Harmonic locus and Calogero-Moser spaces

Alexander Veselov
Loughborough, UK

Integrable Systems and Algebraic Geometry, BIMSA, July 2, 2024
Dedicated to Igor Krichever (1950-2022)

Igor Moiseevich Krichever (8 October 1950 – 1 December 2022)

Teacher

Sergey Petrovich Novikov (20 March 1938 - 6 June 2024)

Scientific family

Ascona-2015: Buchstaber, Veselov, Novikov, Krichever, Dubrovin

Let $V(z)$, $z \in \mathbb{Z}$ be meromorphic potential and $L = -D^2 + V(z)$ be the corresponding Schrödinger operator.

Locus problem: Describe the potentials $V(z)$ such that the corresponding equation

$$(-D^2 + V(z))\psi = \lambda\psi, \quad D = \frac{d}{dz}$$

has all the solutions ψ, which are meromorphic in $z \in \mathbb{C}$ for *all* λ.

Let $V(z)$, $z \in \mathbb{Z}$ be meromorphic potential and $L = -D^2 + V(z)$ be the corresponding Schrödinger operator.

Locus problem: Describe the potentials $V(z)$ such that the corresponding equation

$$(-D^2 + V(z))\psi = \lambda\psi, \quad D = \frac{d}{dz}$$

has all the solutions ψ, which are meromorphic in $z \in \mathbb{C}$ for *all* λ.

Examples: all finite-gap operators are monodromy-free (**S.P. Novikov 1974, Its and Matveev 1975, Krichever 1976**)

Let $V(z)$, $z \in \mathbb{Z}$ be meromorphic potential and $L = -D^2 + V(z)$ be the corresponding Schrödinger operator.

Locus problem: Describe the potentials $V(z)$ such that the corresponding equation

$$(-D^2 + V(z))\psi = \lambda\psi, \quad D = \frac{d}{dz}$$

has all the solutions ψ, which are meromorphic in $z \in \mathbb{C}$ for *all* λ.

Examples: all finite-gap operators are monodromy-free (**S.P. Novikov 1974, Its and Matveev 1975, Krichever 1976**)

History (1870-80s): Study of differential equations in complex domain with "uniform" solutions (**Hermite, Picard, Halphen, Darboux,...**)

Monodromy-free operators and locus problem in 1D

Let $V(z)$, $z \in \mathbb{Z}$ be meromorphic potential and $L = -D^2 + V(z)$ be the corresponding Schrödinger operator.

Locus problem: Describe the potentials $V(z)$ such that the corresponding equation

$$(-D^2 + V(z))\psi = \lambda\psi, \quad D = \frac{d}{dz}$$

has all the solutions ψ, which are meromorphic in $z \in \mathbb{C}$ for *all* λ.

Examples: all finite-gap operators are monodromy-free (**S.P. Novikov 1974, Its and Matveev 1975, Krichever 1976**)

History (1870-80s): Study of differential equations in complex domain with "uniform" solutions (**Hermite, Picard, Halphen, Darboux,...**)

Duistermaat and Grünbaum 1986: the operator $L = -D^2 + V(z)$ is monodromy-free iff Laurent series expansion of its potential near every pole z_0 $V = \sum_{i=-2}^{\infty} c_i (z - z_0)^i$ satisfies *locus (quasi-invariance) conditions*:

$$c_{-2} = m(m+1), m \in \mathbb{N}, \quad c_{2k-1} = 0, k = 0, ..., m.$$

Airault, McKean, Moser 1977: poles of the rational solutions $V(z) = \sum_{i=1}^{N} \frac{2}{(z-z_i)^2}$ of the KdV equation $u_t = 6uu_x - u_{xxx}$ belong to the **locus** given by the algebraic system

$$\sum_{j \neq i}^{N} \frac{1}{(z_i - z_j)^3} = 0, \quad i = 1, \ldots, N,$$

describing the (complex) equilibriums of the Calogero-Moser system with

$$H = \sum_{i=1}^{N} p_i^2 + \sum_{j \neq i}^{N} \frac{2}{(z_i - z_j)^2}.$$

Terminology and modern motivation

Airault, McKean, Moser 1977: poles of the rational solutions $V(z) = \sum_{i=1}^{N} \frac{2}{(z-z_i)^2}$ of the KdV equation $u_t = 6uu_x - u_{xxx}$ belong to the **locus** given by the algebraic system

