Deformed Ruijsenaars-Schneider model: integrability and time discretization

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Plan of the talk

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Integrable many-body systems play a significant role in mathematical physics. The Calogero-Moser (CM) and Ruijsenaars-Schneider (RS) systems are the main examples.

These models exist in rational, trigonometric (or hyperbolic) and elliptic versions, in which the interaction between particles is described by rational, trigonometric (hyperbolic) and elliptic functions respectively.

The elliptic models are most general: the other ones can be obtained from them by appropriate degenerations. We will consider the elliptic models.

The Weierstrass functions

The Weierstrass σ-function is given by the infinite product

$$\sigma(x) = \sigma(x|\omega_1, \omega_2) = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s} + \frac{x^2}{2s^2}}$$

$$s = 2\omega_1 m_1 + 2\omega_2 m_2 \quad \text{with integer } m_1, m_2$$

The Weierstrass ζ- and $\wp$-functions are connected with the σ-function as follows:

$$\zeta(x) = \sigma'(x)/\sigma(x)$$

$$\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$$

The CM model

The equations of motion:

$$\ddot{x}_i = 4 \sum_{j \neq i}^{N} \wp'(x_{ij}), \qquad x_{ij} = x_i - x_j,$$

where dot means the time derivative. The elliptic CM model is Hamiltonian and completely integrable, i.e., it has N independent integrals of motion in involution. Integrability of the model was proved by different methods by Perelomov (1977) and Wojciechowski (1977).

The RS model

The RS model is a deformation of the CM model (a "relativistic extension"). The equations of motion are:

$$\ddot{x}_i + \sum_{j\neq i}^{N} \dot{x}_i \dot{x}_j \Big(\zeta(x_{ij}+\eta) + \zeta(x_{ij}-\eta) - 2\zeta(x_{ij}) \Big) = 0$$

A properly taken limit $\eta \to 0$ leads to equations of motion of the CM model. Integrability of the RS system was proved by Ruijsenaars (1987).

Time discretization of the RS model
(Nijhoff, Ragnisco, Kuznetsov, 1996)

Let x_i^n be coordinate of the ith particle at the nth step of discrete time. The equations of motion are:

$$\prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} - \eta)\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1})$$

$$+ \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1} + \eta) = 0$$

The properly taken continuous time limit yields the equations of motion of the RS model.

The deformed RS model

The RS model admits a deformation (Krichever, A.Z., 2022). The equations of motion are:

$$\ddot{x}_i+\sum_{j\neq i}^{N}\dot{x}_i\dot{x}_j\Big(\zeta(x_{ij}+\eta)+\zeta(x_{ij}-\eta)-2\zeta(x_{ij})\Big)+g(U_i^{-}-U_i^{+})=0,$$

where

$$U_i^{\pm}=\prod_{j\neq i}^{N}U^{\pm}(x_{ij}),\quad U^{\pm}(x_{ij})=\frac{\sigma(x_{ij}\pm 2\eta)\sigma(x_{ij}\mp\eta)}{\sigma(x_{ij}\pm\eta)\sigma(x_{ij})}$$

and g is the deformation parameter. At $g=0$ we have the RS system. It is evident that $g\neq 0$ can be eliminated from the formulas by re-scaling of the time variable $t\to g^{-1/2}t$. In what follows we fix g to be $g=\sigma(2\eta)$.

Time discretization of the deformed RS model

The equations of motion in discrete time are:

$$\mu \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1} - \eta)$$

$$+ \mu \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} + \eta)\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1})$$

$$= \mu^{-1} \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} - \eta)\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1})$$

$$+ \mu^{-1} \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1} + \eta),$$

where μ is a parameter related to the lattice spacing in the time lattice.

The RS model

The N-particle elliptic RS model is a completely integrable Hamiltonian system. The canonical Poisson brackets: $\{x_i, p_j\} = \delta_{ij}$. The integrals of motion in involution:

$$\mathsf{I}_n = \sum_{\mathcal{I}\subset\{1,\dots,N\},\, |\mathcal{I}|=n} \exp\Big(\sum_{i\in\mathcal{I}} p_i\Big) \prod_{i\in\mathcal{I},\, j\notin\mathcal{I}} \frac{\sigma(x_{ij}+\eta)}{\sigma(x_{ij})}$$

