Monoidal functor between $Vec_{G_1}^{\omega_1}$ and $Vec_{G_2}^{\omega_2}$

We shown that it's given by two maps $f, \mu$.

and denoted $F_{f,\mu}$.

where $f: G_1 \to G_2$ morphism of groups

$\mu: G_1 \times G_2 \to k^*$ maps s.t.

$$(*) \quad J_{g,h} = \mu(g,h)\, id_{\delta_{f(gh)}}$$

$$(**) \quad \mu(g,hl) \cdot \mu(h,l)\, \omega_2(f(g), f(h), f(l)) = \omega_1(g,h,l)\, \mu(gh,l)\, \mu(g,h)$$

$\forall g,h,l \in G_1$

Let us investigate the natural monoidal transformations between $F_{f,\nu}$ and $F_{f',\nu'}$. $\quad \eta : F_{f,\nu} \to F_{f',\nu'}$.

As before, we reduce to the simple object $\delta_g$.

For all morphism $\alpha : \delta_g \to \delta_h$, the following commutes

$$
\begin{tikzcd}
F_{f,\nu}(\delta_g) \arrow[r, "\eta_g"] \arrow[d, "F_{f,\nu}(\alpha)"'] \arrow[dr, phantom, "\circlearrowleft"] & F_{f',\nu'}(\delta_g) \arrow[d, "F_{f',\nu'}(\alpha)"] \\
F_{f,\nu}(\delta_h) \arrow[r, "\eta_h"'] & F_{f',\nu'}(\delta_h)
\end{tikzcd}
$$

we can reduce to $\alpha = \delta_{h,g} \cdot \mathrm{id}_{\delta_g}$

$$
\begin{tikzcd}
\delta_{f(g)} \arrow[r, "\eta_g"] \arrow[d, "\delta_{g,h} \cdot \mathrm{id}_{\delta_{f(g)}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \delta_{f'(g)} \arrow[d, "\delta_{g,h} \cdot \mathrm{id}_{\delta_{f'(g)}}"] \\
\delta_{f(h)} \arrow[r, "\eta_h"'] & \delta_{f'(h)}
\end{tikzcd}
$$

Observe that $\eta_g \neq 0 \iff f(g) = f'(g)$

Let show that $\forall g,\ \eta_g \neq 0$. ( so $f = f'$):

We have the relation (see previous lecture):

$$\mu'(g,h)(\eta_g \otimes \eta_h) = \eta_{gh}\, \mu(g,h).$$

Take $h = g^{-1}$. $\qquad e = gg^{-1}$ (neutral)

You get: $\mu'(g,g^{-1})(\eta_g \otimes \eta_{g^{-1}}) = \eta_e\, \mu(g,g^{-1})$.

But $\eta_e \simeq \mathrm{id}_{J_e}$, so as a scalar $\eta_e \neq 0$.

Know $\mu, \mu'$ map to $k^*$. $(\neq 0)$.

$\implies \eta_g \neq 0$.

So we proved that $f = f'$

Recall that by convention, $\eta_e = \mathrm{id}_{\delta_e}$

Now, what about the monoidal structure:

General:

$$
\begin{tikzcd}
F^1(X) \otimes' F^1(Y) \arrow[r, "J^1_{X,Y}"] \arrow[d, "\eta_X \otimes' \eta_Y"'] \arrow[dr, phantom, "\circlearrowright"] & F^1(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\
F^2(X) \otimes F^2(Y) \arrow[r, "J^2_{X,Y}"] & F^2(X \otimes Y)
\end{tikzcd}
$$

here: $\forall g, h \in G$

$$
\begin{tikzcd}
\delta_{f(g)} \otimes \delta_{f(h)} \arrow[r, "\mu(g,h)\,\mathrm{id}"] \arrow[d, "\eta_g \otimes \eta_h"'] \arrow[dr, phantom, "\circlearrowright"] & \delta_{f(gh)} \arrow[d, "\eta_{gh}"] \\
\delta_{f'(g)} \otimes \delta_{f'(h)} \arrow[r, "\mu'(g,h)\,\mathrm{id}"'] & \delta_{f'(gh)}
\end{tikzcd}
$$

Let call $\tilde{\eta} : G \to k^*$ the scalar corresponding to $\eta$ (i.e. $\eta_g = \tilde{\eta}(g)\,\mathrm{id}$).

Summary/theorem: $\tilde{\eta}_{gh} \cdot \mu(g,h) = \mu'(g,h)\,(\tilde{\eta}_g \cdot \tilde{\eta}_h)$ (data + relation) completely characterize such a natural transformation.

We are done with previous investigation.

Exo: By above investigation, the 3-cocycle $\omega$ (for $Vec_G^\omega$) can always be taken "normalized", i.e.

$$\omega(g,1,h)=1 \quad \forall g,h\in G.$$

(W/o loss of generality).

then, $l_{\delta_g}=r_{\delta_g}=id_{\delta_g}$.

Now, let see what $\omega$ (3-cocycle) looks like for $Vec_G^\omega$ with $G=C_n=\langle g \mid g^n=e\rangle$ cyclic grp of order $n$.

Consider the element of $C_n$ as $g^a$, $a\in\mathbb{Z}$ (so $g^{a+n}=g^a$).

