Lemma: Let $\mathcal{C}$ be a monoidal category, $X, Y \in \mathcal{C}$.

Let $a : X \xrightarrow{\sim} X$ iso.

$b : Y \to Y$ (any).

Then: $a \otimes b \overset{(1)}{=} a \otimes id_Y \implies id_X \otimes b = id_X \otimes id_Y$.

Proof: $a \otimes b = (a \otimes id_Y) \circ (id_X \otimes b)$.

$\overset{(1)}{=} (a \otimes id_Y) = (a \otimes id_Y) \circ (id_{X \otimes Y})$ ($id_X \otimes id_Y$.)

but a is iso., so is $a \otimes id_Y$, so we can apply its inverse, and get:

$id_X \otimes b = id_X \otimes id_Y$ ∎

Prop: Same statement as above lemma, but with $X = 1$. Then. $b = id_Y$.

proof: Apply previous lemma with $X = 1$ → $id_1 \otimes b = id_1 \otimes id_Y$
then apply L_1^{-1} → $b = id_Y$ ∎

About yesterday: take $X=Y=1$, $a=b$, then: $b\otimes b = b\otimes id_1 \Rightarrow b = id_1$.

Rk: Above Prop. does NOT extend to $X \neq 1$.

Exercice: Find a counter-example with $X \neq 1$.

Examples of monoidal categories (continued).

Yesterday with mentioned: Sets, k-Vec, R-mod, $\mathrm{Rep}_k(G)$.

Today: First: $(k-)\mathrm{Vec}_G$ the monoidal category of G-graded vector spaces.
(with G is a group, or more generally a monoid)

$$V = \bigoplus_{g\in G} V_g$$

with $\otimes$ as follows:

$$(V\otimes W)_g = \bigoplus_{x,y\in G : xy=g} V_x \otimes V_y$$

here simple object : an object which cannot be decomposed by $\oplus$ (in general, see later)

here they are (up to equi): δ_g $g \in G$ — 1-dim vector space

with $\delta_g \otimes \delta_h = \delta_{gh}$

$1 = h = \delta_e$

e = neutral of G.

$V = \delta_g$, then $V_h = (\delta_g)_h = \delta_{gh} \cdot \delta_g$

(δ_{gh}: Kronecker symbol)

Then: $a, L \ldots$ are objects here.

Let see how to make a non-trivial. we will make a new monoidal cat. called Vec_G^ω by twisting Vec_G.

How to make a non-trivial?

For Vec_G : $a_{\delta_g,\delta_h,\delta_m} = id_{\delta_{ghm}} : (\delta_g \otimes \delta_h)\otimes\delta_m \to \delta_g\otimes(\delta_h\otimes\delta_m)$

(trivial). δ_{ghm} δ_{ghm}.

How to twist that?

We want to get something $\neq id$, but δ_{ghm} is 1-dim.

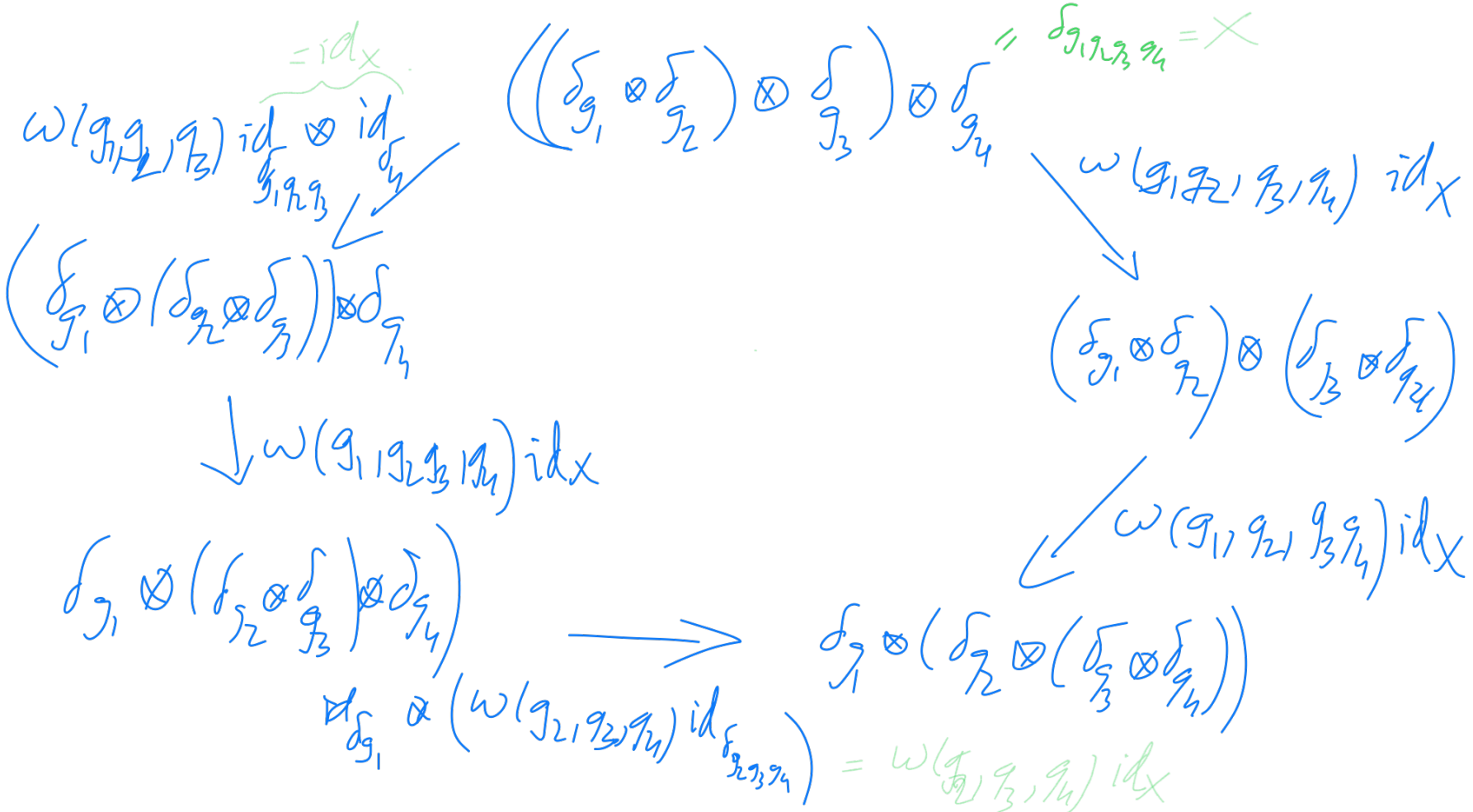
So: $\hom_G(\delta_{ghm}, \delta_{ghm})$ it's a set of k-lin. transf. between two 1-dim. space, so it reduces to k.

