Pedestals:
Polynomial matrices with polynomial eigenvalues

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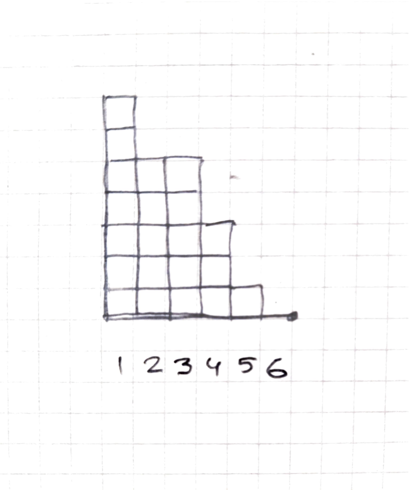
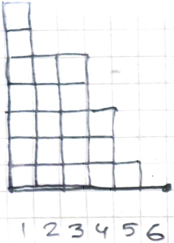
Let us write the integer k as follows:

$$k = \pi(1) + \pi(2) + \ldots + \pi(n),$$

with $\pi(i) \geq 0$, $\pi(i) \geq \pi(i+1)$. We call this function π a *partition of* k *into (at most)* n *parts*. Let $\mathcal{Y}_n$ denote the set of all partitions π-s of arbitrary integers. They are called Young diagrams with at most n columns. We call k to be the volume of the diagram π.


1 2 3 4 5 6



$$21 = 7 + 5 + 5 + 3 + 1 + 0$$

Let g_k be the number of partitions π of k. The generation function of the sequence g_k is given by:

$$G_n(t) = \prod_{l=1}^{n} \frac{1}{1 - t^l}.$$

Young diagrams

Indeed,

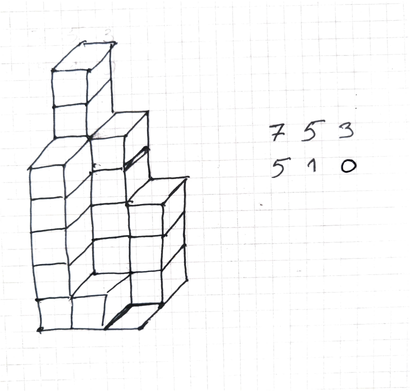
$$\prod_{l=1}^{n} \frac{1}{1-t^l} =$$

$$= (1 + t + t^2 + t^3 + ...)(1 + t^2 + t^4 + ...)(1 + t^3 + t^6 + ...)...$$

$$(1 + t^n + t^{2n} + t^{3n} + ...).$$

3D Young diagrams (Plane partitions)

Consider now the plane partitions, sitting over rectangle $n \times m$.


7 5 3
5 1 0

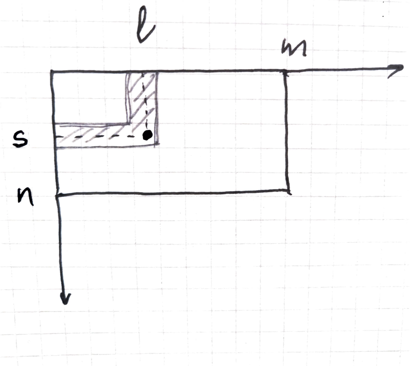
3D Young diagrams (Plane partitions)

Let now g_k be the number of plane partitions of volume k, sitting over rectangle $n \times m$. The generation function of the sequence g_k is given by the Mac-Mahon formula:

$$G_{n\times m}(t) = \prod_{l=1}^{n} \prod_{s=1}^{m} \frac{1}{1 - t^{l+s-1}}.$$

(The sum $l + s - 1$ is the hook length of the cell (l, s).)

The hook length



Young diagrams

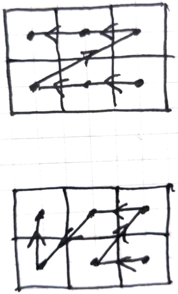
There is a natural map from the set $\Pi_{n,m}$ of plane partitions sitting over rectangle $n \times m$, onto the Young diagrams $\mathcal{Y}_{nm}$.

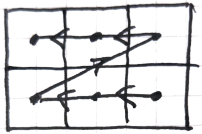
Young diagrams

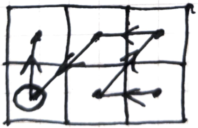
There is a natural map from the set $\mathcal{P}_{n,m}$ of plane partitions sitting over rectangle $n \times m$, onto the Young diagrams $\mathcal{Y}_{nm}$.
How about the inverse map?