Consider the following family of 3-cocycle: (parametrized by $s$).

$$\omega_s(g^a, g^b, g^c) = \exp\left(\frac{2\pi i}{n} \cdot s \cdot a \cdot \left\lfloor \frac{b+c}{n} \right\rfloor\right) \quad 0 \leq s < n$$

↑ integral part (floor)

Exo: • Show that it's a 3-cocycle.

• Classify (categorical) equiv. classes of $Vec_G^\omega$ (Gaelic)

i.e. $Vec_G^\omega \xrightarrow[\otimes]{\sim} Vec_{G'}^{\omega'}$ iff $G \simeq G'$

$\omega_s, \omega_{s'}$ satisfy .... (exo)

(relation between $s$ and $s'$ etc.... (exo))

# Group action of categories and equivariantization.

Let $\mathcal{C}$ be a category.

Consider the monoidal category $\mathrm{Aut}(\mathcal{C})$

object: equivalence of category: $F: \mathcal{C} \to \mathcal{C}$.

morphism: natural isomorphism.

Monoidal structure: $F \otimes G = F \circ G$ (composition)

$\to$ it's a monoidal subcategory of $\mathrm{End}(\mathcal{C})$ $\begin{cases} \text{obj: functor } F: \mathcal{C} \to \mathcal{C} \\ \text{morph: natural trans.} \end{cases}$

If $\mathcal{C}$ is a monoidal category, consider

$\mathrm{Aut}_{\otimes}(\mathcal{C})$: the monoidal cat. of monoidal equiv. of $\mathcal{C}$.

Let $G$ be a group. Consider the monoidal cat. $\mathrm{Cat}(G)$

<u>object:</u> elements of $G$

<u>morphisms:</u> $\hom_{\mathrm{Cat}(G)}(g,h) = \begin{cases} \{id_g\} & \text{if } g=h \\ \emptyset & \text{otherwise} \end{cases}$

<u>monoidal structure:</u> $g \otimes h = gh$.

<u>Def:</u> (i) Action of $G$ on cat. $\mathcal{C}$ is <u>monoidal</u> functor $T: \mathrm{Cat}(G) \to \mathrm{Aut}(\mathcal{C})$

(ii) ________________ <u>monoidal</u> cat $\mathcal{C}$ ________________

$T: \mathrm{Cat}(G) \to \mathrm{Aut}_{\otimes}(\mathcal{C})$.

Let $g \mapsto T_g \in \mathrm{Aut}(\mathcal{C})$ such an action

$\forall g, h$, let $\gamma_{g,h} : T_g \circ T_h \xrightarrow{\sim} T_{gh}$ be the monoidal structure

$\otimes$ of $\mathrm{Aut}(\mathcal{C})$ $\qquad$ $\otimes$ of $\mathrm{Cat}(G)$

(recall that the general notation for that $J_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y)$)

**Def:** $G$-equivariant object in $\mathcal{C}$: it is a pair $(X, u)$ where $X$ is an object of $\mathcal{C}$ and $(g \in G)$.

$u$ is a family of isomorphisms: $u_g : T_g(X) \xrightarrow{\sim} X$ s.t.

$$\begin{tikzcd}
(T_g \circ T_h)(X) = T_g(T_h(X)) \arrow[r, "T_g(u_h)"] \arrow[d, "\gamma_{g,h}(X)"'] & T_g(X) \arrow[d, "u_g"] \\
T_{gh}(X) \arrow[r, "u_{gh}"'] & X
\end{tikzcd}$$

commutes $\forall g, h \in G$

Morphism between $G$-equivariant objects of $\mathcal{C}$.

$\tilde{f} : (X, u) \to (Y, v)$ defined by:

- $f : X \to Y$, morphism.
- the following commuting diagram $(\forall g \in G)$

$$\begin{tikzcd}
T_g(X) \arrow[r, "T_g(f)"] \arrow[d, "u_g"'] \arrow[dr, phantom, "\circlearrowleft"] & T_g(Y) \arrow[d, "v_g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The category of $G$-equivariant objects of $\mathcal{C}$ (defined as above) is called $G$-equivariantization of $\mathcal{C}$, and denoted $\mathcal{C}^G$.

You have the forgetful functor $\mathrm{Forg}: \mathcal{C}^G \to \mathcal{C}$
$$(X, u) \mapsto X.$$

---

Now, if $\mathcal{C}$ is a monoidal category, replace $\mathrm{Aut}(\mathcal{C})$ by $\mathrm{Aut}_{\otimes}(\mathcal{C})$, and you will get a monoidal structure on $\mathcal{C}^G$ and Forg would be a monoidal functor.

<u>Exo:</u> Take $\mathcal{C} = \mathrm{Vec}$, then show that $\mathcal{C}^G \simeq \mathrm{Rep}(G)$.

Def: A monoidal category $\mathcal{C}$ is strict if $\forall X, Y, Z \in \mathcal{C}$

$$\begin{cases} (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \\ X \otimes 1 = X = 1 \otimes X. \end{cases}$$

and the $\begin{cases} \text{associativity} \\ \text{unity} \end{cases}$ constraints are identity maps.

Examples:

① Let $\mathcal{C}$ be a category.

Let $\mathrm{End}(\mathcal{C})$ be the monoidal category of endofunctors:

- objects: functors $F : \mathcal{C} \to \mathcal{C}$
- morphisms: natural transformation $\phi : F_1 \to F_2$.
- $\otimes$ : $F_1 \otimes F_2 = F_1 \circ F_2$

Exo: $\mathrm{End}(\mathcal{C})$ is a strict monoidal category.