$$\sum_{j\neq i}^{N} \frac{1}{(z_i - z_j)^3} = 0, \quad i = 1, \ldots, N,$$

describing the (complex) equilibriums of the Calogero-Moser system with

$$H = \sum_{i=1}^{N} p_i^2 + \sum_{j\neq i}^{N} \frac{2}{(z_i - z_j)^2}.$$

- If $N \neq m(m+1)/2$ is not a triangular number, the locus is empty.
- If $N = m(m+1)/2$ it has dimension m and consists of zeros of the **Burchnall-Chaundy (Adler-Moser) polynomials**

$$P_1 = z,\ \ P_2 = \frac{1}{3}(z^3 + \tau_2),\ \ P_3 = \frac{1}{45}(z^6 + 5\tau_2 z^3 + \tau_3 z - 5\tau_2^2),$$

$$P_4 = \frac{1}{4725}(z^{10} + 15\tau_2 z^7 + 7\tau_3 z^5 - 35\tau_2\tau_3 z^2 + 175\tau_2^3 z - \frac{7}{3}\tau_3^2 + \tau_4 z^3 + \tau_4\tau_2), \ldots$$

Harmonic locus

Harmonic locus $\mathcal{HL}$ consists of monodromy-free potentials of the form

$$V = z^2 + \sum_{i=1}^{N} \frac{m_i(m_i+1)}{(z-z_i)^2}.$$

When all multiplicities $m_i = 1$ (simple part) the poles satisfy the algebraic system

$$\sum_{j \neq i}^{N} \frac{2}{(z_i - z_j)^3} - z_i = 0, \ \ i = 1, \dots, N,$$

describing the (complex) equilibriums of the Calogero-Moser system with

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + U(q), \quad U(q) = \frac{1}{2}\sum_{i=1}^{N} q_i^2 + \sum_{1 \leq i < j \leq n} \frac{1}{(q_i - q_j)^2}.$$

Harmonic locus

Harmonic locus $\mathcal{HL}$ consists of monodromy-free potentials of the form

$$V = z^2 + \sum_{i=1}^{N} \frac{m_i(m_i+1)}{(z-z_i)^2}.$$

When all multiplicities $m_i = 1$ (simple part) the poles satisfy the algebraic system

$$\sum_{j\neq i}^{N} \frac{2}{(z_i - z_j)^3} - z_i = 0, \;\; i = 1, \dots, N,$$

describing the (complex) equilibriums of the Calogero-Moser system with

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + U(q), \quad U(q) = \frac{1}{2}\sum_{i=1}^{N} q_i^2 + \sum_{1\leq i<j\leq n} \frac{1}{(q_i - q_j)^2}.$$

Oblomkov 1999: all such potentials can be explicitly described via the Wronskians of the Hermite polynomials

$$V(z) = z^2 - 2D^2 \log W(H_{k_1}(z), \dots, H_{k_n}(z)), \quad k_1 > k_2 > \dots > k_n > 0,$$

where $H_k(z)$ is the k-th Hermite polynomial.

Hermite polynomials

Hermite polynomials $H_n(z)$ are the classical orthogonal polynomials with Gaussian weight $w(z) = e^{-z^2}$: $H_k(z) = (-1)^k e^{z^2} D^k e^{-z^2}$,

$$H_0(z) = 1,\ H_1(z) = 2z,\ H_2(z) = 4z^2 - 2,\ H_3(z) = 8z^3 - 12z,$$

$$H_4(z) = 16z^4 - 48z^2 + 12,\ H_5(z) = 32z^5 - 160z^3 + 120z, \dots$$

satisfying the recurrence

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z).$$

Hermite polynomials

Hermite polynomials $H_n(z)$ are the classical orthogonal polynomials with Gaussian weight $w(z) = e^{-z^2}$: $H_k(z) = (-1)^k e^{z^2} D^k e^{-z^2}$,