Important particular cases:

$$\mathsf{I}_1 = \sum_{i=1}^{N} e^{p_i} \prod_{j\neq i} \frac{\sigma(x_{ij}+\eta)}{\sigma(x_{ij})}$$

which is the Hamiltonian H_1 of the chiral RS model and

$$\mathsf{I}_N = \exp\Big(\sum_{i=1}^{N} p_i\Big)$$

One can also introduce integrals of motion I_{-n} as

$$\mathsf{I}_{-n} = \mathsf{I}_N^{-1} \mathsf{I}_{N-n}$$

$$= \sum_{\mathcal{I} \subset \{1,\dots,N\},\, |\mathcal{I}|=n} \exp\Big(-\sum_{i \in \mathcal{I}} p_i\Big) \prod_{i \in \mathcal{I}, j \notin \mathcal{I}} \frac{\sigma(x_{ij} - \eta)}{\sigma(x_{ij})}$$

In particular,

$$\mathsf{I}_{-1} = \sum_{i=1}^{N} e^{-p_i} \prod_{j \neq i} \frac{\sigma(x_{ij} - \eta)}{\sigma(x_{ij})}$$

Renormalized integrals of motion:

$$\mathsf{J}_n = \frac{\sigma(|n|\eta)}{\sigma^n(\eta)}\,\mathsf{I}_n, \quad n = \pm 1, \ldots, \pm N.$$

The higher Hamiltonians of the RS model can be obtained from the equation of the spectral curve

$$z^N + \sum_{n=1}^{N} \phi_n(\lambda)\,\mathsf{J}_n\,z^{N-n} = 0, \quad \phi_n(\lambda) = \frac{\sigma(\lambda - n\eta)}{\sigma(\lambda)\sigma(n\eta)}$$

as

$$\mathsf{H}_n = \operatorname*{res}_{z=\infty}\left(z^{n-1}\lambda(z)\right)$$

For example:

$$\mathsf{H}_1 = \mathsf{J}_1,$$

$$\mathsf{H}_2 = \mathsf{J}_2 - \zeta(\eta)\mathsf{J}_1^2,$$

$$\mathsf{H}_3 = \mathsf{J}_3 - (\zeta(\eta) + \zeta(2\eta))\mathsf{J}_1\mathsf{J}_2 + \left(\tfrac{3}{2}\zeta^2(\eta) - \tfrac{1}{2}\wp(\eta)\right)\mathsf{J}_1^3$$

(Prokofev, A.Z., 2021). We also introduce the Hamiltonians

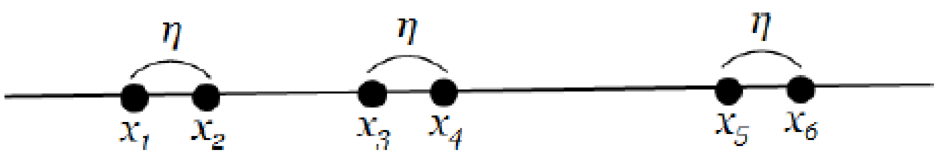
$$\mathsf{H}_n^{\pm} = \mathsf{H}_n \pm \bar{\mathsf{H}}_n$$

On the Toda lattice side, the RS dynamics corresponds to the dynamics of poles of elliptic solutions and the Hamiltonians $\mathsf{H}_n^{\pm}$ generate the flows $\partial_{t_n} \pm \partial_{\bar{t}_n}$, where t_n, $\bar{t}_n$ are canonical higher times of the Toda lattice hierarchy.

The Ruijsenaars-Schneider dynamics is the same as dynamics of poles of elliptic solutions to the 2D Toda equation in the Toda times t_1, $\bar{t}_1$ (Krichever, A.Z., 1995).

Moreover, this correspondence extends to a complete isomorphism between the elliptic Ruijsenaars-Schneider model (with higher Hamiltonian flows) and elliptic solutions to the whole 2D Toda lattice hierarchy (Prokofev, A.Z., 2021).

The deformed RS model as a dynamical system for pairs of RS particles



The restriction of the RS dynamics of $2N$ particles to the subspace $\mathcal{P}$ in which the particles stick together in N pairs such that

$$x_{2i} - x_{2i-1} = \eta, \qquad i = 1, \ldots, N$$

leads to the equations of motion of the deformed RS system for coordinates of the pairs (Krichever, A.Z., 2022). It is natural to introduce the variables

$$X_i = x_{2i-1}, \qquad i = 1, \ldots, N$$

which are coordinates of the pairs.

