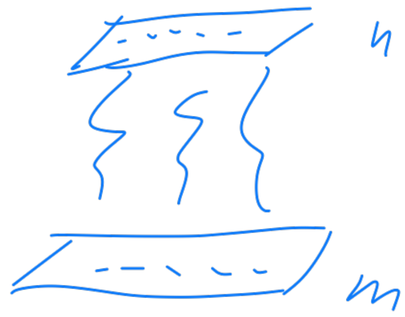
Few clarification about the tangle category $\mathcal{T}$:

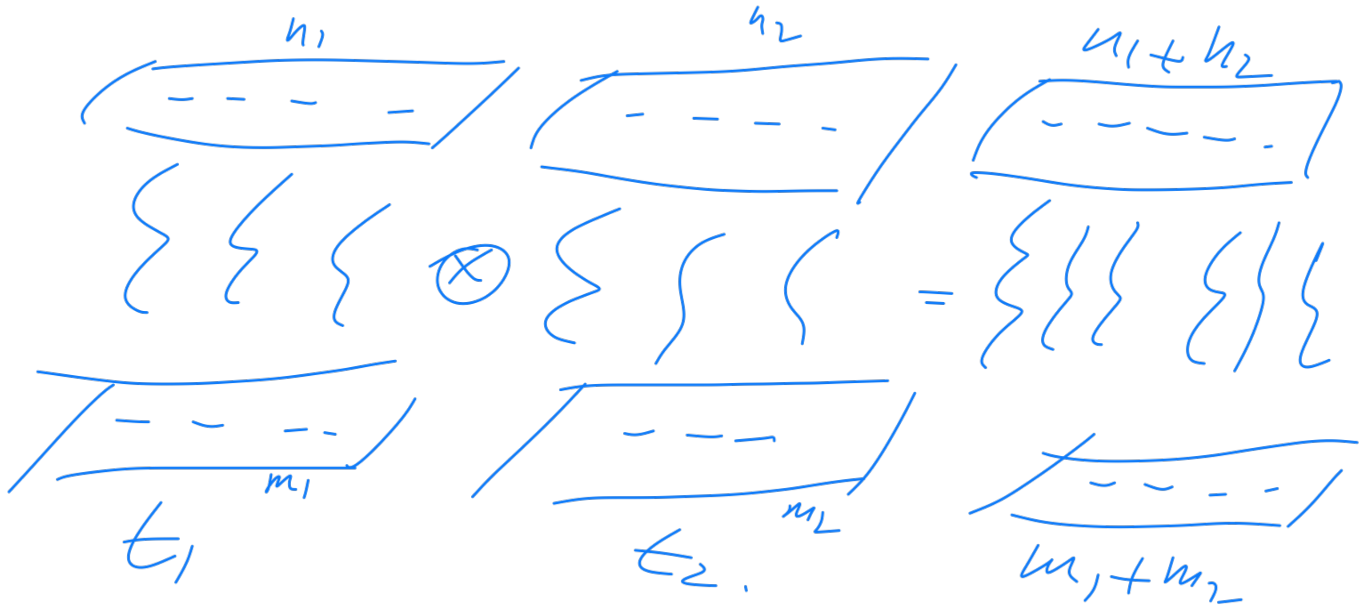
As I wrote before, the objects are the natural numbers

the morphism are tangles 

$\in \mathrm{hom}_{\mathcal{T}}(n,m)$

Monoidal structure:

- $n \otimes m = n+m$.
- $t_1 \otimes t_2 = ?$



Monoidal functors: Consider two monoidal categories

$$(\mathcal{C}, \otimes, 1, a, \iota), \quad (\mathcal{C}', \otimes', 1', a', \iota').$$

A monoidal functor between the above two monoidal categories is a couple (F, J), where $F: \mathcal{C} \to \mathcal{C}'$ is a functor, and J is a natural isomorphism

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} \to \mathcal{C}' & \xrightarrow{J} & \mathcal{C} \times \mathcal{C} \to \mathcal{C}' \\ (X, Y) \mapsto F(X) \otimes' F(Y) & & (X, Y) \mapsto F(X \otimes Y). \end{array}$$

Then:

$\forall X, Y \in \mathcal{C}$, $J_{X,Y}: F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y)$ isomorphism.

satisfying the following two axioms:

(1) $F(1) \sim 1'$ (which means that $\exists\, \varphi : 1' \xrightarrow{\sim} F(1)$ iso)

(2)
$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[rr, "a'_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y} \otimes' id_{F(Z)}"'] & & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "id_{F(X)} \otimes J_{Y,Z}"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & & F(X) \otimes' F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[rr, "F(a_{X,Y,Z})"'] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

commutes $\forall X, Y, Z \in \mathcal{C}$. (denoted the "monoidal structure axiom")

Remark: For a functor to be monoidal, it should satisfies these extra assumptions (1),(2), which can be reformulated as equations admitting possibly zero, one or several solutions (up to equivalence). So a functor can have zero monoidal structure, or one or several (up to equiv). We will see examples later having several non-equiv. monoidal structure.

