Macdonald-level extension of beta ensembles and multivariate hypergeometric polynomials

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Contents

I will describe some models of random particle systems.

The origin of these models is representation theory of infinite-dimensional classical groups, but I will focus on algebraico-combinatorial aspects of the theory.

Basic terminology

A collection X of N points on the real line $\mathbb{R}$ is called an N-particle configuration.

$\mathrm{Conf}_N(\mathbb{R})$ is the space of N-particle configurations.

An N-particle ensemble on $\mathbb{R}$ is given by a probability measure M on $\mathrm{Conf}_N(\mathbb{R})$. Then we may speak about random configurations.

Problem to be discussed (for concrete models): how to construct ensembles with ∞ many particles?

It is non-trivial, because in general it's not easy to deal with probability measures on 'big' spaces, such as $\mathrm{Conf}_\infty(\mathbb{R})$.

A possible way: use a large-N limit transition.

1 Preliminaries: Discrete beta-ensembles on $\mathbb{Z}$ of representation-theoretic origin

1.1 Dyson's circular beta-ensembles

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle with center at 0. Let $\mathrm{Conf}_N(\mathbb{T})$ denote the set of N-particle configurations $(u_1, \dots, u_N)$ on $\mathbb{T}$.

Let $\mathrm{Prob}(\cdot)$ denote the set of probability measures on a given space.

We are interested in probability measures $M_N \in \mathrm{Prob}(\mathrm{Conf}_N(\mathbb{T}))$. Given such a measure, we may speak of an ensemble of random N-particle configurations on $\mathbb{T}$.

Let $\beta > 0$ be a parameter. The N-particle Dyson's circular beta-ensemble is given by the probability measure

$$\boxed{M_{N,\beta}(du) := \frac{1}{C_{N,\beta}} \prod_{1 \le i < j \le N} |u_i - u_j|^{\beta} \cdot \mu_{\mathbb{T}^N}(du).}$$

Here $u = (u_1, \dots, u_N) \in \mathrm{Conf}_N(\mathbb{T})$, $\mu_{\mathbb{T}^N}(du)$ is the Lebesgue measure on the torus $\mathbb{T}^N = \mathbb{T}\times\cdots\times\mathbb{T}$, and $C_{N,\beta}$ is the normalization constant.

This concept is due to Dyson [J. Math. Phys. 1962].

The origin. For three special values $\beta = 1, 2, 4$ (corresponding to $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$), the Dyson ensembles admit a simple matrix/Lie group interpretation. Namely, we consider three infinite series $G(N)/K(N)$ of compact symmetric spaces

$$U(N)/O(N), \quad \big(U(N)\times U(N)\big)/\operatorname{diag} U(N), \quad U(2N)/Sp(N).$$

Consider the double cosets $K(N)gK(N)$, $g \in G(N)$. In the case $\beta = 2$, the double cosets are the same as conjugation clases in $U(N)$.

In all three cases, the double cosets are parametrized by configurations $u = (u_1, \dots, u_N) \in \mathrm{Conf}_N(\mathbb{T})$.

So, we have a natural projection

$$G(N) \to K(N)\setminus G(N)/K(N) = \mathrm{Conf}_N(\mathbb{T}).$$

It turns out that the pushforward, under this projection, of the normalized Haar measure is just $M_{N,\beta}$ for $\beta = 1, 2, 4$ (a particular case of Elie Cartan's formula).

For general $\beta > 0$ — extrapolation.

1.2 Dual picture: problem of harmonic analysis for ∞-dimensional symmetric spaces

Here 'dual picture' means that instead of a compact symmetric space $G(N)/K(N)$ we consider the Hilbert space $L^2\big(G(N)/K(N)\big)$ and the natural unitary representation T_N of the group $G(N)$ on this space.

Its decomposition is well known: this is an example of a (relatively simple) problem of spherical noncommutative harmonic analysis.

Question ($\beta = 1, 2, 4$):

(i) Is it possible to give a sense to the large-N limit

$$T_\infty := \lim_{N\to\infty} T_N$$

as a unitary representation of the direct limit group $G(\infty) := \bigcup_N G(N)$?

(ii) How to decompose T_∞ into irreducibles (harmonic analysis)?

The question is nontrivial, because there is no invariant measure on $G(\infty)/K(\infty)$, so one cannot extend the definition of $L^2(G(N)/K(N))$ directly.

Answer:

1) Yes, the limit representation T_∞ can be defined. It turns out that its construction involves additional continuous parameters (which is not a defect but a bonus!).