$$H_0(z) = 1,\ H_1(z) = 2z,\ H_2(z) = 4z^2 - 2,\ H_3(z) = 8z^3 - 12z,$$

$$H_4(z) = 16z^4 - 48z^2 + 12,\ H_5(z) = 32z^5 - 160z^3 + 120z, \dots$$

satisfying the recurrence

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z).$$

The operators $\tilde{L} = -D^2 + x^2 - 2D^2 \log W(H_{k_1}(x), \dots, H_{k_n}(x))$ are the integrable (Darboux) transformations of the harmonic oscillator $L = -D^2 + x^2$ with the eigenfunctions $\psi_n = H_n(x)e^{-x^2/2}$:

$$L\psi_n = (n + 1/2)\psi_n,\ , n = 0, 1, \dots$$

Felder, Hemery, Veselov 2012: Label these Wronskians by the partitions $\lambda = (\lambda_1, \dots, \lambda_l)$: $W_\lambda(z) := W(H_{\lambda_1+l-1}, H_{\lambda_2+l-2}, \dots, H_{\lambda_l})$, then

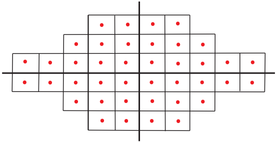
1. *$W_\lambda(z)$ is a polynomial in z of degree $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_l$,*
2. *$W_\lambda(-z) = (-1)^{|\lambda|} W_\lambda(z)$,*
3. *$W_{\lambda^*}(z) = (-i)^{|\lambda|} W_\lambda(iz)$, where λ^* is the conjugate of λ.*

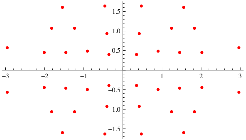
Harmonic locus and Young diagrams

Felder, Hemery, Veselov 2012: Label these Wronskians by the partitions $\lambda = (\lambda_1, \dots, \lambda_l)$: $W_\lambda(z) := W(H_{\lambda_1+l-1}, H_{\lambda_2+l-2}, \dots, H_{\lambda_l})$, then

1. *$W_\lambda(z)$ is a polynomial in z of degree $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_l$,*
2. *$W_\lambda(-z) = (-1)^{|\lambda|} W_\lambda(z)$,*
3. *$W_{\lambda^*}(z) = (-i)^{|\lambda|} W_\lambda(iz)$, where λ^* is the conjugate of λ.*

For the doubled partitions $(\lambda_1, \lambda_1, \dots, \lambda_l, \lambda_l)$ there is a surprising (empirical) relation between the Young (Ferrers) diagram and zero set of $W_\lambda(z)$:



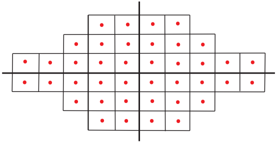


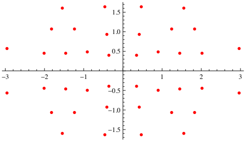
Harmonic locus and Young diagrams

Felder, Hemery, Veselov 2012: Label these Wronskians by the partitions $\lambda = (\lambda_1, \dots, \lambda_l)$: $W_\lambda(z) := W(H_{\lambda_1+l-1}, H_{\lambda_2+l-2}, \dots, H_{\lambda_l})$, then

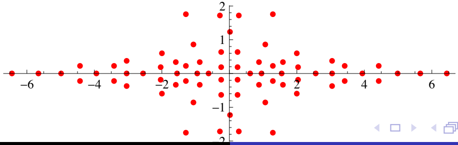
1. *$W_\lambda(z)$ is a polynomial in z of degree $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_l$,*
2. *$W_\lambda(-z) = (-1)^{|\lambda|} W_\lambda(z)$,*
3. *$W_{\lambda^*}(z) = (-i)^{|\lambda|} W_\lambda(iz)$, where λ^* is the conjugate of λ.*

For the doubled partitions $(\lambda_1, \lambda_1, \dots, \lambda_l, \lambda_l)$ there is a surprising (empirical) relation between the Young (Ferrers) diagram and zero set of $W_\lambda(z)$:





but, in general, it is not so clear: e.g. for $\lambda = (28, 16, 10, 6, 4, 4, 3, 1)$ we have



Inverse problem for harmonic locus

Question: Given the zero set of W_λ, how to recover the partition λ?

