Examples of monoidal functors.

Forgetful functor (forgetting structure)

for monoidal categories: (let fix a field k),
(G finite group)

Usually, a forgetful functor is of the form

$\mathcal{G} \longrightarrow Sets$

groups, Topological space ...

① $Rep(G) \xrightarrow{F} Vec$: this usual forgetful functor be made as a monoidal functor.

② let $H < G$ (subgroup), consider the functor $Rep(G) \to Rep(H)$ by restriction of rep. Then ① corresponds to $H = \{e\}$.

③ let $f: H \longrightarrow G$ be a morphism of (finite) groups.
Let V be a rep. of G. Then we get a rep. of H as the restriction of V to $f(H) < G$. It's denoted $f^*: Rep(G) \to Rep(H)$ (pull back functor.)

<u>Exercice</u>: The above functor admits a structure of monoidal functor.

(4) Let $f: H \to G$ morphism of groups.

Let V be a H-graded vector space: $V = \bigoplus_{h \in H} V_h$

Then, we get a G-graded vector space as:

$$V = \bigoplus_{g \in G} \left(\bigoplus_{h \in f^{-1}(\{g\})} V_h \right)$$

This provides a functor denoted $f_*: Vec_H \to Vec_G$

You get a forgetful functor by taking $G = \{e\}$, then $f_*: Vec_H \to Vec$.

<u>Exo</u>: It admits a structure of monoidal functor.

(5) Let S be a monoid, and take $\mathcal{C} = \mathrm{Vec}_S$.

The identity functor $\mathrm{id}_{\mathcal{C}}$ is a monoidal functor (with $J_{X,Y} = \mathrm{id}_{X\otimes Y}$)

Let investigate the natural transformation of monoidal functor

$$\eta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$$

First: η is a natural transformation, i.e. $\forall\, X \xrightarrow{f} Y$

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}}(X) \arrow[r, "\eta_X"] \arrow[d, "\mathrm{id}_{\mathcal{C}}(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{id}_{\mathcal{C}}(X) \arrow[d, "\mathrm{id}_{\mathcal{C}}(f)"] \\
\mathrm{id}_{\mathcal{C}}(Y) \arrow[r, "\eta_Y"] & \mathrm{id}_{\mathcal{C}}(Y)
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\
Y \arrow[r, "\eta_Y"'] & Y
\end{tikzcd}$$

reduction to simple object: $\longleftrightarrow$ $\forall s, s' \in S$

$$\begin{tikzcd}
\delta_s \arrow[r, "\eta_{\delta_s}"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \delta_s \arrow[d, "f"] \\
\delta_{s'} \arrow[r, "\eta_{\delta_{s'}}"'] & \delta_{s'}
\end{tikzcd}$$

but $\hom_{\mathcal{C}}(\delta_s, \delta_{s'}) = \Bbbk\, \mathrm{id}_{\delta_s}\, \delta_{s,s'}$

if $s \neq s' \Rightarrow f = 0$ OK.

If $s = s'$, η_{δ_s} and f are given by elts $k, k' \in \Bbbk$ $\left(\text{but } kk' = k'k \ \forall k, k' \in \Bbbk\right)$ OK.

As a natural transformation, η is completely given by a map

$$\tilde{\eta} : S \to k.$$

Now let consider the axiom for <u>monoidal</u> natural transformation:

$\forall X, Y \in \mathcal{C}$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "id"] \arrow[d, "\eta_X \otimes \eta_Y"'] & X \otimes Y \arrow[d, "\eta_{X\otimes Y}"] \\
X \otimes Y \arrow[r, "id"'] & X \otimes Y
\end{tikzcd}$$

So $\eta_X \otimes \eta_Y = \eta_{X \otimes Y}$

If we restrict to the simple objects, this means that

$$\tilde{\eta}(s s') = \tilde{\eta}(s)\, \tilde{\eta}(s')$$

($\tilde{\eta}$ is multiplicative

Moreover: by convention $\eta_1 = id_1$; so $\tilde{\eta}(e) = 1$.

Conclusion: $\tilde{\eta}$ is a morphism of monoid, and conversely for every morphism of monoid, you have a monoidal natural transformation.

$\longrightarrow$ it's a complete characterization.

$\left\{ \omega_i : G_i^3 \to \Bbbk . \right.$

<u>Monoidal</u> functors between $Vec_{G_1}^{\omega_1}$ and $Vec_{G_2}^{\omega_2}$.

Recall that, $\forall i$, $\forall g_1, g_2, g_3, g_4 \in G_i$, we have the <u>3-cocycle</u> property equivalent to the Pentagon axiom (: associativity constraints).

$$\omega_i(g_1 g_2, g_3, g_4)\, \omega_i(g_1, g_2, g_3 g_4) = \omega_i(g_1, g_2, g_3)\, \omega_i(g_1, g_2 g_3, g_4)\, \omega_i(g_2, g_3, g_4)$$

Let us denote $\mathcal{C}_i = Vec_{G_i}^{\omega_i}$, $i = 1, 2$.

We will consider monoidal ($\Bbbk$-linear) - functor $F : \mathcal{C}_1 \to \mathcal{C}_2$

<u>Proposition</u>: The functor F induces a group morphism

$$f : G_1 \to G_2 ,$$

by restricting to the simple objects.

