Rigid monoidal categories.

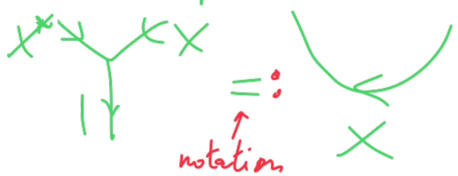
A monoidal category $\mathcal{C}$ is called *rigid* if every object admits a left and a right *dual object*.

Def: An object $X^* \in \mathcal{C}$ is called a *left dual* of $X \in \mathcal{C}$ if there are:

$ev_X : X^* \otimes X \to 1$ (called evaluation morphism)

$coev_X : 1 \to X \otimes X^*$ (called coevaluation morphism)

Pictures for these morphisms:

ev_X : 

$coev_X$: 

such that the following compositions are identity morphisms:

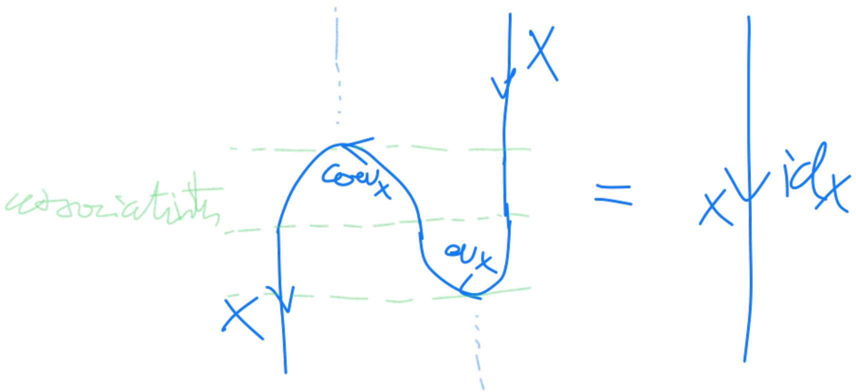
$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X = X \otimes 1$$

$$X = 1 \otimes X$$

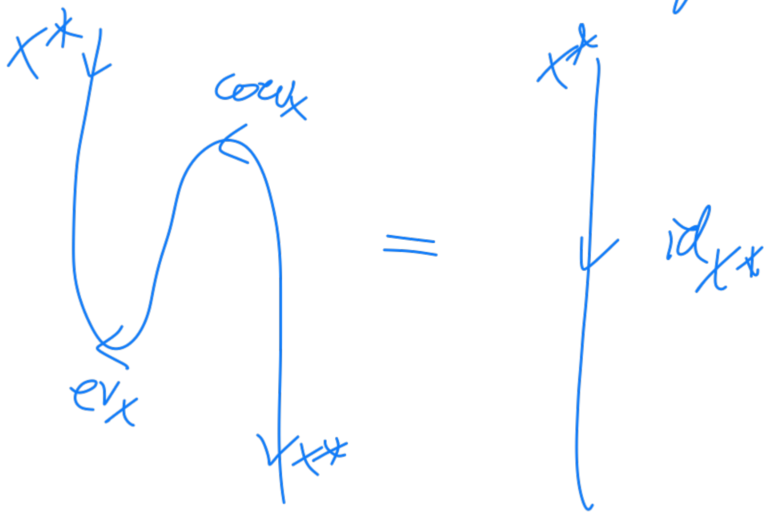
$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^* = 1 \otimes X^*$$

$$X^* = X^* \otimes 1$$

Pictorially, these compositions are as follows: (zigzag/snake equations)



and



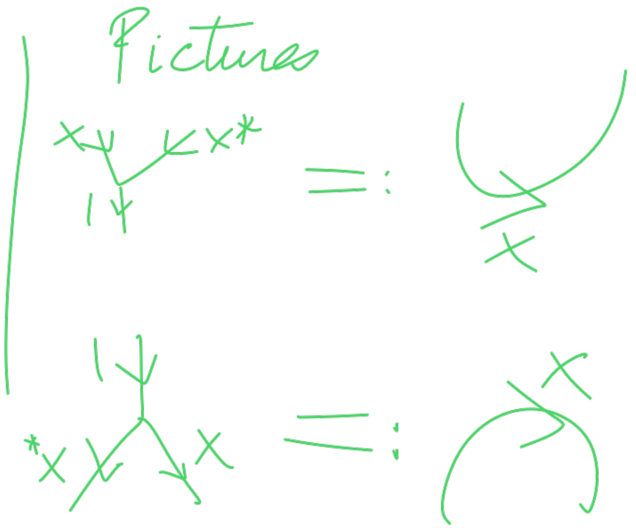
Idem for right dual:

Def: An object ${}^*X$ in $\mathcal{C}$ is called a <u>right dual</u> of X

if there are morphisms:

$$ev'_X : X \otimes {}^*X \longrightarrow 1$$

$$coev'_X : 1 \longrightarrow {}^*X \otimes X$$



such that

$$\underset{\overset{\shortparallel}{X\otimes 1}}{X} \xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{ev'_X \otimes id_X} X$$

$$\underset{\overset{\shortparallel}{1\otimes {}^*X}}{{}^*X} \xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} {}^*X$$

are identity morphisms.

Remark: If X^* is a left dual of X, then X is a right dual of X^* (in other words, you can take ${}^*(X^*) = X$).

proof: take ${}^*(X^*) := X$, $ev'_{X^*} := ev_X$, $coev'_{X^*} := coev_X$.

Then we need to check that those two compositions are identities

$$X^* \xrightarrow{id_{X^*}\otimes coev'_{X^*}} X^*\otimes\left({}^*(X^*)\otimes X^*\right) \xrightarrow{a^{-1}_{X^*,{}^*(X^*),X^*}} \left(X^*\otimes {}^*(X^*)\right)\otimes X^* \xrightarrow{ev'_{X^*}\otimes id_{X^*}} X^*$$

$$X \dashrightarrow \cdots \cdots \dashrightarrow X$$

They reformulate by the def. as

$$X^* \xrightarrow{id_{X^*}\otimes coev_X} X^*\otimes(X\otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^*\otimes X)\otimes X^* \xrightarrow{ev_X\otimes id_{X^*}} X^*$$

$$X \dashrightarrow \cdots$$

which are identities by def of the fact that X^* is a left dual of X $\blacksquare$

Idem: If $^*X$ is the right dual of X, then X is a left dual of $^*X$

In other words, you can take $(^*X)^* := X$.

Remark: $1^* = {}^*1 = 1$ (by convention $1 \otimes 1 = 1$, so just take ev. ad coev. to be identities
(without convention, just take ι or ι^{-1}))

added part

Left and right adjoint of a functors:

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories (locally small, i.e. $\hom_{\mathcal{C}}(X,Y)$ is a set)

Consider two functors $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$

such that for all $X \in \mathcal{C}$, $Y \in \mathcal{D}$

$$\hom_{\mathcal{C}}(F(Y), X) \simeq \hom_{\mathcal{D}}(Y, G(X))$$

where $\simeq$ is natural in the following sense: there are natural iso. α_X, β_Y

$$\begin{array}{ccc} \hom_{\mathcal{C}}(F(-), X) : \mathcal{D} \to Set & & \hom_{\mathcal{C}}(F(Y), -) : \mathcal{C} \to Set \\ \Downarrow \alpha_X & \text{and} & \Downarrow \beta_Y \\ \hom_{\mathcal{D}}(-, G(X)) : \mathcal{D} \to Set & & \hom_{\mathcal{D}}(Y, G(-)) : \mathcal{C} \to Set \end{array}$$

It looks like the notion of adjoint matrix/operators defined as
V: vector space (or Hilbert space), $M \in End(V)$ matrix.
$\forall u, v \in V$ $\langle Mu, v \rangle = \langle u, M^* v \rangle$ / where M^* = adjoint of M.

F is called a left adjoint functor of G

G ——— right ——— F.

Exercice: Let $\mathcal{C}$ be a category and $End(\mathcal{C})$ the monoidal category of functors from $\mathcal{C}$ to $\mathcal{C}$ (endofunctors).