For the simple zeros case with $m_i = 1$ we have the following answer (conjectured in 2012).

Question: Given the zero set of W_λ, how to recover the partition λ?

For the simple zeros case with $m_i = 1$ we have the following answer (conjectured in 2012).

Recall that the *content* $c(\Box)$ of the box $\Box = (i, j)$ from the Young diagram λ is defined as $j - i$. The multiset $C(\lambda) := \{c(\Box),\ \Box \in \lambda\}$ determines λ uniquely:

0	1	2	3
-1	0	1	
-2			

Inverse problem for harmonic locus

Question: Given the zero set of W_λ, how to recover the partition λ?

For the simple zeros case with $m_i = 1$ we have the following answer (conjectured in 2012).

Recall that the *content* $c(\square)$ of the box $\square = (i,j)$ from the Young diagram λ is defined as $j - i$. The multiset $C(\lambda) := \{c(\square),\ \square \in \lambda\}$ determines λ uniquely:

0	1	2	3
-1	0	1	
-2			

Felder, Veselov 2024: For a simple locus configuration $(z_1, \dots, z_n)$ the corresponding partition λ is uniquely determined by the property that the contents of λ coincide with the eigenvalues of Moser's matrix M :

$$C(\lambda) = Spec\, M, \quad M_{ij} = \begin{cases} -\frac{1}{(z_i - z_j)^2} & i \neq j \\ \sum_{k \neq j}^{n} \frac{1}{(z_k - z_j)^2} & i = j. \end{cases}$$

Proof uses the theory of Calogero-Moser systems and Calogero-Moser spaces.

Moser 1975: Lax form $\dot{L} = [L, M]$ for the (now called CM) system with

$$H_{CM} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{1\le i<j\le n} \frac{\gamma^2}{(q_i - q_j)^2} :$$

$$L_{ij} = p_i\delta_{ij} + \frac{i\gamma}{q_i - q_j}(1 - \delta_{ij}), \quad M_{ij} = \sum_{k\neq i}^{n} \frac{i\gamma}{(q_k - q_i)^2}\delta_{ij} + \frac{i\gamma}{(q_i - q_j)^2}(1 - \delta_{ij}).$$

Moser 1975: Lax form $\dot{L} = [L, M]$ for the (now called CM) system with

$$H_{CM} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{1\le i<j\le n} \frac{\gamma^2}{(q_i - q_j)^2}:$$

$$L_{ij} = p_i\delta_{ij} + \frac{i\gamma}{q_i - q_j}(1-\delta_{ij}), \quad M_{ij} = \sum_{k\neq i}^{n} \frac{i\gamma}{(q_k - q_i)^2}\delta_{ij} + \frac{i\gamma}{(q_i - q_j)^2}(1-\delta_{ij}).$$

Kazhdan, Kostant and Sternberg 1978: CM system as a symplectic reduction of free motion on Lie algebra of $U(n)$: for the moment map

$$\mu : (P, Q) \to [P, Q] = i\gamma(1-\delta_{ij}), \ (P, Q) \in T^*u(n)$$

$$Q = q_i\delta_{ij}, \ P = L, \ H_{CM} = \frac{1}{2} tr\, P^2 = \frac{1}{2} tr\, L^2.$$

Moser 1975: Lax form $\dot{L} = [L, M]$ for the (now called CM) system with

$$H_{CM} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{1\le i<j\le n} \frac{\gamma^2}{(q_i - q_j)^2}:$$

$$L_{ij} = p_i\delta_{ij} + \frac{i\gamma}{q_i - q_j}(1-\delta_{ij}), \quad M_{ij} = \sum_{k\neq i}^{n} \frac{i\gamma}{(q_k - q_i)^2}\delta_{ij} + \frac{i\gamma}{(q_i - q_j)^2}(1-\delta_{ij}).$$

Kazhdan, Kostant and Sternberg 1978: CM system as a symplectic reduction of free motion on Lie algebra of $U(n)$: for the moment map

$$\mu : (P, Q) \to [P, Q] = i\gamma(1-\delta_{ij}), \; (P, Q) \in T^*u(n)$$

$$Q = q_i\delta_{ij}, \;\; P = L, \;\; H_{CM} = \frac{1}{2} tr\, P^2 = \frac{1}{2} tr\, L^2.$$

