Formally:

$$\begin{tikzcd}
& (X_1^* \otimes (X \otimes X_2^*)) \otimes 1 \arrow[dr, "(id_{X_1^*} \otimes id_{X \otimes X_2^*}) \otimes c_1"] & \\
(X_1^* \otimes 1) \otimes 1 \arrow[ur, "(id_{X_1^*} \otimes c_2) \otimes id_1"] \arrow[dr, "(id_{X_1^*} \otimes id_1) \otimes c_1"'] & & (X_1^* \otimes (X \otimes X_2^*)) \otimes (X \otimes X_1^*) \\
& (X_1^* \otimes 1) \otimes (X \otimes X_1^*) \arrow[ur, "(id_{X_1^*} \otimes c_2) \otimes id_{X \otimes X_1^*}"'] &
\end{tikzcd}$$

These two compositions are equal because (just need to write it):

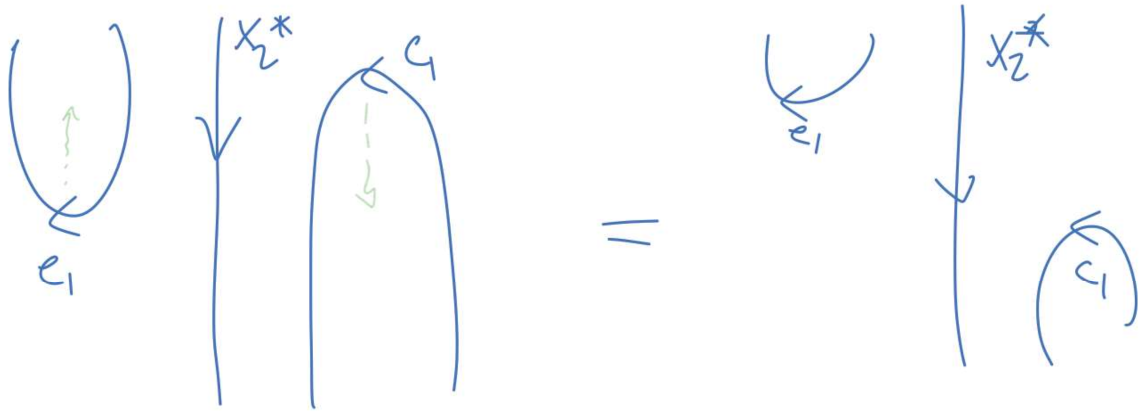
$$\left[(id_{X_1^*} \otimes id_{X \otimes X_2^*}) \otimes c_1\right] \circ \left[(id_{X_1^*} \otimes c_2) \otimes id_1\right] = (id_{X_1^*} \otimes c_2) \otimes c_1$$

$$\|$$

$$\left[(id_{X_1^*} \otimes id_1) \otimes c_1\right] \circ \left[(id_{X_1^*} \otimes c_2) \otimes id_{X \otimes X_1^*}\right] = (id_{X_1^*} \otimes c_2) \otimes c_1$$

So the subdiagram 1 commutes.

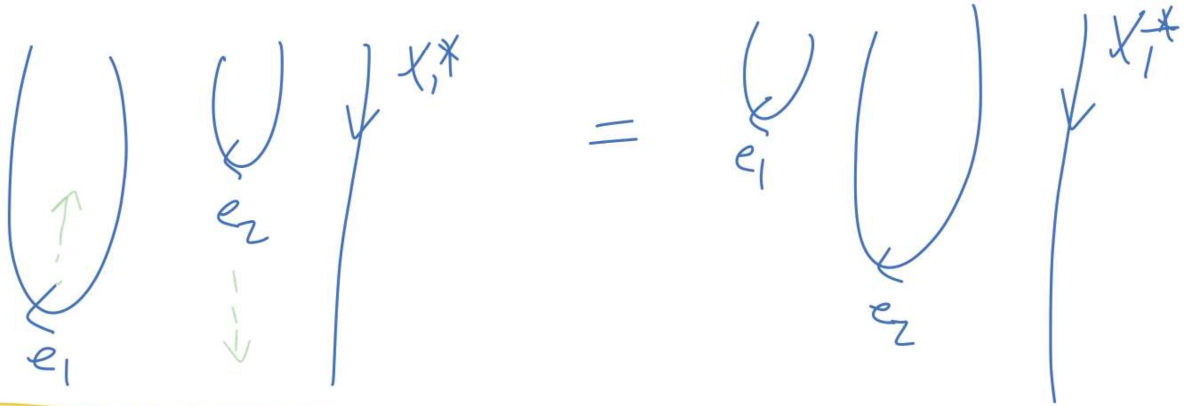
2



commutation

OK.