Pedestals

The rectangle $n \times m$ has its natural partial order. Let us fix some linear order P on it, which extends the partial order. It is just a map from the rectangle $n \times m$ onto the segment $[1, nm]$. And let Q be any other linear order on it.
We call the node $Q^{-1}(k)$ a (P, Q)-disagreement node (or descent) iff $P\left(Q^{-1}(k-1)\right) > P\left(Q^{-1}(k)\right)$.







Pedestals

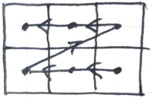
Given the pair P, Q, we define the function q_{PQ} on the rectangle $n \times m$ by

$$q_{PQ}\left(Q^{-1}(k)\right) =$$

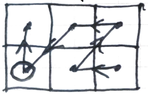
$$= \#\left\{l : l \leq k, Q^{-1}(l) \text{ is a } (P, Q)\text{-disagreement node}\right\}.$$

Clearly, the function q_{PQ} is non-decreasing on the rectangle. It is called the pedestal of Q with respect to P. Let $\mathcal{E}_P$ denotes the set of all pedestals q_{PQ}.

P



Q



q^{PQ}

1	0	0
1	0	0

Pedestals

Theorem

There is a bijection between the set $\mathcal{P}_{n,m}$ of nondecreasing functions (i.e. 3D diagrams) and the direct product $\mathcal{E}_P \times \mathcal{Y}_{nm}$, respecting the volumes.

It is given by the following construction: to each pedestal q_{PQ} and each partition π (i.e. 2D diagram) it corresponds the following function p on $n \times m$:

$$p\left(Q^{-1}(k)\right) = q_{PQ}\left(Q^{-1}(k)\right) + \pi(k), \;\; k = 1, ..., mn.$$

Clearly, the function thus defined is non-decreasing on the rectangle $n \times m$.

Pedestals

Therefore we have to fix some ordering P on the rectangle $n \times m$, consider all the pedestals q_{PQ}, and take the generating function

$$\Pi_P(t) = \sum_Q t^{v(q_{PQ})}$$

(in fact, generating polynomial) of the sequence of the number of pedestals with a given volume. Then we have the identity:

$$G_{n\times m}(t) = \Pi_P(t)\, G_{nm}(t) \equiv \Pi_P(t) \prod_{l=1}^{nm} \frac{1}{1-t^l}.$$

In particular, the polynomial $\Pi_P(t)$ does not depend on P, and thus can be denoted by $\Pi_{n\times m}(t)$.

Pedestals

In particular, we have

$$\Pi_{n\times m}(t) =$$

$$= \frac{\prod_{l=1}^{nm}(1-t^l)}{\prod_{l=1}^{n}\prod_{s=1}^{m}(1-t^{l+s-1})},$$

i.e. we see fine cancellations here.

Partition (3,2)

The standard tableaux are

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\ , \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\ , \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\ , \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\ , \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\ .$$

Pedestal matrix

$$\begin{pmatrix} 1 & q^3 & q & q^4 & q^2 \\ q^3 & 1 & q^4 & q & q^2 \\ q & q^4 & 1 & q^3 & q^2 \\ q^4 & q & q^3 & 1 & q^2 \\ q^4 & q & q^3 & q^2 & 1 \end{pmatrix}$$

Eigenvalues

$$-(-1+q)(1+q)$$
$$(-1+q)^2\left(1+q+q^2\right)$$
$$-(-1+q)(1+q)\left(1+q+q^2\right)$$
$$-(-1+q)(1+q)\left(1-q+q^2\right)$$

Pedestal polynomials

The fact that the function $\Pi_P(t)$ does not depend on the order P on our rectangle has the following generalization. Instead of characterizing the pedestal q_{PQ} just by its volume let us associate with it the monomial

$$m_{PQ}(x_1, x_2, x_3, \ldots) = x_1^{l_1-1} x_2^{l_2-l_1} \ldots x_r^{l_r-l_{r-1}} x_{r+1}^{n-l_r+1},$$

where r is the number of (P, Q)-disagreement nodes, and $l_1, \ldots, l_r$ are their locations. Note that $m_{PQ}(1, t, t^2, \ldots) = t^{v(q_{PQ})}$.

Pedestal polynomials

We have shown with Oleg Ogievetsky that the polynomial

$$\mathfrak{h}_P(x_1, x_2, x_3, \ldots) = \sum_Q m_{PQ}(x_1, x_2, x_3, \ldots)$$

is also independent of P, so it can be denoted as $\mathfrak{h}_{n\times m}(x_1, x_2, x_3, \ldots)$.