Perelomov 1978: Harmonic version $\dot{L}_\pm = [L_\pm, M] \pm L_\pm,\; L_\pm = L \pm Q,$

$$H_{CM}^{\mathcal{H}} = \frac{1}{2} tr\, L^2 + \frac{1}{2} tr\, Q^2 = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \sum_{1\le i<j\le n} \frac{\gamma^2}{(q_i - q_j)^2}.$$

G. Wilson 1998: *Calogero-Moser space* $\mathcal{C}_n$ is the quotient space

$$\mathcal{C}_n = \{(X, Z, v, w) : [X, Z] + I = vw\}/GL_n(\mathbb{C}),$$

where X and Z are n by n complex matrices, v and w are n-dimensional vector and covector (considered as $n \times 1$ and $1 \times n$ matrices respectively), and an element $g \in GL_n$ acts as

$$(X, Z, v, w) \mapsto (gXg^{-1}, gZg^{-1}, gv, wg^{-1}).$$

G. Wilson 1998: *Calogero-Moser space* $\mathcal{C}_n$ is the quotient space

$$\mathcal{C}_n = \{(X, Z, v, w) : [X, Z] + I = vw\}/GL_n(\mathbb{C}),$$

where X and Z are n by n complex matrices, v and w are n-dimensional vector and covector (considered as $n \times 1$ and $1 \times n$ matrices respectively), and an element $g \in GL_n$ acts as

$$(X, Z, v, w) \mapsto (gXg^{-1}, gZg^{-1}, gv, wg^{-1}).$$

Wilson: $\mathcal{C}_n$ is a smooth irreducible affine algebraic variety of dimension $2n$, which can be viewed as a quantisation of the Hilbert scheme of n points in $\mathbb{C}^2$.

G. Wilson 1998: *Calogero-Moser space* $\mathcal{C}_n$ is the quotient space

$$\mathcal{C}_n = \{(X, Z, v, w) : [X, Z] + I = vw\}/GL_n(\mathbb{C}),$$

where X and Z are n by n complex matrices, v and w are n-dimensional vector and covector (considered as $n \times 1$ and $1 \times n$ matrices respectively), and an element $g \in GL_n$ acts as

$$(X, Z, v, w) \mapsto (gXg^{-1}, gZg^{-1}, gv, wg^{-1}).$$

Wilson: $\mathcal{C}_n$ is a smooth irreducible affine algebraic variety of dimension $2n$, which can be viewed as a quantisation of the Hilbert scheme of n points in $\mathbb{C}^2$.

There is a natural symplectic action of $\mathbb{C}^\times = \mathbb{C} \setminus 0$ on $\mathcal{C}_n$ defined by

$$X \mapsto \mu X, \;\; Z \mapsto \mu^{-1} Z, \;\; v \mapsto v, \;\; w \mapsto w, \;\; \mu \in \mathbb{C}^\times.$$

Let $\mathcal{C}_n^{\mathbb{C}^\times}$ be the fixed point subset of $\mathcal{C}_n$ under this action. Wilson identified it with the set $\mathcal{P}_n$ of all partitions of n.

Modified Calogero-Moser spaces

Felder, Veselov 2024: *The modified Calogero-Moser space* $\mathcal{CM}_n$ is the quotient

$$\mathcal{CM}_n = \{\Pi = (L, Q, M, v, w)\}/GL_n(\mathbb{C}),$$

where L, Q, M are n by n complex matrices, v and w are a vector and covector as before, which satisfy the following relations

$$\begin{aligned} (\mathrm{I}): &\quad [L, Q] = I - vw, \\ (\mathrm{II}): &\quad [M, Q] = L, \\ (\mathrm{III}): &\quad [M, L] = Q, \\ (\mathrm{IV}): &\quad Mv = 0, \quad wM = 0. \end{aligned}$$

The group GL_n acts by conjugation on L, Q, M and on v, w as before.

