ring $\mathcal{F}$, defined as follows:

- finite basis $B = \{b_1, \dots, b_r\}$ 5 $(r<\infty)$

It is a generalization of the product on a finite group G, or of the $\otimes$ of its irr. rep.

- fusion rules: $b_i b_j = \sum_{k=1}^{r} N_{ij}^{k} b_k$ 2 with $N_{ij}^{k} \in \mathbb{Z}_{\geq 0}$. 6

$(N_{ij}^{k} = \dim(\mathrm{Hom}(X_i \otimes X_j, X_k))$

* Associativity: $b_i (b_j b_k) = (b_i b_j) b_k$ 4
* Neutral: $b_1 b_i = b_i b_1 = b_i$ 7 (Neutral element is simple)
* Inverse/Adjoint/Dual: $\forall i \ \exists! \ i^*$ with $N_{ik}^{1} = N_{ki}^{1} = \delta_{i^*,k}$ 3
* Frobenius reciprocity: 1 $N_{ij}^{k} = N_{i^*k}^{j} = N_{kj^*}^{i}$

(this means that the fusion matrix of dual = adjoint matrix)

We will see that we can associate a fusion ring to a fusion cat. (called its Grothendieck ring). Note that not all fusion rings are Grothendieck (i.e. they are not all categorifiable).

(every monoidal cat. is equiv. to its skeleton).

A (skeletal) fusion category is just given by a fusion ring and a solution of its Pentagon Equation (which is a system of polynomial equations of degree 3 with several variables, corresponding to the Pentagon Axiom of a monoidal cat.).

Thm: The classification of fusion categories reduces to classify the fusion rings and their solution (if exist) to their Pentagon system of equations. (P.E).

Main Problem in the theory of fusion cat: see whether a given fusion ring admits a solution for its P.E. (i.e. categorifiable?)

VERY HARD (in general) because the system P.E is HUGE.

Tomorrow, we will see the notion of Frobenius_Perron dim. for a fusion ring, then, we will define all the new notions required in the def of multitensor cat. (coming from ch 1).