Let $F, G \in End(\mathcal{C})$ be endofunctors.

Show that F is a left adjoint functor of G in $\mathcal{C}$. (def as above with $\mathcal{D} = \mathcal{C}$)

iff F is a left dual of G in $End(\mathcal{C})$.

Idea: G _____ right __________ F __________

__________ G _____ right __________ F _______________

Then: the notion of left (resp. right) dual object generalizes the notion of ____________________ adjoint functor.

end of added part.

Proposition: If $X \in \mathcal{C}$ (monoidal category) has a left (resp. right) dual object, then it is unique up to a unique isomorphism preserving the ev. and coev. morphisms.

proof: Let X_1^*, X_2^* be two left dual objects of X.

Let e_1, c_1, e_2, c_2 be corresponding evaluation and coev. morphisms.

Consider the following morphism $\alpha : X_1^* \to X_2^*$ defined as

$$\underset{\substack{\shortparallel \\ X_1^* \otimes 1}}{X_1^*} \xrightarrow{id_{X_1^*} \otimes c_2} X_1^* \otimes (X \otimes X_2^*) \xrightarrow{a^{-1}_{X_1^*, X, X_2^*}} (X_1^* \otimes X) \otimes X_2^* \xrightarrow{e_1 \otimes id_{X_2^*}} \underset{\substack{\shortparallel \\ 1 \otimes X_2^*}}{X_2^*}.$$

Similarly, consider $\beta : X_2^* \to X_1^*$.

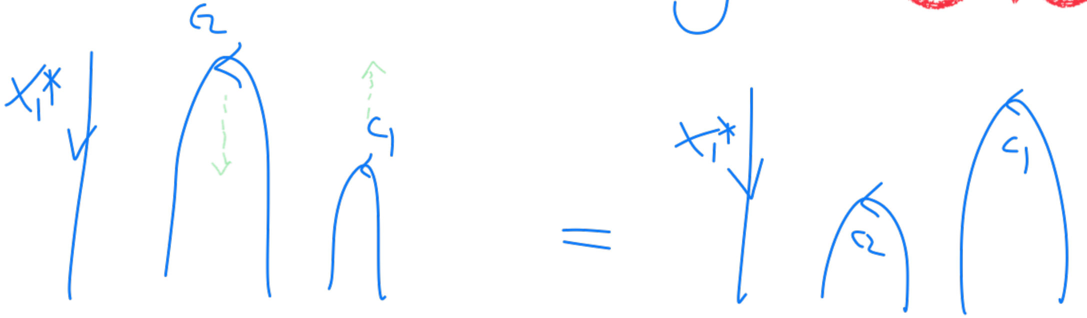
We will show that $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity morphisms (so that α is iso.)

Consider the following diagram (where the associativity constraints and parenthesize are suppressed) ← OK by coherence theorem (convention also).

$$\begin{tikzcd}
X_1^* \arrow[r, "id\otimes c_1"] \arrow[d, "id\otimes c_2"'] \arrow[dr, phantom, "1"] & X_1^*\otimes X\otimes X_1^* \arrow[d, "id\otimes c_2\otimes id"] \arrow[dr, "id"] \arrow[dr, phantom, "4", shift right=4ex] & \\
X_1^*\otimes X\otimes X_2^* \arrow[r, "id\otimes c_1"] \arrow[d, "e_1\otimes id"'] \arrow[dr, phantom, "2"] & X_1^*\otimes X\otimes X_2^*\otimes X\otimes X_1^* \arrow[r, "id\otimes e_2\otimes id"'] \arrow[d, "e_1\otimes id"] \arrow[dr, phantom, "3"] & X_1^*\otimes X\otimes X_1^* \arrow[d, "e_1\otimes id"] \\
X_2^* \arrow[r, "id\otimes c_1"'] & X_2^*\otimes X\otimes X_1^* \arrow[r, "e_2\otimes id"'] & X_1^*
\end{tikzcd}$$

Let us show that the local diagrams 1, 2, 3, 4 commute.

1 Picture:



(we will see the formal proof tomorrow).