Felder, Veselov 2024: *The modified Calogero-Moser space* $\mathcal{CM}_n$ is the quotient

$$\mathcal{CM}_n = \{\Pi = (L, Q, M, v, w)\}/GL_n(\mathbb{C}),$$

where L, Q, M are n by n complex matrices, v and w are a vector and covector as before, which satisfy the following relations

$$\begin{aligned} (\mathrm{I}): &\quad [L, Q] = I - vw, \\ (\mathrm{II}): &\quad [M, Q] = L, \\ (\mathrm{III}): &\quad [M, L] = Q, \\ (\mathrm{IV}): &\quad Mv = 0, \quad wM = 0. \end{aligned}$$

The group GL_n acts by conjugation on L, Q, M and on v, w as before.

FV 2024: Modified Calogero–Moser space $\mathcal{CM}_n$ can be identified with the harmonic locus and with set of partitions of n via the map $\chi : \mathcal{CM}_n \to \mathcal{HL}_n$,

$$\chi(\Pi) = z^2 - 2D^2 \log\det(zI - Q).$$

The proof

Step 1. The modified CM space $\mathcal{CM}_n$ can be identified with the fixed set $\mathcal{C}_n^{\mathbb{C}^\times}$:

$$X = \frac{1}{2}(L + Q), \ Z = L - Q.$$

The proof

Step 1. The modified CM space $\mathcal{CM}_n$ can be identified with the fixed set $\mathcal{C}_n^{\mathbb{C}^\times}$:

$$X = \frac{1}{2}(L + Q), \;\; Z = L - Q.$$

Step 2. M is the generator of $\mathbb{C}^\times$-action:

$$[M, X] = X, \;\; [M, Z] = -Z.$$

The proof

Step 1. The modified CM space $\mathcal{CM}_n$ can be identified with the fixed set $\mathcal{C}_n^{\mathbb{C}^\times}$:

$$X = \frac{1}{2}(L + Q), \; Z = L - Q.$$

Step 2. M is the generator of $\mathbb{C}^\times$-action:

$$[M, X] = X, \; [M, Z] = -Z.$$

Step 3. Use Wilson's identification of $\mathcal{C}_n^{\mathbb{C}^\times}$ with the set of partitions $\mathcal{P}_n$ and his formula

$$\det(X_\lambda - \sum_{i \geq 1} p_i (-Z_\lambda)^{i-1}) = B(\lambda) s_\lambda,$$

where p_i and s_λ are power sums and Schur symmetric functions respectively.

The proof

Step 1. The modified CM space $\mathcal{CM}_n$ can be identified with the fixed set $\mathcal{C}_n^{\mathbb{C}^\times}$:

$$X = \frac{1}{2}(L + Q), \;\; Z = L - Q.$$

Step 2. M is the generator of $\mathbb{C}^\times$-action:

$$[M, X] = X, \;\; [M, Z] = -Z.$$

Step 3. Use Wilson's identification of $\mathcal{C}_n^{\mathbb{C}^\times}$ with the set of partitions $\mathcal{P}_n$ and his formula

$$\det(X_\lambda - \sum_{i \geq 1} p_i (-Z_\lambda)^{i-1}) = B(\lambda) s_\lambda,$$

where p_i and s_λ are power sums and Schur symmetric functions respectively.

Step 4. Use the theory of Appell polynomials and generating function of Hermite polynomials to derive that up to a constant multiple

$$W_\lambda(z) = \det(X_\lambda - zI - \frac{1}{2} Z_\lambda) = \det(Q_\lambda - zI).$$

Step 1. The modified CM space $\mathcal{CM}_n$ can be identified with the fixed set $\mathcal{C}_n^{\mathbb{C}^\times}$:

$$X = \frac{1}{2}(L+Q), \; Z = L - Q.$$

Step 2. M is the generator of $\mathbb{C}^\times$-action:

$$[M, X] = X, \; [M, Z] = -Z.$$

Step 3. Use Wilson's identification of $\mathcal{C}_n^{\mathbb{C}^\times}$ with the set of partitions $\mathcal{P}_n$ and his formula

$$\det(X_\lambda - \sum_{i \geq 1} p_i (-Z_\lambda)^{i-1}) = B(\lambda) s_\lambda,$$

where p_i and s_λ are power sums and Schur symmetric functions respectively.

