Recall that the morphisms $l_1, r_1, \iota : 1\otimes 1 \to 1$.

Corollary: $l_1 = r_1 = \iota$

proof:
In general, we already proved that $\forall\ Y, Z \in \mathcal{C}$,

$$\begin{tikzcd}
(Y\otimes Z) & & ((1\otimes Y)\otimes Z) \arrow[ll, "(l_Y\otimes id_Z)"'] \arrow[dll, "a_{1,Y,Z}"] \\
(1\otimes(Y\otimes Z)) \arrow[u, "l_{Y\otimes Z}"] & &
\end{tikzcd}$$

commutes.

Take $Y=Z=1$, then: $l_1 \otimes id_1 = l_{1\otimes 1} \circ a_{1,1,1}$

We know that $l_{1\otimes 1} = id_1 \otimes l_1$ (we proved $l_{1\otimes X} = id_1 \otimes l_X$)

Then: $l_1 \otimes id_1 = (id_1 \otimes l_1)\circ a_{1,1,1}$.

Here is another triangle: $\forall X, Y \in \mathcal{C}$

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

commutes.

Again, take $X = Y = 1$, then: $(\mathrm{id}_1 \otimes l_1) \circ a_{1,1,1} = r_1 \otimes \mathrm{id}_1$

Now: by def, we have:

$$L_1(l_1) = (l \otimes \mathrm{id}_1) \circ a^{-1}_{1,1,1}$$

$$\begin{array}{c} L_1(l_1) \\ \| \\ l_{1 \otimes 1} = \mathrm{id}_1 \otimes l_1. \end{array}$$

Then: $(\mathrm{id}_1 \otimes l_1) \circ a_{1,1,1} = l \otimes \mathrm{id}_1.$

Now: $l_1 \otimes id_1 = (id_1 \otimes l_1) \circ a_{1,1,1} = \iota \otimes id_1$.

$$= r_1 \otimes id_1$$

Finally: $R_1 : X \mapsto X \otimes 1$. (autoequivalence)

So we proved that an equivalence of category is a bijection on the morphisms, so admits an inverse on.

$$\begin{tikzcd}
l_1 \arrow[r, phantom, "="] \arrow[d, "R_1"'] & r_1 \arrow[r, phantom, "="] & \iota \\
l_1 \otimes id_1 \arrow[r, phantom, "="] & r_1 \otimes id_1 \arrow[r, phantom, "="] & \iota \otimes id_1 \arrow[u, "R_1^{-1}"']
\end{tikzcd}$$

$\blacksquare$

Recall that the unit object denotes $(1, \iota)$.

<u>Proposition</u>: The unit object in a monoidal category is unique, up to a unique isomorphism.

<u>proof</u>: Let $(1, \iota)$ and $(1', \iota')$ be two unit objects.

Let (r, l), (r', l') be the corresponding left and right unit constraints.

$$X \xrightarrow{(r'_X)^{-1}} X \otimes 1' \quad \text{(by def)}$$

So we made an isomorphism from 1 to $1'$.

Take $X = 1$

$$1 \xrightarrow{(r'_1)^{-1}} 1 \otimes 1' \xrightarrow{l_{1'}} 1'$$

$$\boxed{1 \xrightarrow{\eta} 1' \qquad \eta = l_{1'} \circ (r'_1)^{-1}.}$$

Exercise: Prove that the following diagram commutes:

$$\begin{tikzcd}
1 \otimes 1 \arrow[r, "\iota"] \arrow[d, "\gamma \otimes \gamma"'] & 1 \arrow[d, "\gamma"] \\
1' \otimes 1' \arrow[r, "\iota'"] & 1'
\end{tikzcd}$$

It remains to prove that such γ is unique.

Let $\gamma_1, \gamma_2 : 1 \xrightarrow{\sim} 1'$ be two such isomorphism

then consider $b = \gamma_1^{-1} \circ \gamma_2 : 1 \to 1$.

Next, see that the following commutes:

$$
\begin{tikzcd}
1 \otimes 1 \arrow[r, "b \otimes b"] \arrow[d, "\iota"'] & 1 \otimes 1 \arrow[d, "\iota"] \\
1 \arrow[r, "b"'] & 1
\end{tikzcd}
\qquad (A)
$$

because it decompose as follows:

$$
\begin{tikzcd}
1 \otimes 1 \arrow[r, "b_2 \otimes b_2"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\circlearrowleft"] & 1' \otimes 1' \arrow[r, "b_1' \otimes b_1'"] \arrow[d, "\iota'"] \arrow[dr, phantom, "\circlearrowleft"] & 1 \otimes 1 \arrow[d, "\iota"] \\
1 \arrow[r, "b_2"'] & 1' \arrow[r, "b_1'"'] & 1
\end{tikzcd}
$$

both commutes by the previous exercise.