3



OK

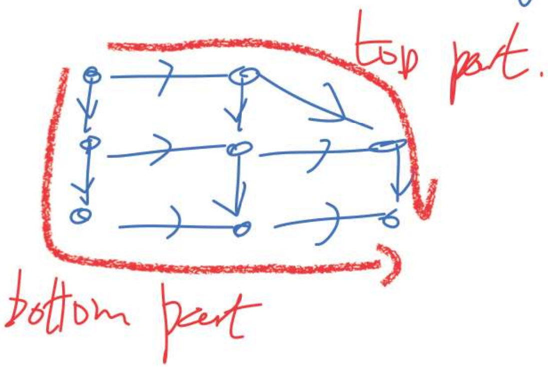
we can ignore these parts (just id here)

4

$$
\begin{tikzcd}
X_1^* \otimes X \otimes X_1^* \arrow[d, "id \otimes c_2 \otimes id"'] \arrow[drr, "id"] & & \\
X_1^* \otimes X \otimes X_2^* \otimes X \otimes X_1^* \arrow[rr, "id \otimes e_2 \otimes id"'] & & X_1^* \otimes X \otimes X_1^*
\end{tikzcd}
$$

The commutation here is exactly the first axiom for X_2^* to be a left dual of X. OK.

By the commutation of 1, 2, 3, 4 the big diagram above commutes. It looks like the following:



Observe that the top part is exactly the second axiom for X_1^* to be a left dual of X: so it's the identity.

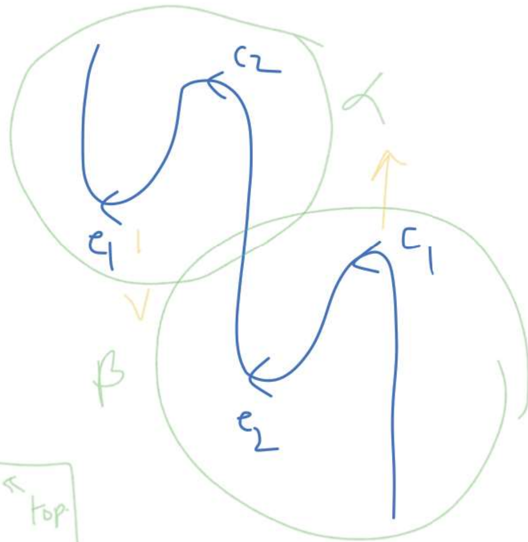
Then the bottom part is also the identity.

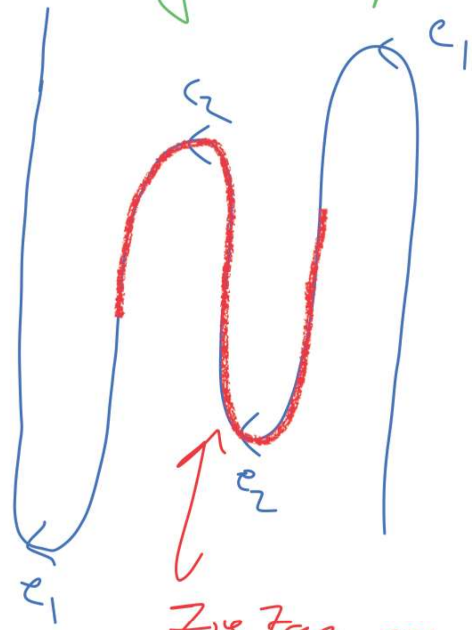
But the bottom part is exactly $\beta\circ\alpha$

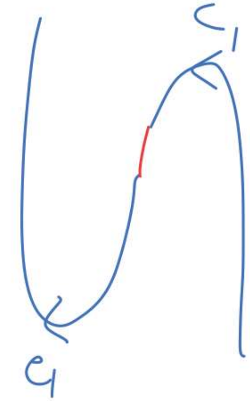
So $\beta\circ\alpha=\mathrm{id}$. We can prove similarly that $\alpha\circ\beta=\mathrm{id}$

Conclusion: α is an isomorphism.

Pictorial proof of $\beta \circ \alpha = \mathrm{id}$ (idem for $\alpha \circ \beta = \mathrm{id}$).