We pass from the initial $4N$-dimensional phase space $\mathcal{F}$ with coordinates $(\{x_i\},\{p_i\})$ to the $2N$-dimensional subspace $\mathcal{P}\subset\mathcal{F}$ of pairs defined by the constraints

$$\left\{\begin{array}{l} x_{2i}-x_{2i-1}=\eta, \quad x_{2i-1}=X_i, \\ \\ p_{2i-1}+p_{2i}=2\log\sigma(\eta)+\sum\limits_{j\neq i}\log\dfrac{\sigma(X_{ij}-2\eta)}{\sigma(X_{ij}+2\eta)} \end{array}\right.$$

The coordinates in $\mathcal{P}$ are $(\{X_i\},\{P_i\})$. The subspace $\mathcal{P}$ is preserved by the H_1^--flow $\partial_t=\partial_{t_1}-\partial_{\bar{t}_1}$ but is destroyed by the H_1^+-flow $\partial_{t_1}+\partial_{\bar{t}_1}$. Therefore, to define the dynamical system we should fix $T_1^+=\frac{1}{2}(t_1+\bar{t}_1)$ to be 0, i.e. put $\bar{t}_1=-t_1$, and consider the evolution with respect to the time $t=T_1^-=\frac{1}{2}(t_1-\bar{t}_1)$.

Moreover, the subspace $\mathcal{P}$ is invariant not only with respect to the H_1^--flow but also with respect to all higher H_k^--flows. This gives the possibility to obtain integrals of motion J_n of the deformed RS model by restriction of the RS integrals of motion J_n, J_{-n} to the subspace $\mathcal{P}$.

We denote the restriction of J_k by J_k:

$$J_k((\{X_i\}_{N_0}, \{P_i\}_{N_0}) = \mathsf{J}_k(\{x_\ell\}_N, \{p_\ell\}_N)\Big|_{\mathcal{P}}$$

Our main result is the explicit expressions for integrals of motion of the deformed RS system:

$$J_n = \frac{1}{2}\sum_{m=0}^{[n/2]} \frac{\sigma(n\eta)\sigma^{2m-n}(\eta)}{m!\,(n-2m)!} \sum_{[i_1\dots i_{n-m}]}^{N} \dot{x}_{i_{m+1}}\dots\dot{x}_{i_{n-m}} \prod_{\substack{\alpha,\beta=m+1\\ \alpha<\beta}}^{n-m} V(x_{i_\alpha i_\beta})$$

$$\times \left[\prod_{\gamma=1}^{m}\prod_{\ell\neq i_1,\dots,i_{n-m}}^{N} U^{+}(x_{i_\gamma \ell}) + \prod_{\gamma=1}^{m}\prod_{\ell\neq i_1,\dots,i_{n-m}}^{N} U^{-}(x_{i_\gamma \ell})\right],$$

where

$$V(x_{ij}) = \frac{\sigma^2(x_{ij})}{\sigma(x_{ij}+\eta)\sigma(x_{ij}-\eta)}$$

$$U^{\pm}(x_{ij}) = \frac{\sigma(x_{ij}\pm 2\eta)\sigma(x_{ij}\mp\eta)}{\sigma(x_{ij}\pm\eta)\sigma(x_{ij})}$$

Examples:

$$J_1 = \sum_{i=1} \dot{x}_i,$$

$$J_2 = \frac{\sigma(2\eta)}{2\sigma^2(\eta)} \left[\sum_{i \neq j} \dot{x}_i \dot{x}_j V(x_{ij}) + \sigma^2(\eta) \sum_i \left(\prod_{\ell \neq i} U^+(x_{i\ell}) + \prod_{\ell \neq i} U^-(x_{i\ell}) \right) \right],$$

$$J_3 = \frac{\sigma(3\eta)}{6\sigma^3(\eta)} \left[\sum_{i \neq j,k,\, j \neq k} \dot{x}_i \dot{x}_j \dot{x}_k V(x_{ij}) V(x_{ik}) V(x_{jk}) \right.$$
$$\left. + 3\sigma^2(\eta) \sum_{i \neq j} \dot{x}_j \left(\prod_{\ell \neq i,j} U^+(x_{i\ell}) + \prod_{\ell \neq i,j} U^-(x_{i\ell}) \right) \right].$$

The generating function of the integrals of motion:

$$R(z,\lambda)=\det_{1\le i,j\le N}\Big(z\delta_{ij}-\dot{x}_i\phi(x_{ij}-\eta,\lambda)-\sigma(2\eta)z^{-1}U_i^-\phi(x_{ij}-2\eta,\lambda)\Big),$$

where

$$\phi(x,\lambda):=\frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)}$$

The equation $R(z,\lambda)=0$ defines the spectral curve which is an integral of motion.