Now consider the auto equivalence R_1, we know that

$$R_1 \xrightarrow[\sim]{r} id_{\mathcal{C}} \quad \text{natural isomorphism}$$

So by def: take $X \xrightarrow{f} Y$.

$$\begin{tikzcd}
R_1(X) \arrow[r, "r_X"] \arrow[d, "R_1(f)"'] & Id_{\mathcal{C}}(X) \arrow[d, "Id_{\mathcal{C}}(f)"] \\
R_1(Y) \arrow[r, "r_Y"'] & Id_{\mathcal{C}}(Y)
\end{tikzcd}$$

then take $X = Y = 1$, and any $1 \xrightarrow{c} 1$

$$\begin{tikzcd}
1 \otimes 1 \arrow[r, "r_1"] \arrow[d, "c \otimes id_1"'] & 1 \arrow[d, "c"] \\
1 \otimes 1 \arrow[r, "r_1"'] & 1
\end{tikzcd}$$

commutes.

(B)

But we know that $l = r_1$

So for $c = b$ + (A) + (B)

we get that

$$b \otimes id_1 = b \otimes b$$

because: $b \otimes b = (b \otimes id_1) \circ (id_1 \otimes b)$

$= b \otimes id_1$

$= (b \otimes id_1) \circ (id_1 \otimes id_1)$

but $b \otimes id_1$ is isomorphism, so after simplification:

$id_1 \otimes b = id_1 \otimes id_1$

$\Downarrow (l^{-1})$

$b = id_1$

Then $b = id_1$

Then: Apply R_1^{-1}, you get $b = id_1$.

So: $\eta_1^{-1} \circ \eta_2 = id_1$, i.e. $\eta_1 = \eta_2$ ☒

Proposition: Let $\mathcal{C}$ be a monoidal category. Then $End_{\mathcal{C}}(1)$ is a commutative monoid under composition, and $f \otimes g = \iota^{-1} \circ (f \circ g) \circ \iota \quad \forall f, g \in End_{\mathcal{C}}(1)$.

proof: Recall that if $\phi : F_1 \to F_2$ natural transformation then

$(\forall X \xrightarrow{f} Y) \quad F_2(f) \circ \phi_X = \phi_Y \circ F_1(f)$ { Moreover, if ϕ is a natural iso.
$\boxed{F_1(f) = \phi_Y^{-1} \circ F_2(f) \circ \phi_X}$

Apply above equality to $r: R_1 \to id_{\mathcal{C}}$ and $l: L_1 \to Id_{\mathcal{C}}$, with $X = Y = 1$ and $f, g: 1 \to 1$.

$$R_1(f) = r_1^{-1} \circ f \circ r_1 \quad , \quad L_1(g) = l_1^{-1} \circ g \circ l_1$$

$$\overset{\|}{f \otimes id_1} \qquad\qquad \overset{\|}{id_1 \otimes g}$$

We know that $r_1 = l_1 = l$, so:

$$f \circ g = (f \otimes id_1) \circ (id_1 \otimes g) = (l^{-1} \circ f \circ l) \circ (l^{-1} \circ g \circ l)$$

$$= l^{-1} \circ (f \circ g) \circ l$$

$$= (id_1 \otimes g) \circ (f \otimes id_1) = \cdots = l^{-1} \circ (g \circ f) \circ l$$

$\Rightarrow f \circ g = g \circ f$

$\blacksquare$

Example of monoidal categories:

- Sets: morphism = function (any)

  $\otimes = \times$ (direct product)

  $1 = \{0\}$ (unit object).

  $\iota: 1 \otimes 1 \to 1$ trivial

  associativity a: trivial also.

- $\Bbbk$-Vec with $\Bbbk$ a field.

  morphisms = $\Bbbk$-linear function.

  $\otimes = \otimes_{\Bbbk}$.

  $1 = \Bbbk$.

  If no confusion, $\Bbbk$-Vec will be denoted Vec.

Again: $(a, \iota) \to$ take the obvious one.

Remark: You can generalize above replacing field by unital commutative ring R

$\rightarrow$ you get the monoidal category of R-modules (R-mod)

- $\mathrm{Rep}_{\Bbbk}(G)$ with G a group.

  object: $\Bbbk$-vector space V on which G acts.

  $$\rho_V : G \to GL(V).$$

  morphism: $\Bbbk$-linear function preserving the action of G.

  $1 = \Bbbk$.

  $\otimes$: $\rho_{V \otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$.

Again, when no confusion, it is denoted $\mathrm{Rep}(G)$.