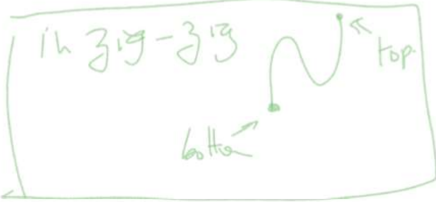
$\beta \circ \alpha =$ 

$=$ 

$=$ 

$=$  $= \mathrm{id}$

Zig zag equation  $=$  $= \mathrm{id}$.



It remains to prove that α preserves ev. and coev., and that it's the only such iso. from X_1^* to X_2^*.

Preservation:

commutation of:

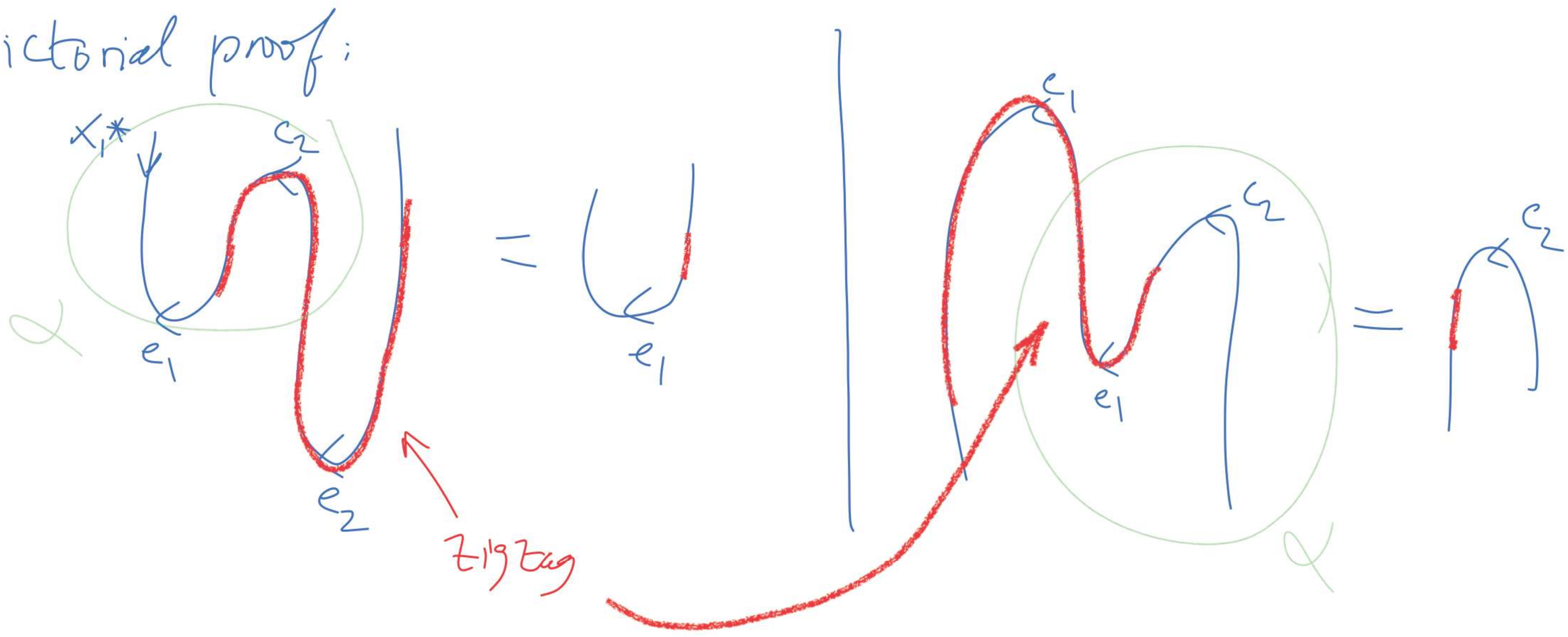
$$\begin{tikzcd}
X_1^* \otimes X \arrow[dr, "e_1"] \arrow[dd, "\alpha \otimes \mathrm{id}"'] & & X \otimes X_1^* \arrow[dd, "\mathrm{id} \otimes \alpha"] \\
\circlearrowright & 1 \arrow[ur, "c_1"] \arrow[dr, "c_2"'] & \circlearrowright \\
X_2^* \otimes X \arrow[ur, "e_2"'] & & X \otimes X_2^*
\end{tikzcd}$$

In other words:

$e_2 \circ (\alpha \otimes \mathrm{id}_X) = e_1$

$(\mathrm{id}_X \otimes \alpha) \circ c_1 = c_2$

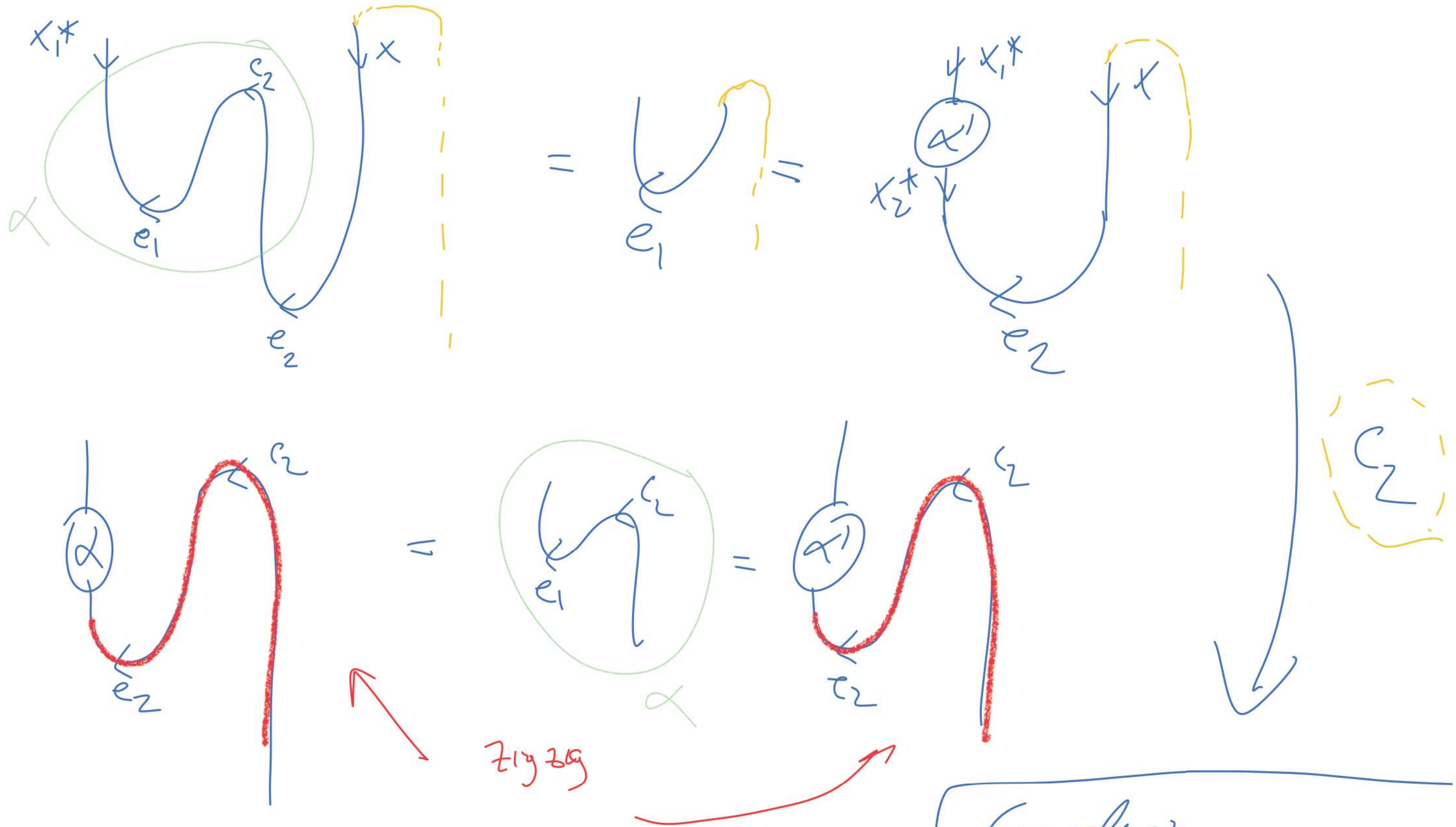
Pictorial proof:



Finally, let us prove the unicity (by pictures).
Let α' be an other iso. such that

$$e_2 \circ (\alpha \otimes id_X) = e_1 = e_2 \circ (\alpha' \otimes id_X)$$

note that this only equality is enough for the proof



Then

Conclusion:

$\alpha' = \alpha$

The proof for right dual is similar...