Proof: By semi-simplicity (see later), we can restrict to the simple objects δ_g. The functor F maps the simple objects to the simple objects, because

$$FPdim(F(\delta_g)) = FPdim(\delta_g) = 1.$$

{ we will see later the notion of Frobenius-Perron dim (FPdim) }

The FPdim is determined by the monoidal structure, in fact it can be get directly from the fusion ring (which means that we don't need to care about J).

(we will see later)

So F must keep FPdim.

But $FPdim(X) = 1 \Rightarrow X$ simple

(need semi-simple assumption)

(of course the converse is not true)

So $F(\delta_g)$ is simple.

Then $\exists g' \in G_2$ such that $F(\delta_g) \simeq \delta_{g'}$

{ so we have the expected map: $G_1 \to G_2$, $g \mapsto g'$, let call it f.

$\boxed{F(\delta_g) \simeq \delta_{f(g)}}$

Now, let see why f is a morphism of groups:

By def. of monoidal functor, $\forall X, Y$

- $F(X) \otimes F(Y) \xrightarrow[J_{X,Y}]{\sim} F(X \otimes Y)$.

by restriction to simple object:

$$F(\delta_{gh}) = F(\delta_g \otimes \delta_h) \simeq F(\delta_g) \otimes F(\delta_h)$$

$$\delta_{f(gh)} \qquad\qquad \delta_{f(g)} \otimes \delta_{f(h)} = \delta_{f(g)f(h)}$$

$f(gh) = f(g) f(h)$. $\Leftarrow$

- $F(id) = id \Rightarrow F(\delta_e) = \delta_e$, i.e. $f(e_{G_1}) = e_{G_2}$.

So (because G is finite group), f is a morphism of group $\blacksquare$

Now let consider the monoidal structure: we need isomorphism

$$J_{g,h} : F(\delta_g) \otimes F(\delta_h) \xrightarrow{\sim} F(\delta_{gh}) \quad , g, h \in G,$$

it's an element of $\hom_{\mathcal{C}_2}(\delta_{f(gh)}, \delta_{f(gh)}) = k \, id_{\delta_{f(gh)}}$ { so it's given by $k \in k$. and $k \neq 0$ iso.

So: (*) $\begin{array}{l} J_{g,h} = \mu(g,h)\, \mathrm{id}_{\delta_{f(gh)}} \\ \quad \| \\ J_{\delta_g,\delta_h}. \end{array}$ with $\mu : G_1 \times G_1 \to k^*$

Now, we can reformulate the monoidal structure axiom (using ω_i, f, μ)

General:

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[rr, "a'_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y} \otimes' \mathrm{id}_{F(Z)}"] & & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes' J_{Y,Z}"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "J_{X \otimes Y, Z}"] & & F(X) \otimes' F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[rr, "F(a_{X,Y,Z})"] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

But here, $X=\delta_g$, $Y=\delta_h$, $Z=\delta_\ell$

$J_{X,Y}=\mu(g,h)\,id_{\delta_{f(gh)}}$, $F(X)=\delta_{f(g)}$, $a'_{F(X),F(Y),F(Z)}=\omega_2(f(g),f(h),f(\ell))\,id$

F is $\mathbb{k}$_linear

$$F(a_{X,Y,Z})=F(\omega_1(g,h,\ell)\,id_{\delta_{gh\ell}})\stackrel{\downarrow}{=}\omega_1(g,h,\ell)\,F(id_{\delta_{gh\ell}})=\omega_1(g,h,\ell)\,id_{\delta_{f(gh\ell)}}$$

Warning: here $F(\delta_g)\otimes F(\delta_h)=F(\delta_{gh})$ but in general we chose a $J_{g,h}\neq id$, it's id up to a mult. cte $(\mu(g,h))$.

$\delta_{f(ghl)} =$

$$\begin{tikzcd}
(\delta_{f(g)} \otimes \delta_{f(h)}) \otimes \delta_{f(l)} \arrow[rr, "\omega_2(f(g),f(h),f(l))\, id \ldots"] \arrow[d, "\mu(g,h)\, id \ldots"'] & & \delta_{f(g)} \otimes (\delta_{f(h)} \otimes \delta_{f(l)}) \arrow[d, "id\, \delta_{f(g)} \otimes \mu(h,l)\, id\, \delta_{f(hl)}"] \\
\delta_{f(gh)} \otimes \delta_{f(l)} \arrow[d, "\mu(gh,l)\, id \ldots"'] \arrow[rr, phantom, "\circlearrowleft"] & & \delta_{f(g)} \otimes \delta_{f(hl)} \arrow[d, "\mu(g,hl)\, id \ldots"] \\
\delta_{f((gh)l)} \arrow[rr, "\omega_1(g,h,l)\, id \ldots"'] & & \delta_{f(g(hl))}
\end{tikzcd}$$

in other: $\forall\, g,h,l \in G,$

$$\mu(g,hl)\,\mu(h,l)\,\omega_2(f(g),f(h),f(l)) = \omega_1(g,h,l)\,\mu(gh,l)\,\mu(g,h) \qquad (**)$$

As a summary, we get:

<u>Theorem</u>: Such couple (f, μ) provides a complete characterization of a monoidal (k-linear) functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ with $(*)$, $(**)$ and $F(\delta_g) = \delta_{f(g)}$.

<u>Exo</u>: F is an equivalence of cat. iff f is an isomorphism.

Above functor F will be denoted $F_{f,\mu}$.

Next week: we will start by investigating the natural monoidal transformat. from $F_{f,\mu}$ to $F_{f',\mu'}$.