2) The decomposition of T_∞ into irreducibles is governed by an ensemble with infinitely many particles on $\mathbb{R}$.

3) This ensemble in turn is obtained as a large-N limit of certain discrete ensembles on the lattice $\mathbb{Z}$, which look as a discrete analog of Dyson beta ensembles.

4) Moreover, the whole construction admits a purely combinatorial interpretation which makes sense for all $\beta > 0$.

Thus, we can convert our problem of harmonic analysis for infinite-dimensional symmetric spaces into a problem of algebraic combinatorics with a slight probabilistic flavor.

1.3 Discrete beta-ensembles on the lattice $\mathbb{Z}$

In what follows we will assume

$$\boxed{\beta = 2\tau, \quad \tau \in \{1, 2, 3, \dots\}}$$

so that β will be a positive even integer. It's for the sake of simplicity only; the results hold for any $\beta > 0$.

The discrete ensembles in question live on the lattice $\mathbb{Z}$ (which is the dual to the circle $\mathbb{T}$!). So, instead of the continuous space $\operatorname{Conf}_N(\mathbb{T})$, we are dealing with the countable set $\operatorname{Conf}_N(\mathbb{Z})$. Its elements are N-particle configurations on $\mathbb{Z}$:

$$\mathscr{L} = (\ell_1 > \dots > \ell_N) \subset \mathbb{Z}.$$

These are in fact veiled highest weights $\lambda = (\lambda_1 \ge \dots \ge \lambda_N)$:

$$\lambda \to \mathscr{L}, \qquad \ell_i = \lambda_i + (N - i)\tau, \quad 1 \le i \le N.$$

I am going to introduce a probability measure

$$M_{N,2\tau}^{z,z',w,w'} \in \operatorname{Prob}\big(\operatorname{Conf}_N(\mathbb{Z})\big), \qquad \tau \in \{1,2,3,\dots\}.$$

Here (z, z', w, w') is a quadruple of continuous parameters subject to some constraints. For instance, sufficient conditions are

$$z, w \in \mathbb{C}, \quad z' = \overline{z}, \quad w' = \overline{w}, \quad \operatorname{Re}(z+w) > -\frac{1}{2}.$$

Definition (Probability measure on $\operatorname{Conf}_N(\mathbb{Z})$). We define the weight of a configuration $\mathscr{L} = (\ell_1, \dots, \ell_N) \in \operatorname{Conf}_N(\mathbb{Z})$ as

$$\boxed{M_{N,2\tau}^{z,z',w,w'}(\mathscr{L}) := \frac{1}{C_{N,2\tau}} \prod_{i=1}^{N} F_N(\ell_i) \cdot V_{N,2\tau}(\mathscr{L}),}$$

Here

$$F_N(\ell) := \frac{\Gamma(-z-(N-1)\tau+\ell)\Gamma(-z'-(N-1)\tau+\ell)}{\Gamma(w+\ell+1)\Gamma(w'+\ell+1)}, \quad \ell \in \mathbb{Z},$$

and

$$V_{N,2\tau}(\mathscr{L}) := \prod_{1\le i<j\le N} (\ell_i - \ell_j)^2$$

$$\times \left| \prod_{1\le i\neq j\le N} (\ell_i - \ell_j - 1)(\ell_i - \ell_j - 2)\dots(\ell_i - \ell_j - (\tau - 1)) \right|$$

Remark. 1) For $\tau = 1$, the second double product disappears.

2) If $\tau > 1$, $V_{N,2\tau}(\mathscr{L})$ vanishes whenever $\ell_i - \ell_{i+1} < \tau - 1$. Thus, the measure $M_{N,2\tau}^{z,z',w,w'}$ lives on the subset of 'τ-sparse configurations': any two particles are separated by at least $\tau - 1$ holes.

3) For large distances between the ℓ_i's,

$$V_{N,2\tau}(\mathscr{L}) \approx \prod_{i<j} (\ell_i - \ell_j)^{2\tau}.$$

4) As in Dyson's context, $V_{N,2\tau}(\mathscr{L})$ is responsible for pair interaction between the particles, of 'log-gas type'. Only now we have a lattice model.

Theorem. *Let the parameters τ and z, z', w, w' be fixed. After a scaling and yet another transformation, the measures $M_{N,2\tau}^{z,z',w,w'}$ converge, as $N \to \infty$, to a probability measure $M_{\infty,2\tau}^{z,z',w,w'}$ that lives on a space of infinite particle configurations on the real line $\mathbb{R}$.*

References: Borodin-Olshanski [Ann. Math. 2005], Olshanski [J. Funct. Anal. 2003] and [Funct. Anal. Appl. 2003].