Pedestal matrices

We will go to the general case. Instead of the partially ordered set – the rectangle $n \times m$ – we will consider any finite poset X. We will denote by Tot_X the set of all possible linear orders on X. One way of expressing the property that $\mathfrak{h}_P$, $P \in \mathrm{Tot}_X$ depends only on X is to say that the matrix M_X of size $|\mathrm{Tot}_X| \times |\mathrm{Tot}_X|$, with entries $(M_X)_{PQ} = m_{PQ}(x_1, x_2, x_3, \ldots)$ is *stochastic*, i.e. the vector $(1, 1, \ldots, 1)$ is the right eigenvector, with the eigenvalue $\mathfrak{h}_X(x_1, x_2, x_3, \ldots)$. The matrix M_X is the **pedestal matrix**.

Pedestal matrices

Theorem

(Richard Kenyon, Maxim Konsevich, Oleg Ogievetsky, Cosmin Pohoata, Will Sawin, S.S.)
For every poset X, all the eigenvalues of the $|\mathrm{Tot}_X| \times |\mathrm{Tot}_X|$-matrix M_X with entries $(M_X)_{PQ} = m_{PQ}(x_1, x_2, x_3, \ldots)$ are polynomials in $x_1, x_2, x_3, \ldots$ with integer coefficients.

Partition (3,2,1):

□□□

□□

□

The 16 standard tableaux (i.e. the orders P, Q on our Young tableau) are

$$\{1,4,6,2,5,3\}\ ,\ \{1,3,6,2,5,4\}\ ,\ \{1,2,6,3,5,4\}\ ,\ \{1,3,6,2,4,5\}\ ,$$
$$\{1,2,6,3,4,5\}\ ,\ \{1,4,5,2,6,3\}\ ,\ \{1,3,5,2,6,4\}\ ,\ \{1,2,5,3,6,4\}\ ,$$
$$\{1,3,4,2,6,5\}\ ,\ \{1,2,4,3,6,5\}\ ,\ \{1,2,3,4,6,5\}\ ,\ \{1,3,5,2,4,6\}\ ,$$
$$\{1,2,5,3,4,6\}\ ,\ \{1,3,4,2,5,6\}\ ,\ \{1,2,4,3,5,6\}\ ,\ \{1,2,3,4,5,6\}\ .$$

Pedestal matrices

To save the space we write down the pedestal matrix in which the replacement

$$(x_1^6, x_1^5x_2, x_1^4x_2^2, x_1^4x_2x_3, x_1^3x_2^3, x_1^3x_2^2x_3, x_1^2x_2^4, x_1^2x_2^3x_3, x_1^2x_2^2x_3^2, x_1^2x_2^2x_3x_4) \to$$

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$$

is made.

The matrix

$$
\begin{pmatrix}
a_1 & a_5 & a_7 & a_3 & a_9 & a_2 & a_6 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\
a_5 & a_1 & a_7 & a_3 & a_9 & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\
a_5 & a_7 & a_1 & a_9 & a_3 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_8 & a_2 & a_{10} & a_4 & a_6 \\
a_5 & a_3 & a_9 & a_1 & a_7 & a_6 & a_2 & a_8 & a_4 & a_{10} & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 \\
a_5 & a_9 & a_3 & a_7 & a_1 & a_6 & a_8 & a_2 & a_{10} & a_4 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 \\
a_2 & a_6 & a_8 & a_3 & a_9 & a_1 & a_5 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\
a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_1 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\
a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_7 & a_1 & a_9 & a_3 & a_5 & a_{10} & a_4 & a_8 & a_2 & a_6 \\
a_6 & a_2 & a_8 & a_4 & a_{10} & a_5 & a_3 & a_9 & a_1 & a_7 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 \\
a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_1 & a_5 & a_9 & a_3 & a_8 & a_2 & a_6 \\
a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_5 & a_1 & a_9 & a_3 & a_8 & a_6 & a_2 \\
a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_4 & a_{10} & a_2 & a_8 & a_6 & a_1 & a_7 & a_3 & a_9 & a_5 \\
a_5 & a_9 & a_3 & a_8 & a_2 & a_6 & a_{10} & a_4 & a_8 & a_2 & a_6 & a_7 & a_1 & a_9 & a_3 & a_5 \\
a_6 & a_4 & a_{10} & a_2 & a_8 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_3 & a_9 & a_1 & a_7 & a_5 \\
a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_9 & a_2 & a_6 & a_9 & a_3 & a_7 & a_1 & a_5 \\
a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_8 & a_6 & a_2 & a_9 & a_3 & a_7 & a_5 & a_1
\end{pmatrix}
$$