The generating function $R(z,u)$ is given by

$$R(z,u) = z^N + z^{-N}\frac{\sigma(u-2N\eta)}{\sigma(u)}$$

$$+ \sum_{k=1}^{N} z^{N-k}\frac{\sigma(u-k\eta)}{\sigma(u)\sigma(k\eta)}\,J_k + \sum_{k=1}^{N-1} z^{k-N}\frac{\sigma(u-2N\eta+k\eta)}{\sigma(u)\sigma(k\eta)}\,J_{-k}$$

It can be proved that $J_{-k} = J_k$.

The spectral curve

The characteristic equation $R(z,u)=0$ defines a Riemann surface $\tilde{\Gamma}$ which is a $2N$-sheet covering of the u-plane. Any point of it is $P=(z,u)$, where z,u are connected by equation $R(z,u)=0$. There are $2N$ points above each point u. The Riemann surface $\tilde{\Gamma}$ is invariant under the simultaneous transformations

$$u \mapsto u+2\omega, \quad z \mapsto e^{-2\zeta(\omega)\eta}z$$

$$u \mapsto u+2\omega', \quad z \mapsto e^{-2\zeta(\omega')\eta}z$$

The factor of $\tilde{\Gamma}$ over these transformations is an algebraic curve Γ which covers the elliptic curve with periods $2\omega, 2\omega'$. It is the spectral curve of the deformed RS model.

The spectral curve Γ admits a holomorphic involution ι with two fixed points. Indeed, the equation $R(z, u) = 0$ is invariant under the involution

$$\iota : (z, u) \mapsto (z^{-1}, 2N\eta - u)$$

The fixed points lie above the points u_* such that $u_* = 2N\eta - u_*$ modulo the lattice with periods $2\omega, 2\omega'$, i.e. $u_* = N\eta - \omega_\alpha$, where ω_α is either 0 or one of the three half-periods $\omega_1 = \omega$, $\omega_2 = \omega'$, $\omega_3 = \omega + \omega'$. Substituting this into the equation of the spectral curve and taking into account that $J_{-k} = J_k$, we conclude that the fixed points are $(\pm 1, N\eta)$ and there are no fixed points above $u_* = N\eta - \omega_\alpha$ with $\omega_\alpha \neq 0$.

Commutation representation (Manakov’s triple)

There is no Lax representation for the deformed RS system. Instead, it admits the commutation representation in the form of the Manakov's triple.

Lame-Hermite function:

$$\Phi(x,\lambda) = \frac{\sigma(x+\lambda)}{\sigma(x)\sigma(\lambda)} e^{-\zeta(\lambda)x}$$

Recall that

$$U_i^{\pm} = \prod_{j\neq i} \frac{\sigma(x_{ij} \pm 2\eta)\sigma(x_{ij} \mp \eta)}{\sigma(x_{ij} \pm \eta)\sigma(x_{ij})}$$

Introduce $N \times N$ matrices L, M, R:

$$L_{ij}(z,\lambda) = \dot{x}_i \Phi(x_{ij} - \eta, \lambda) + z^{-1}\sigma(2\eta)U_i^{-}\Phi(x_{ij} - 2\eta, \lambda)$$

$$M_{ij}(z,\lambda) = \dot{x}_i(1-\delta_{ij})\Phi(x_{ij},\lambda) + z^{-1}\sigma(2\eta)U_i^{+}\Phi(x_{ij}-\eta,\lambda)$$
$$-\delta_{ij}\left(\sum_k \dot{x}_k \zeta(x_{ik}+\eta) - \sum_{k\neq i}\zeta(x_{ik})\right)$$

$$R_{ij}(z,\lambda) = \sigma(2\eta)z^{-1}\Phi(x_{ij} - \eta, \lambda)$$

The equations of motion of the deformed RS model are equivalent to the following commutation relation:

$$\dot{L} + [L, M] = R(L - zI)$$

(Manakov's triple representation).

From $\mathrm{tr}\, R = 0$ it follows that

$$\det(zI - L(z, \lambda))$$

is conserved in time. It is the generating funcion of integrals of motion.

The Backlund transformation

It is known that the integrable many-body systems of CM and RS type are dynamical systems for poles of singular solutions to nonlinear integrable differential and difference equations. The nonlinear integrable equations are known to serve as compatibility conditions for linear differential or difference equations for the “wave function” ψ. Poles of solutions to the nonlinear equations (zeros of the tau-function) are simultaneously poles of the ψ-function, so the latter are subject to equations of motion of the CM or RS type. In fact zeros of the ψ-function are subject to the same equations, and this leads to the idea to obtain the Bäcklund transformation of the CM or RS system as passage from poles to zeros.

