Let us first prove that the functor L is essentially surjective.

<u>key idea is:</u> $F(Y) = F(1 \otimes Y) \overset{c}{\simeq} F(1) \otimes Y$ by def of $(F, c) \in \mathcal{C}'$

Now all the details:

Let us write the diagram that c checks, but restricted to $X = 1$:

$$\begin{tikzcd}
 & (F(1)\otimes Y)\otimes Z \arrow[dl, "c_{1,Y}\otimes id_Z"'] \arrow[dr, "a_{F(1),Y,Z}"] & \\
F(1\otimes Y)\otimes Z \arrow[d, "c_{1\otimes Y, Z}"'] & & F(1)\otimes(Y\otimes Z) \arrow[d, "c_{1,Y\otimes Z}"] \\
F((1\otimes Y)\otimes Z) \arrow[rr, "F(a_{1,Y,Z})"] & & F(1\otimes(Y\otimes Z))
\end{tikzcd}$$

By convention, $a_{1,Y,Z} = id$, $F(id) = id$.

$$(1 \otimes Y) \otimes Z = Y \otimes Z = 1 \otimes (Y \otimes Z)$$

Then we get the following commuting square :

$$\begin{tikzcd}
(F(1\otimes Y)\otimes Z =) & (F(1)\otimes Y)\otimes Z \arrow[r, "a_{F(1),Y,Z}"] \arrow[d, "c_{1,Y}\otimes id_Z"'] & F(1)\otimes(Y\otimes Z) \arrow[d, "c_{1,Y\otimes Z}"] & \\
& F(Y)\otimes Z \arrow[r, "c_{Y,Z}"'] & F(Y\otimes Z) & (= F(1\otimes(Y\otimes Z)))
\end{tikzcd} \qquad (***)$$

Consider $F(1)\otimes Y \xrightarrow[\sim]{c_{1,Y}} F(1\otimes Y) = F(Y)$.

Take $\tilde{F} : Y \longmapsto F(1)\otimes Y$, $\tilde{a}_{Y\otimes Z} = a_{F(1),Y,Z}$.

$(\tilde{F}, \tilde{a}) \xrightarrow{\theta} (F, c)$ by $\tilde{F}(Y) \xrightarrow{\theta_Y} F(Y)$ $\quad \theta_Y = c_{1,Y}$

θ is a nat. iso. because $\theta_Y = c_{1,Y}$ and $c = (c_{X,Y})$ is nat. iso.

But θ is really a morphism in this cat, exactly because of $(***)$.

But $(\tilde{F}, \tilde{a}) = L(F(1))$, so L is essentially surj.

Now, it remains to prove that L is full and faithful.

Let us start with full: (i.e. surj. on the hom-set).

Let $\theta : L(X) \to L(X')$ be a morphism of C'

we want to make $f : X \to X'$ s.t. $L(f) = \theta$.

Define f as follows: $X = X \otimes 1 \xrightarrow{\theta} X' \otimes 1 = X'$

up to convention we have $f = \theta$,

We need to prove that $\theta_Z = f \otimes id_Z$. $\left\{ \begin{array}{l} \text{i.e. } \theta = L(f) \quad \underline{\text{full}} \\ \text{because recall that by def} \\ L(f) = f \otimes id\_ . \end{array} \right.$

By def of morphism for θ:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "\theta_Y \otimes id_Z"'] & X \otimes (Y \otimes Z) \arrow[d, "\theta_{Y \otimes Z}"] \\
(X' \otimes Y) \otimes Z \arrow[r, "a_{X',Y,Z}"'] & X' \otimes (Y \otimes Z)
\end{tikzcd}$$

Then take $Y = 1$.

But by convention (again) $a_{X,1,Z} = id\_$
$a_{X',1,Z} = id\_$

$\underline{\text{So}}$: $\theta_1 \otimes id_Z = \theta_{1 \otimes Z} = \theta_Z$
$\quad\quad \| $
$\quad f \otimes id_Z$.

$\underline{\text{So}}$: $\theta_Z = f \otimes id_Z$ $\quad \theta = L(f)$

full ok

It's ok for L full.