Theorem: There is a canonical isomorphism $\varphi : 1' \to F(1)$, such that:

$$\begin{tikzcd} 1' \otimes' F(1) \arrow[r, "l'_{F(1)}"] \arrow[d, "\varphi \otimes' id_{F(1)}"'] & F(1) \arrow[d, "F(l_1)^{-1}"] \\ F(1) \otimes' F(1) \arrow[r, "J_{1,1}"] & F(1 \otimes 1) \end{tikzcd}$$

commutes.

Lemma: Let F be an equivalence of category $F: \mathcal{C} \to \mathcal{C}'$ and $F \xrightarrow{\Phi} G$ be a natural isomorphism.

Then G is also an equivalence of category.

Proof: $\exists F' \ldots$ $F \circ F' \overset{\varphi'}{\xrightarrow{\sim}} id_{\mathcal{C}'}$, $F' \circ F \overset{\varphi}{\xrightarrow{\sim}} id_{\mathcal{C}}$ and

$\forall\, X \xrightarrow{f} Y$,

$$\begin{tikzcd} F(X) \arrow[r, "\Phi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\Phi_Y"'] & G(Y) \end{tikzcd}$$

commutes.

Next: Apply F' to above square:

$$
\begin{tikzcd}
\mathrm{id}_{\mathcal{C}}(X) \arrow[d, "\mathrm{id}_{\mathcal{C}}(f)"'] & F'\circ F(X) \arrow[l, dashed, "\varphi_X"'] \arrow[r, "F'(\phi_X)"] \arrow[d, "F'\circ F(f)"] & F'\circ G(X) \arrow[d, "F'\circ G(f)"] \\
\mathrm{id}_{\mathcal{C}}(Y) & F'\circ F(Y) \arrow[l, dashed, "\varphi_Y"] \arrow[r, "F'(\phi_Y)"'] & F'\circ G(Y)
\end{tikzcd}
$$

Take $G' = F'$ and the natural isomorphism

$G'\circ G \overset{\beta}{\xrightarrow{\sim}} \mathrm{id}_{\mathcal{C}}$ with $\beta_X = \varphi_X \circ F'(\phi_X)^{-1}$

Iden: $G\circ G' \overset{\beta'}{\xrightarrow{\sim}} \mathrm{id}_{\mathcal{C}'}$ $\Big\}$ So G is an equivalence of category. $\blacksquare$

Lemma: There is a natural isomorphism $R_{1'} \xrightarrow[\sim]{\Phi} R_{F(1)}$

Proof: By def, we can pick $\tilde{\varphi} : 1' \xrightarrow{\sim} F(1)$ iso.

Observe that the following diagram commutes:

$$
\begin{tikzcd}
X \otimes' 1' = R_{1'}(X) \arrow[r, "\mathrm{id}_X \otimes' \tilde{\varphi}"] \arrow[d, "R_{1'}(f) = f \otimes' \mathrm{id}_{1'}"'] & X \otimes' F(1) = R_{F(1)}(X) \arrow[d, "R_{F(1)}(f) = f \otimes' \mathrm{id}_{F(1)}"] \\
Y \otimes' 1' = R_{1'}(Y) \arrow[r, "\mathrm{id}_Y \otimes \tilde{\varphi}"'] & Y \otimes' F(1) = R_{F(1)}(Y)
\end{tikzcd}
$$

because (just by checking the diagram):

① $(f \otimes' id_{F(1)}) \circ (id_X \otimes' \tilde{\varphi}) = (f \circ id_X) \otimes (id_{F(1)} \circ \tilde{\varphi}) = f \circ \tilde{\varphi}$

$\shortparallel$

② $(id_Y \otimes' \tilde{\varphi}) \circ (f \otimes' id_1) = (id_Y \circ f) \otimes' (\tilde{\varphi} \circ id_1) = f \circ \tilde{\varphi}$

So we can take the natural isomorphism ϕ s.t

$$\phi_X = id_X \otimes \tilde{\varphi}$$

$\blacksquare$

Corollary: $R_{F(1)}$ is an equivalence of category

Proof: Apply the two previous lemma. $\blacksquare$

Proof of above Theorem:

we want: $\varphi \otimes' id_{F(1)} = J_{1,1} \circ F(l_1)^{-1} \circ l'_{F(1)}$

$= R_{F(1)}(\varphi)$.

we want to make such a φ,

But we know that $R_{F(1)}$ is an equivalence of cat.