The eigenvalues

$$(a_1 - a_4 - a_7 + a_{10})_3\ ,\ a_1 - a_4 + a_7 - a_{10}\ ,\ (a_1 + a_2 - a_5 - a_6)_2\ ,\ (a_1 - a_2 - a_5 + a_6)$$

$$(a_1 - a_2 - a_3 + a_4 + a_7 - a_8 - a_9 + a_{10})_2\ ,\ (a_1 - a_2 - a_3 + a_4 - a_7 + a_8 + a_9 - a_{10})_2\ ,$$

$$(a_1 - a_4 + a_5 - a_6 + a_7 - a_{10})_2\ ,\ a_1 + 2a_2 + 2a_3 + a_4 - a_7 - 2a_8 - 2a_9 - a_{10}\ ,$$

$$a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_6 + a_7 + 2a_8 + 2a_9 + a_{10}\ .$$

Pedestal matrices

The plan of the proof :

1. We introduce the class of certain matrices M_F, such that the matrix M_X can be written as a linear combination of M_F-s with integer coefficients.
2. We show that all M_F-s can be made upper-triangular via the conjugation with the **same** matrix, and the resulting upper-triangular matrices have integer entries on the diagonal.

The filters F

A filter F is a surjective map $F : X \to [1, 2, ..., k]$, $k \leq n$, such that if $\alpha_i \preccurlyeq \alpha_j$ then $F(\alpha_i) \leq F(\alpha_j)$. (For $k = n$ a filter is the same as a linear order.) For $b_1, ..., b_r$ being integers summing up to n we denote by $\mathcal{F}_{b_1,...,b_r}$ the set of all filters $F : X \to [1, 2, ..., r]$ such that $\left|F^{-1}(i)\right| = b_i$ for all $i = 1, ..., r$.

The filters F

Let P be a linear order on X and F a filter on X. We define a linear order $Q(P,F)$ by the relations:

1. for α_i, α_j in the same strata, i.e. $F(\alpha_i) = F(\alpha_j)$ we have $Q(\alpha_i) < Q(\alpha_j)$ iff $P(\alpha_i) < P(\alpha_j)$.
2. for α_i, α_j in different stratas we have $Q(\alpha_i) < Q(\alpha_j)$ iff $F(\alpha_i) < F(\alpha_j)$.

The filters F

We define the matrix M_F by

$$(M_F)_{PQ} = \begin{cases} 1 & \text{if } Q = Q(P, F) \\ 0 & \text{in all other cases} \end{cases} .$$

In particular, the matrix M_F has exactly one non-zero entry in every row.

The filter semigroup and the face semigroup

The central real hyperplane arrangement A_n (braid arrangement) of hyperplanes $\{H_{ij} : 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$ defined by

$$H_{ij} = \{(x_1, ..., x_n) : x_i = x_j\}$$

A chamber is an open connected component of

$$\mathbb{R}^n \setminus \{\cup H_{ij}\}$$

A *cone* is a union of closures of chambers, which is *convex*.
$\mathfrak{O}(n)$ is the set of all different cones.

Let a poset X of n elements be given, with a binary relation $\preccurlyeq$. To every pair $i, j \in X$ such that

$$i \preccurlyeq j$$

there corresponds a half-space $K_{ij} = \{x_i \leq x_j\} \subset \mathbb{R}^n$.
Consider the cone

$$A(X, \preccurlyeq) = \left\{ \bigcap_{i,j : i \preccurlyeq j} K_{ij} \right\} \in \mathfrak{O}(n)$$

where the intersection is taken over all pairs i, j such that $i \preccurlyeq j$.

The above defined correspondence $(X, \preccurlyeq) \to A(X, \preccurlyeq)$ is a one-to-one correspondence between the set of all partial orders on $\{1, 2, ..., n\}$ and the set of all (convex) cones $\mathfrak{O}(n)$.