<u>Dual of morphisms:</u> Let X, Y be two objects of the monoidal category $\mathcal{C}$, which have left dual X^* and Y^*.

Let $f: X \to Y$ be a morphism.

We define $f^*: Y^* \to X^*$ as follows:

$$f^* := \begin{tikzcd}
Y^* \arrow[r, "\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X"] \arrow[dr, "f^*"'] & Y^* \otimes (X \otimes X^*) \arrow[r, "a^{-1}_{Y^*, X, X^*}"] & (Y^* \otimes X) \otimes X^* \arrow[d, "(\mathrm{id}_{Y^*} \otimes f) \otimes \mathrm{id}_{X^*}"] \\
 & X^* & (Y^* \otimes Y) \otimes X^* \arrow[l, "\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}"']
\end{tikzcd}$$

Similarly, for ${}^*X, {}^*Y$, ${}^*f : {}^*Y \to {}^*X$

$$
\begin{tikzcd}
{}^*f := {}^*Y \arrow[r, "\mathrm{coev}'_X \otimes \mathrm{id}_{{}^*Y}"] \arrow[drr, "{}^*f"'] & ({}^*X \otimes X) \otimes {}^*Y \arrow[r, "a_{{}^*X, X, {}^*Y}"] & {}^*X \otimes (X \otimes {}^*Y) \arrow[d, "\mathrm{id}_{{}^*X} \otimes (f \otimes \mathrm{id}_{{}^*Y})"] \\
& {}^*X & {}^*X \otimes (Y \otimes {}^*Y) \arrow[l, "\mathrm{id}_{{}^*X} \otimes \mathrm{ev}'_Y"']
\end{tikzcd}
$$

Exercise 1 (monoidal functor preserve left dual): Let $\mathcal{C}, D$ be monoidal cat. Let $F = (F, J, \varphi)$ be a monoidal functor $F : \mathcal{C} \to D$. Let X be an object of $\mathcal{C}$ with a left dual X^*. Prove that $F(X^*)$ is a left dual of $F(X)$ with ev. and coev. as follows:

$$\mathrm{ev}_{F(X)} : F(X^*) \otimes F(X) \xrightarrow{J_{X^*, X}} F(X^* \otimes X) \xrightarrow{F(\mathrm{ev}_X)} F(1) \xrightarrow{\varphi^{-1}} 1$$

$$\mathrm{coev}_{F(X)} : 1 \xrightarrow{\varphi} F(1) \xrightarrow{F(\mathrm{coev}_X)} F(X \otimes X^*) \xrightarrow{J^{-1}_{X, X^*}} F(X) \otimes F(X^*)$$

in other words, you can take $F(X)^* := F(X^*)$

Exercice 2 (dual morphisms and composition and tensor product.): Let $\mathcal{C}$ be a monoidal cat., let U, V, W be objects of $\mathcal{C}$ and let

$f: V \to W$, $g: U \to V$ be morphisms.

Show that:

(a) If U, V, W have left (resp. right) duals then

$$(f \circ g)^* = g^* \circ f^* \quad \left(\text{resp. } {}^*(f \circ g) = {}^*g \circ {}^*f\right)$$

(b) If U, V have left (resp. right) duals then $U \otimes V$ has a left dual $V^* \otimes U^*$ (resp. ${}^*V \otimes {}^*U$).

Exercice 3 (left tensor functor, dual and adjoint functor):

Let $\mathcal{C}$ be a monoidal cat. and let V be an object in $\mathcal{C}$.

(a) If V has a left dual V^*, show that there are natural "adjunction" isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}}(U \otimes V, W) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(U, W \otimes V^*)$$

$$\mathrm{Hom}_{\mathcal{C}}(V^* \otimes U, W) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(U, V \otimes W)$$

(b) If V has a right dual ${}^*V$, show that ______ ______ :

$$\mathrm{Hom}_{\mathcal{C}}(U \otimes {}^*V, W) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(U, W \otimes V)$$

$$\mathrm{Hom}_{\mathcal{C}}(V \otimes U, W) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(U, {}^*V \otimes W)$$

<u>Hint</u>: For the first, consider iso $f \mapsto (f \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_U \otimes \mathrm{coev}_V)$ with inverse $g \mapsto (\mathrm{id}_W \otimes \mathrm{ev}_V) \circ (g \otimes \mathrm{id}_V$

Remark: Previous exercise says that when a left dual of V exists then the functor $V^* \otimes \_$ is a left adjoint of $V \otimes \_$ (idem for right).