We can only twist by a multiplication etc.

For Vec_G^ω : $a_{\delta_g,\delta_h,\delta_m} = \omega(g,h,m)\, id_{\delta_{ghm}}$,

with $\omega : G\times G\times G \to k^*$.

But the Pentagon axiom put a constraint on ω, as follows:



Pentagon axiom reduces (here) to the following equation:

$$\omega(g_1g_2,g_3,g_4)\,\omega(g_1,g_2,g_3g_4) = \omega(g_1,g_2,g_3)\,\omega(g_1,g_2g_3,g_4)\,\omega(g_2,g_3,g_4)$$

Such an ω is called a 3-cocycle $\omega : G\times G\times G \to k^*$ { such a ω provides a monoidal cat. vec_G^ω. }

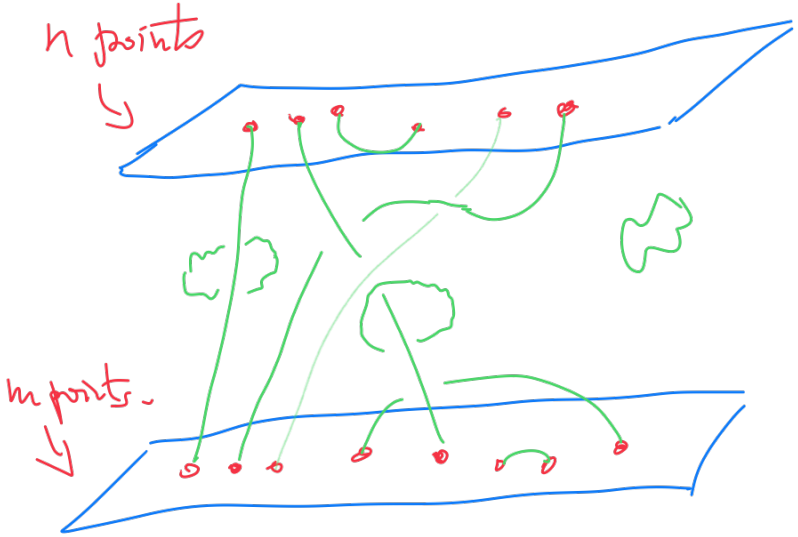
Now let provide an examples where the objects are NOT vector space or set (and it's not a twist also ..-).

→ Something really new :

Category of tangles : $\mathcal{C}$.

objects : natural numbers $n \in \mathbb{N}_{\geq 0}$. (a number is not a set in a usual sense),

morphism : $\hom_{\mathcal{C}}(n, m) = T(n, m)$ set of tangles.