$n = 4$

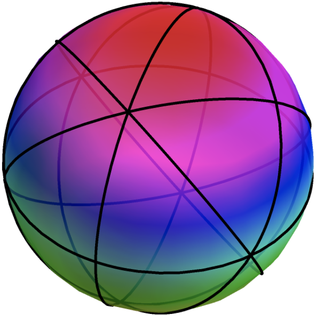


Figure: The central real hyperplane arrangement A_4 in $\mathbb{R}^4$, projected to $\mathbb{R}^3$ along the line $x = y = z = t$ and intersected with the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

$n = 4$

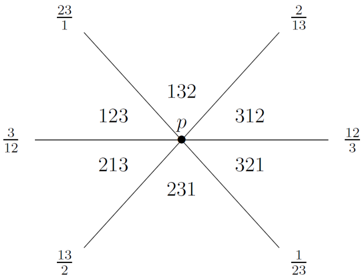
A partition of $\mathbb{S}^2$ into 24 equal triangles $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}\right)$. Convex unions are: the sphere, the hemisphere, the moon, an elementary triangle (e-triangle), a pair of e-riangles with a common side, a triangle made from three e-triangles, a ‘square’ formed by four e-triangles, a triangle made from a ‘square’ and a fifth adjacent e-triange, a triangle formed by six e-triangles with a common $\frac{\pi}{3}$-vertex. Their numbers are 1, 12, 60, 24, 36, 48, 6, 24, 8, totally 219. This is precisely the number of partial orders on the set of four distinct elements.

Face-product

Let f', f'' be two faces in $A(X) = A(X, \preccurlyeq)$. Define the face $f = f''(f') \in A(X)$ – or the *face-product* $f''f'$:
Choose points $x' \in f'$, $x'' \in f''$ in general position.
Let $s_{x'x''} : [0,1] \to \mathbb{R}^n$ be a linear segment, $s_{x'x''}(0) = x'$, $s_{x'x''}(1) = x''$.
Consider the face $f \in A(X)$ containing all the points $s_{x'x''}(1-\varepsilon)$ of the segment for $\varepsilon > 0$ small enough.
By definition, $f''(f') = f$.

Face-product

This is the picture of the projection of the braid arrangement in $\mathbb{R}^3$ to the plane orthogonal to the main diagonal:



Face-product

If f'' is a chamber then $f''f' = f''$.
If f'' is a chamber then $f'f''$ is also a chamber, so the faces are acting on chambers.

Semigroup

The face-product is associative.
For every choice of faces $f, g, h \in A(X, \preccurlyeq)$ we have

$$f(gh) = (fg)h,$$

Left-regular bands

The semigroup $A(X, \preccurlyeq)$ is a *left-regular band* :

$$ff = f,$$

$$fgf = fg.$$

The semigroup $A(X, \preccurlyeq)$ defines back the poset X.

Filters and Faces

Let F be a filter on X of rank k, i.e. a surjective map $F : X \to \{1, ..., k\}$, preserving the partial order, and let

$$\{b_1, ..., b_{j_1}\}, \{b_{j_1+1}, ..., b_{j_2}\}, ..., \{b_{j_{k-1}+1}, ..., b_{j_k}\} \subset X$$

be its 'floors':

$$\{b_{j_{r-1}+1}, ..., b_{j_r}\} = F^{-1}(r), \ r = 1, ..., k.$$

Consider the face $f_F \in A(X, \preccurlyeq)$, defined by the equations

$$x_{b_{j_{r-1}+1}} = ... = x_{b_{j_r}}, \ r = 1, ..., k$$

and inequalities

$$x_{b_{j_1}} < x_{b_{j_2}} < ... < x_{b_{j_k}}.$$

This is a one-to-one correspondence between faces and filters. The filters of the highest rank n, i.e. the linear extensions of $\preccurlyeq$, correspond to the chambers.

Filters and Faces

The corresponding filter-product looks as follows. For F', F'' being two filters of X, the filter $F = F''F'$ on X is uniquely defined by the following properties:

- For u, v with $F''(u) < F''(v)$ we have $F(u) < F(v)$.
- For u, v with $F''(u) = F''(v)$ we have $F(u) < F(v)$ iff $F'(u) < F'(v)$.

Filters and Faces

Let F be a filter on X, and P is some filter of rank n, i.e. a linear order on X. Then the filter FP is again a filter of rank n. Consider the square matrix $M_F^X = \left\| \left(M_F^X \right)_{P,Q} \right\|$ where P, Q are linear orders on X :

$$\left(M_F^X \right)_{P,Q} = \begin{cases} 1 & \text{if } Q = FP \\ 0 & \text{if } Q \neq FP \end{cases} .$$

The operators M_F^X play a central role in our proof.