Step 4. Use the theory of Appell polynomials and generating function of Hermite polynomials to derive that up to a constant multiple

$$W_\lambda(z) = \det(X_\lambda - zI - \frac{1}{2} Z_\lambda) = \det(Q_\lambda - zI).$$

Step 5. Use Oblomkov's theorem to make link with harmonic locus.

Inversion of the Wronskian map

Felder, Veselov 2024: The subset of $\mathcal{CM}_n$ with diagonalisable Q with simple spectrum can be identified with the simple part of the harmonic locus $\mathcal{HL}_n$.

The spectrum of the corresponding Moser's matrix M is integer and coincides with the content multiset $C(\lambda)$ of the corresponding partition λ.

Felder, Veselov 2024: The subset of $\mathcal{CM}_n$ with diagonalisable Q with simple spectrum can be identified with the simple part of the harmonic locus $\mathcal{HL}_n$.

The spectrum of the corresponding Moser's matrix M is integer and coincides with the content multiset $C(\lambda)$ of the corresponding partition λ.

Indeed, Moser's matrices obviously satisfy the relations (I), (II) and (IV) (with $w = (1, 1, \dots, 1) = v^T$), while (III) is equivalent to the locus conditions:

$$
\begin{aligned}
\text{(I)}: &\quad [L, Q] = I - vw, \\
\text{(II)}: &\quad [M, Q] = L, \\
\text{(III)}: &\quad [M, L] = Q, \\
\text{(IV)}: &\quad Mv = 0, \quad wM = 0.
\end{aligned}
$$

Felder, Veselov 2024: The subset of $\mathcal{CM}_n$ with diagonalisable Q with simple spectrum can be identified with the simple part of the harmonic locus $\mathcal{HL}_n$.

The spectrum of the corresponding Moser's matrix M is integer and coincides with the content multiset $C(\lambda)$ of the corresponding partition λ.

Indeed, Moser's matrices obviously satisfy the relations (I), (II) and (IV) (with $w = (1, 1, \ldots, 1) = v^T$), while (III) is equivalent to the locus conditions:

$$\begin{aligned} \text{(I)}: &\quad [L, Q] = I - vw, \\ \text{(II)}: &\quad [M, Q] = L, \\ \text{(III)}: &\quad [M, L] = Q, \\ \text{(IV)}: &\quad Mv = 0, \quad wM = 0. \end{aligned}$$

The proof of the formula $Spec\, M_\lambda = C(\lambda)$ follows from an explicit description of X_λ, Z_λ from **Wilson 1998**, who used the Frobenius parametrisation of λ.

Felder, Veselov 2024: The subset of $\mathcal{CM}_n$ with diagonalisable Q with simple spectrum can be identified with the simple part of the harmonic locus $\mathcal{HL}_n$.

The spectrum of the corresponding Moser's matrix M is integer and coincides with the content multiset $C(\lambda)$ of the corresponding partition λ.

Indeed, Moser's matrices obviously satisfy the relations (I), (II) and (IV) (with $w = (1, 1, \dots, 1) = v^T$), while (III) is equivalent to the locus conditions:

$$\begin{aligned}
(\mathrm{I}): &\quad [L, Q] = I - vw, \\
(\mathrm{II}): &\quad [M, Q] = L, \\
(\mathrm{III}): &\quad [M, L] = Q, \\
(\mathrm{IV}): &\quad Mv = 0, \quad wM = 0.
\end{aligned}$$

The proof of the formula $Spec\, M_\lambda = C(\lambda)$ follows from an explicit description of X_λ, Z_λ from **Wilson 1998**, who used the Frobenius parametrisation of λ.

This agrees with **Calogero et al 1970s**: matrix M defined by zeros of Hermite polynomial $H_n(z)$ has eigenvalues $0, \dots, n-1$ (which are contents of $\lambda = (n)$).

Consider now the matrix

$$K_{ij}(\lambda) = \delta_{ij} \left(1 + \sum_{l \neq j} \frac{6}{(z_l - z_j)^4}\right) - (1 - \delta_{ij}) \frac{6}{(z_i - z_j)^4},$$

where $z_i = z_i(\lambda)$ are the roots of the corresponding Hermite Wronskian $W_\lambda(z)$, which are assumed to be simple. This is the Hessian matrix of $U(q) = \frac{1}{2} \sum_{i=1}^n q_i^2 + \sum_{1 \leq i < j \leq n} \frac{1}{(q_i - q_j)^2}$ at the equilibrium points $q_i = z_i$.