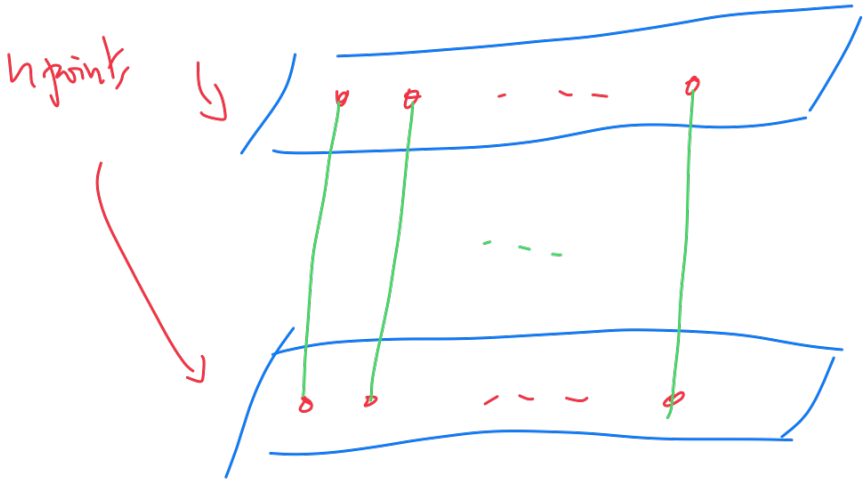


here $n = 6$, $m = 8$.

it's an element of $T(6, 8) = \hom_{\mathcal{C}}(6, 8)$.

it's defined up to isotopy

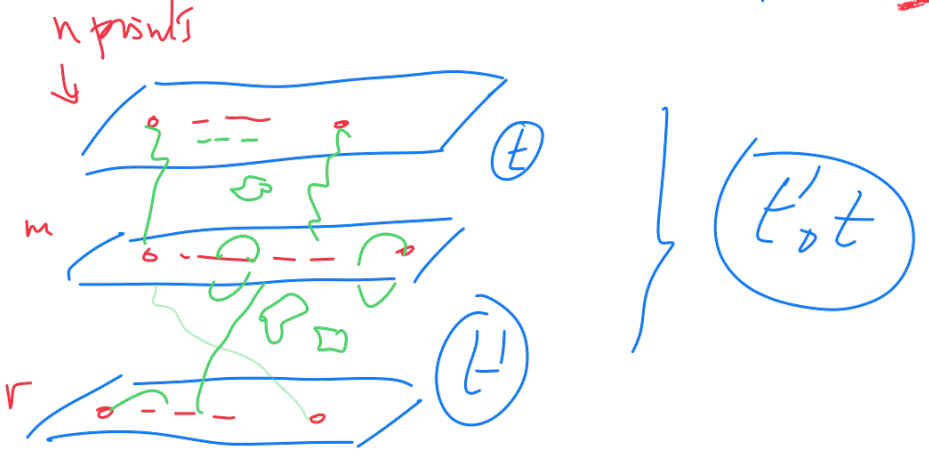
The identities in this category.

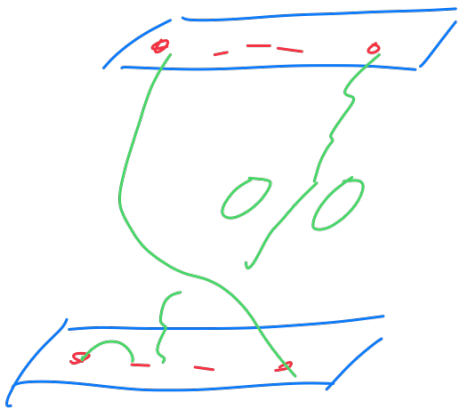


this is the identity of $T(n,n)$.

Composition of morphism/tangle.

$t \in T(n,m)$, $t' \in T(m,r)$ - To make $t' \circ t$





you get a tangle in $T(n,r)$.

$\otimes$: $n \otimes m = n+m$.



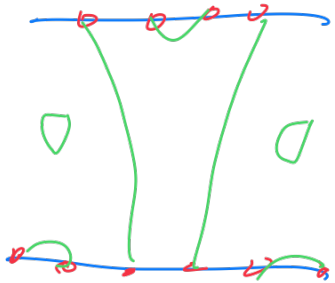
unit: 0. $0 \otimes 0 = 0$.

objects (a, c) ....



Let mention the 2D analogous of above. (called Temperley-Lieb category).

→ the only difference is that everything must be on a plane. (2D)

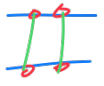
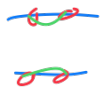


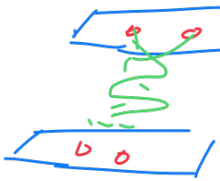
must be 2-dim: - no crossing 

$\in TL(4,6)$. - no  (because it's 3D).

"Rk": If you don't consider the closed loops 

$\tilde{T}(n,m)$ is infinite in general

but $\tilde{TL}(n,m)$ is finite. → $\{$  ,  $\} = \tilde{TL}(2,2)$



$\tilde{T}(4,4)$ ∞ many such tangle up to isotopy.