The filters F – back to pedestals

Let us rewrite our pedestal matrix M_X as the sum over all monomials,

$$M_X = \sum_{r=1}^{n} \sum_{\substack{a_1,\ldots,a_r \geq 1 \\ a_1+\ldots+a_r=n}} x_1^{a_1} \ldots x_r^{a_r} B_{a_1,\ldots,a_r},$$

where the entries of each matrix $B_{a_1,\ldots,a_r}$ are 0 or 1.

The filters F

We claim that if $B_{a_1,\ldots,a_r} \neq 0$ then the following inclusion-exclusion identity holds:

$$B_{a_1,\ldots,a_r} = \sum_{F \in \mathcal{F}_{a_1,\ldots,a_r}} M_F - \left[\sum_{\substack{F \in \mathcal{F}_{a_1+a_2,a_3,\ldots,a_r} \cup \\ \cup \mathcal{F}_{a_1,a_2+a_3,\ldots,a_r} \cup \ldots}} M_F \right] + \left[\sum_{\substack{F \in \mathcal{F}_{a_1+a_2+a_3,a_4,\ldots,a_r} \cup \\ \cup \mathcal{F}_{a_1+a_2,a_3+a_4,\ldots,a_r} \cup \ldots}} M_F \right] - \ldots \tag{1}$$

where the sums are taken over all possible mergers of neighboring indices a_i, and the signs are $(-1)^{\#mergers}$.

The filters F

Indeed, an order Q from the row P from the lhs, agrees with P over the first $a_1 - 1$ locations, then disagrees once, then agrees again over next $a_2 - 1$ locations, then disagrees once again, etc. An order Q from the row P which appears in the rhs and corresponds to the first sum, agrees with P over the first $a_1 - 1$ locations, then it **agrees or disagrees** once, then agrees again over next $a_2 - 1$ locations, then **agrees or disagrees** once again, etc. Therefore we have to remove all these Q-s which agrees with P over the first $a_1 - 1$ locations, then **agrees** once again, then agrees also over next $a_2 - 1$ locations, etc.

Conjugation

Our matrices $M_{F,X}$ are of the size $|\mathrm{Tot}_X| \times |\mathrm{Tot}_X|$. Let us now abolish all order relations on X, getting the poset $\bar{X}$ with $|\mathrm{Tot}_{\bar{X}}| = n!$. Of course, $M_{F,X}$ is a submatrix of $M_{F,\bar{X}}$.
Let it be an upper-left submatrix.
To the right of it all matrix elements of $M_{F,\bar{X}}$ are zero, and so $M_{F,X}$ is a block of $M_{F,\bar{X}}$. Indeed, each row of $M_{F,\bar{X}}$ has exactly one 1, and the rest are 0-s. But each row of $M_{F,X}$ already has one 1. So it is sufficient to know that the spectrum of $M_{F,\bar{X}}$ consists of integers.

Conjugation

We will deal only with 'totally unordered' poset $\bar{X}$.

Let us consider an even bigger matrices, $M_{F,T}$, of size $2^{n(n-1)/2}$. Here T stays for *tournaments* between n entries.

A tournament is an assignment of the order $\preccurlyeq$ to every pair $i \neq j$ of the elements of the set $[1, ..., n]$, independently for each pair.

Conjugation

If we have a tournament $\preccurlyeq$ and a filter F then we define a new tournament $\preccurlyeq_F$ by the rule:

1. If $F(i) = F(j)$ then $i \preccurlyeq_F j$ iff $i \preccurlyeq j$,
2. If $F(i) < F(j)$ then $i \preccurlyeq_F j$.

Any linear order defines a tournament in an obvious way, so our matrices $M_{F,X}$ are blocks of $M_{F,T}$-s, and it is sufficient to study $M_{F,T}$-s.

Conjugation

The key observation: $M_{F,T}$ is a tensor product of $n(n-1)/2$ two-by-two matrices, corresponding to all pairs (i,j), since the orders $\preccurlyeq$ can be assigned to the pairs independently.

Since the tensor product of upper triangular matrices is upper triangular, it is sufficient to check our claim just for the filters and tournaments in the case $n = 2$.

Conjugation

The three possible $M_{F,T}$-s in this case are
$M_1 := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $M_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $M_3 := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Conjugation

Conjugating them by the discrete Fourier transform matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ brings them to the triple of upper tirangular matrices: $UM_1U^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $UM_2U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $UM_3U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. That finishes the proof.

Based on:

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