The first linear problem for the Toda lattice with constraint of type B:

$$\partial_t \psi(x) = v(x)\Big(\psi(x+\eta) - \psi(x-\eta)\Big)$$

(Krichever, A.Z., 2022), where $v(x)$ is expressed through the tau-function $\tau(x)$ as

$$v(x) = \frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}.$$

For elliptic solutions

$$\tau(x) = \prod_{i=1}^{N} \sigma(x - x_i)$$

We can represent solutions for ψ in the form

$$\psi(x) = \mu^{x/\eta} e^{(\mu-\mu^{-1})t} \frac{\widehat{\tau}(x)}{\tau(x)},$$

where μ is a parameter and

$$\widehat{\tau}(x) = \prod_{i=1}^{N} \sigma(x - y_i)$$

with some y_i's.

The zeros y_i and the poles x_i of the ψ-function obey the system of equations

$$\dot{x}_i = \mu\sigma(-\eta)\prod_{j\neq i}\frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}\prod_k\frac{\sigma(x_i - y_k + \eta)}{\sigma(x_i - y_k)}$$

$$+ \mu^{-1}\sigma(-\eta)\prod_{j\neq i}\frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}\prod_k\frac{\sigma(x_i - y_k - \eta)}{\sigma(x_i - y_k)},$$

$$\dot{y}_i = \mu\sigma(-\eta)\prod_{j\neq i}\frac{\sigma(y_i - y_j + \eta)}{\sigma(y_i - y_j)}\prod_k\frac{\sigma(y_i - x_k - \eta)}{\sigma(y_i - x_k)}$$

$$+ \mu^{-1}\sigma(-\eta)\prod_{j\neq i}\frac{\sigma(y_i - y_j - \eta)}{\sigma(y_i - y_j)}\prod_k\frac{\sigma(y_i - x_k + \eta)}{\sigma(y_i - x_k)}$$

$x_j \to y_j$ is the Bäcklund transformation.

The discrete time dynamics

The Bäcklund transformation

$$x_j \longrightarrow y_j$$

can be regarded as a time evolution by one step of the discrete time.

Denoting the discrete time variable by n, we then write

$$x_i = x_i^n, \qquad y_i = x_i^{n+1}$$

Then the Bäcklund transformations can be read as the equations of motion in discrete time:

$$\mu \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1} - \eta)$$

$$+ \mu \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} + \eta)\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1})$$

$$= \mu^{-1} \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} - \eta)\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1})$$

$$+ \mu^{-1} \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1} + \eta)$$

The continuous time limits

These equations admit different continuum limits. For one of them, we introduce the variables

$$X_j^n = x_j^n - n\eta$$

and assume that these variables behave smoothly when the time changes, i.e., $X_j^{n+1} = X_j^n + O(\varepsilon)$ as $\varepsilon \to 0$, where we introduce the lattice spacing ε in the time lattice, so that the continuous time variable is $t = n\varepsilon$.

We should expand in powers of ε taking into account that

$$X_j^{n\pm1} = X_j \pm \varepsilon \dot{X}_j + \frac{1}{2}\varepsilon^2 \ddot{X}_j + O(\varepsilon^3)$$

as $\varepsilon \to 0$. It is enough to expand up to the order ε. For consistency one should require that μ^{-1} is of order ε. Putting $\mu^{-1} = \varepsilon$, one obtains (in the leading order ε) the equations of motion of the deformed RS system.

Another possibility is to assume that the original variables x_j^n are smooth when the time changes, i.e.,

$$x_j^{n\pm 1} = x_j \pm \varepsilon \dot{x}_j + \frac{1}{2}\varepsilon^2 \ddot{x}_j + O(\varepsilon^3)$$

In general position, i.e. if $\mu^{-2} - 1 = O(1)$ as $\varepsilon \to 0$, the leading order is ε and the expansion gives the RS equations. However, if $\mu^{-2} = 1 + \alpha\varepsilon$, then the first order gives the identity $0 = 0$ and one should expand up to the second order in ε. In this case one obtains the equations for dynamics of poles of elliptic solutions to the semi-discrete BKP equation (Rudneva, A.Z., 2020).

Concluding remarks

We proved integrability of the deformed RS system by presenting all integrals of motion in explicit form. We also obtained the discrete time version of the deformed RS system by considering the Bäcklund transformations.

The connection between the standard RS system and the deformed one is not trivial. On one hand, the latter is an extension of the former and includes it as a particular case. However, on the other hand, the deformed RS system is contained in the RS system since it can be regarded as its reduction in the sense that its equations of motion are obtained by restriction of the RS dynamics to the subspace $\mathcal{P}$ of pairs.