$\Rightarrow$ it admits an inverse on the morphism.

$$\varphi := R_{F(1)}^{-1}\left(J_{1,1} \circ F(l_1)^{-1} \circ l'_{F(1)}\right) \qquad \blacksquare$$

We proved the existence of iso $\varphi : 1' \to F(1)$ s.t.

$$\begin{tikzcd}
1' \otimes' F(1) \arrow[r, "l'_{F(1)}"] \arrow[d, "\varphi \otimes' \mathrm{id}_{F(1)}"'] & F(1) \arrow[d, "F(l_1)^{-1}"] \\
F(1) \otimes' F(1) \arrow[r, "J_{1,1}"] & F(1 \otimes 1)
\end{tikzcd}$$

commutes.

<u>Exercice</u>: We can replace 1 by X (i.e. above diagram can be generalize) and have the analogous with r (instead of l), in other words:

$$\begin{tikzcd}
1' \otimes' F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "\varphi \otimes' \mathrm{id}_{F(X)}"'] & F(X) \arrow[d, "F(l_X)^{-1}"] \\
F(1) \otimes' F(X) \arrow[r] & F(1 \otimes X)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(X) \otimes' 1' \arrow[r, "r'_{F(X)}"] \arrow[d, "\mathrm{id}_{F(X)} \otimes' \varphi"'] & F(X) \arrow[d, "F(r_X)^{-1}"] \\
F(X) \otimes' F(1) \arrow[r, "J_{X,1}"] & F(X \otimes 1)
\end{tikzcd}$$

commute. $\forall X \in \mathcal{C}$.

Remark: A monoidal functor can alternatively be defined as a triple (F, J, φ) satisfying the

- monoidal structure axiom.
- commuting diagrams in previous exercise.

This is the usual way to define a monoidal category, but the one given at the begging (following the book) is simpler, because we don't need to assume the existence of φ (it automatically exists as we just proved).

Convention: we can "safely" identify $1'$ with $F(1)$. Then: $\begin{cases} F(1) = 1' \\ \varphi = id_{1'} \end{cases}$ and $J_{1,x} = J_{x,1} = id_{x}$.

Natural transformation of monoidal functors:

$$\begin{tikzcd}[column sep=huge]
(\mathcal{C}, \otimes, 1, a, \iota) \arrow[r, bend left=40, "{(F^1, J^1)}"] \arrow[r, bend right=40, "{(F^2, J^2)}"'] \arrow[r, phantom, "\Downarrow \eta"] & (\mathcal{C}', \otimes', 1', a', \iota')
\end{tikzcd}$$

- natural transformation $\eta : F^1 \to F^2$

(i.e. $\forall X \in \mathcal{C}$, $\exists\ \eta_X$ such that

$\forall\ X \xrightarrow{f} X$

$$\begin{tikzcd}
F^1(X) \arrow[r, "\eta_X"] \arrow[d, "F^1(f)"] & F^2(X) \arrow[d, "F^2(f)"] \\
F^1(Y) \arrow[r, "\eta_Y"] & F^2(Y)
\end{tikzcd}$$

- η_1 isomorphism: $F^1(1) \xrightarrow{\eta_1} F^2(1)$ ← convention.

$$\begin{tikzcd}
F^1(X) \otimes' F^1(Y) \arrow[r, "J^1_{X,Y}"] \arrow[d, "\eta_X \otimes' \eta_Y"'] & F^1(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\
F^2(X) \otimes' F^2(Y) \arrow[r, "J^2_{X,Y}"] & F^2(X \otimes Y)
\end{tikzcd}$$

commutes.

Before the use of the convention, we have $\varphi_i : 1' \to F^i(1)$, $i = 1, 2$

Exo: $\eta_1 \circ \varphi_1 = \varphi_2$.

$$\begin{tikzcd}
1' \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & F^1(1) \arrow[dl, "\eta_1"] \\
F^2(1) &
\end{tikzcd}$$

commutes

Then, After convention: $\varphi_1 = \varphi_2 = id_{1'}$, and $\eta_1 = id_{1'}$.

Exercise: Let F be an equivalence of monoidal cat.

Then $\exists F'$ and natural isomorphism of monoidal cat.

$$F \circ F' \xrightarrow{\sim} \mathrm{id}, \quad F' \circ F \xrightarrow{\sim} \mathrm{id}.$$