Let us now prove that L is faithful (i.e. injective on the hom-sets):

If $L(f) = L(g)$ for some morphisms f, g of $\mathcal{C}$

So: (by def of L) $f \otimes \mathrm{id}_- = g \otimes \mathrm{id}_-$ | in particular: $\underset{f}{f \otimes \mathrm{id}_1} = \underset{g}{g \otimes \mathrm{id}_1}$ convention.

So: $f = g$ and L is faithful.

We already proved that L is (essentially surjective, full, faithful.)

So: (by previous result) L is an equivalence of category.

Now, it remains to prove that L is a monoidal functor:

unit of $\mathcal{C}'$: $1_{\mathcal{C}'} = (\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{-\otimes -})$

it's really an object of $\mathcal{C}'$

$\mathrm{id}_{\mathcal{C}}(X) \otimes Y \to \mathrm{id}_{\mathcal{C}}(X \otimes Y)$

$X \otimes Y \xrightarrow{\mathrm{id}_{X\otimes Y}} X \otimes Y$.

convention

(1) $L(1_{\mathcal{C}}) = (1_{\mathcal{C}} \otimes -, a_{1_{\mathcal{C}}, -, -}) = (\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{-\otimes -})$

(2) We are looking for an iso. $J_{X,Y} : L(X)\otimes L(Y) \to L(X\otimes Y)$.

Let us show that $a^{-1}_{X,Y,-}$ maps $L(X)\otimes L(Y)$ to $L(X\otimes Y)$.

First: what is $L(X)\otimes L(Y) = (X\otimes -, a_{X,-,-})\otimes(Y\otimes -, a_{Y,-,-})$

$= (X\otimes(Y\otimes -),$  $)$

What about 

The general formula is: $F^1(c^2_{-,-})\circ c^1_{F^2(-),-}$

<u>Here</u>: $F^1 = X\otimes -$

$F^2 = Y\otimes -$

$c^1 = a_{X,-,-}$

$c^2 = a_{Y,-,-}$

$(id_X\otimes a_{Y,-,-})\circ a_{X,Y\otimes -,-} =$ 

Then: Need to see that

$$J_{X,Y} = a^{-1}_{X,Y,-} : \Big(\underbrace{X \otimes (Y \otimes -)}, \underbrace{(id_X \otimes a_{Y,-,-}) \circ a_{X, Y\otimes -, -}}_{\alpha} \Big) = L(X) \otimes L(Y)$$

is a morphism in $\mathcal{C}^s$

$$\downarrow \wr$$

$$\Big(\underbrace{(X \otimes Y) \otimes -}, \underbrace{a_{X\otimes Y, -, -}}_{\beta} \Big) = L(X \otimes Y)$$

It's ok for the first components. What about the second components?

The diagram for being a morphism of $\mathcal{C}^s$ reduces to the Pentagon axiom: take (Z,T) for $(-,-)$

$$\begin{tikzcd}
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "a_{X, Y\otimes Z, T}"] \arrow[d, dashed, "a^{-1}_{X,Y,Z} \otimes id_T"'] \arrow[drr, bend left, "\alpha_{Z,T}"] & & X \otimes ((Y \otimes Z) \otimes T) \arrow[d, "id_X \otimes a_{Y,Z,T}"] \\
((X \otimes Y) \otimes Z) \otimes T \arrow[dr, "a_{X\otimes Y, Z, T}", "\beta_{Z,T}"'] & \circlearrowleft & X \otimes (Y \otimes (Z \otimes T)) \arrow[dl, dashed, "a^{-1}_{X,Y,Z\otimes T}"] \\
& (X \otimes Y) \otimes (Z \otimes T) &
\end{tikzcd}$$

it's exactly the diagram of what a morphism should satisfy in $\mathcal{C}^s$

and, $J_{X,Y}$ satisfies the monoidal structure axiom (again by the pentagon axiom)

<u>Conclusion</u>: L is a monoidal equiv. of cat. $\boxed{\times}$

This is the end of the proof of MacLane strictness theorem.

<u>Rk</u>: This theorem is <u>not</u> trivial:

A monoidal cat. $\mathcal{C}$ is always monoidally equiv. to a strict one $\mathcal{C}^s$ (ok)

BUT not always isomorphic to $\mathcal{C}^s$.

Generic counter-example: If $\mathcal{C}$ is <u>not</u> strict, then it is <u>not iso.</u> to $\mathcal{C}^s$.