Consider now the matrix

$$K_{ij}(\lambda) = \delta_{ij}\left(1 + \sum_{l\neq j}\frac{6}{(z_l - z_j)^4}\right) - (1-\delta_{ij})\frac{6}{(z_i - z_j)^4},$$

where $z_i = z_i(\lambda)$ are the roots of the corresponding Hermite Wronskian $W_\lambda(z)$, which are assumed to be simple. This is the Hessian matrix of $U(q) = \frac{1}{2}\sum_{i=1}^n q_i^2 + \sum_{1\leq i<j\leq n}\frac{1}{(q_i-q_j)^2}$ at the equilibrium points $q_i = z_i$.

Felder, Veselov 2024:

$$Spec\, K(\lambda) = \{(\lambda_{l(\square)+1} - c(\square))^2, \quad \square \in \lambda\},$$

which is equivalent to a conjecture of **Conti and Masoero 2021**.

Consider now the matrix

$$K_{ij}(\lambda) = \delta_{ij}\left(1 + \sum_{l \neq j} \frac{6}{(z_l - z_j)^4}\right) - (1 - \delta_{ij})\frac{6}{(z_i - z_j)^4},$$

where $z_i = z_i(\lambda)$ are the roots of the corresponding Hermite Wronskian $W_\lambda(z)$, which are assumed to be simple. This is the Hessian matrix of $U(q) = \frac{1}{2}\sum_{i=1}^n q_i^2 + \sum_{1\le i<j\le n} \frac{1}{(q_i-q_j)^2}$ at the equilibrium points $q_i = z_i$.

Felder, Veselov 2024:

$$Spec\, K(\lambda) = \{(\lambda_{l(\Box)+1} - c(\Box))^2, \quad \Box \in \lambda\},$$

which is equivalent to a conjecture of **Conti and Masoero 2021**.

Perelomov 1978: the frequencies of small oscillations of CM system near the equilibrium given by the zeros of the Hermite polynomial $H_n(z)$ are $1, 2, \dots, n$ (which are also the exponents of the Lie algebra $u(n)$).

Concluding remarks

There are many related questions still open. Here are some of them.

Concluding remarks

There are many related questions still open. Here are some of them.

- Inverse problem for non-simple part of the harmonic locus and for the trigonometric locus for $V(z) = \sum_{i=1}^{N} m_i(m_i+1)\sin^{-2}(z-z_i)$.
- Description of the elliptic locus for $V(z) = \sum_{i=1}^{N} m_i(m_i+1)\wp(z-z_i)$.
- Description of the *monster potentials*

$$V(z) = \frac{L}{z^2} + z^{2\alpha} - 2D^2 \sum_{k=1}^{n} \log(z^{2\alpha+2} - z_k)$$

  introduced by **Bazhanov, Lukyanov and Zamolodchikov 2003.**
- Multidimensional case in relation with Hadamard problem and Huygens' principle (**Chalykh, Feigin, Veselov 1999**).

Concluding remarks

There are many related questions still open. Here are some of them.

- Inverse problem for non-simple part of the harmonic locus and for the trigonometric locus for $V(z) = \sum_{i=1}^{N} m_i(m_i+1)\sin^{-2}(z-z_i)$.
- Description of the elliptic locus for $V(z) = \sum_{i=1}^{N} m_i(m_i+1)\wp(z-z_i)$.
- Description of the *monster potentials*

$$V(z) = \frac{L}{z^2} + z^{2\alpha} - 2D^2 \sum_{k=1}^{n} \log(z^{2\alpha+2} - z_k)$$

  introduced by **Bazhanov, Lukyanov and Zamolodchikov 2003.**
- Multidimensional case in relation with Hadamard problem and Huygens' principle (**Chalykh, Feigin, Veselov 1999**).

Reference

G. Felder, A.P. Veselov *Harmonic locus and Calogero-Moser spaces.* arXiv 2404.18471