2 Macdonald-level hypergeometric ensembles

2.1 Notation

• q and t are the two parameters of Macdonald polynomials. We assume:

$$0 < q < 1, \qquad 0 < t < 1, \qquad t = q^{\tau}, \quad \tau = 1, 2, 3, \dots$$

The last assumption is for simplicity only.

• Two additional parameters $\zeta_\pm$:

$$\zeta_- < 0 < \zeta_+.$$

• The two-sided q-lattice $\mathbb{L} = \mathbb{L}_- \cup \mathbb{L}_+ \subset \mathbb{R}$:

$$\mathbb{L}_- := \zeta_- q^{\mathbb{Z}} = \{\zeta_- q^m : m \in \mathbb{Z}\}, \quad \mathbb{L}_+ := \zeta_+ q^{\mathbb{Z}} = \{\zeta_+ q^m ; m \in \mathbb{Z}\}.$$

Because $0 < q < 1$, the lattice nodes accumulate near 0 and diverge in the direction of $\pm\infty$:

$$\dots \quad \zeta_- q^{-1} \qquad \zeta_- \qquad \zeta_- q \quad \dots \qquad \dots \quad \zeta_+ q \qquad \zeta_+ \qquad \zeta q \quad \dots$$

• $\mathrm{Conf}_N(\mathbb{L})$ is the set of N-particle configurations on $\mathbb{L}$, which are τ-sparse (as before, this means that any two particles are separated by at least $\tau - 1$ holes).

2.2 N-particle hypergeometric ensembles

Below we use a standard notation from q-calculus:

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n), \quad x \in \mathbb{C}.$$

It is closely related to the notion of q-Gamma function:

$$\Gamma_q(A) := \frac{(q;q)_\infty}{(q^A;q)_\infty}(1-q)^{1-A}.$$

Definition. We fix a quadruple $(\alpha,\beta,\gamma,\delta)$ is of parameters subject to conditions specified below.

The following formula determines a 'hypergeometric' probability measure $M_{N;q,t}^{\alpha,\beta,\gamma,\delta}$ on the set $\operatorname{Conf}_N(\mathbb{L})$.

If $X=(x_1>\cdots>x_N)\in\operatorname{Conf}_N(\mathbb{L})$, then

$$\boxed{M_{N;q,t}^{\alpha,\beta,\gamma,\delta}(X) := \frac{1}{C(N;q,t;\alpha,\beta,\gamma,\delta)} \cdot \prod_{i=1}^{N} F_{N;q,t}^{\alpha,\beta,\gamma,\delta}(x_i) \cdot V_{N;q,t}(X),}$$

where $C(N;q,t;\alpha,\beta,\gamma,\delta)$ is a normalization constant;

$$\boxed{V_{N,q,t}(X) := \prod_{1\le i\ne j\le N} \prod_{r=0}^{\tau-1} |x_i - x_j q^r|}$$

is a (q,t)-analog of $\prod_{i<j}(x_i - x_j)^{2\tau}$ (or rather its version on $\mathbb{Z}$);

$$\boxed{F_{N;q,t}^{\alpha,\beta,\gamma,\delta}(x) := (1-q)|x| \frac{(\alpha x;q)_\infty (\beta x;q)_\infty}{(\gamma t^{1-N} x;q)_\infty (\delta t^{1-N} x;q)_\infty}, \quad x \in \mathbb{L},}$$

is a (q,t)-analog of the previously defined function

$$F_N(\ell) = \frac{\Gamma(-z-(N-1)\tau+\ell)\Gamma(-z'-(N-1)\tau+\ell)}{\Gamma(w+\ell+1)\Gamma(w'+\ell+1)}, \quad \ell \in \mathbb{Z}.$$

The parameters $(\alpha,\beta,\gamma,\delta)$ should be such that for each N:

- $F_{N;q,t}^{\alpha,\beta,\gamma,\delta}(x) \ge 0$ for any $x \in \mathbb{L}$,
- the normalization is possible.

2.3 Conditions on $(\alpha,\beta,\gamma,\delta)$: principal and degenerate series

There are two variants of sufficient conditions on $(\alpha,\beta,\gamma,\delta)$ which guarantee that the hypergeometric measures are well-defined for all N:

1) Principal series:

$$\alpha=\overline{\beta}\in\mathbb{C}\setminus\mathbb{R},\quad \gamma=\overline{\delta}\in\mathbb{C}\setminus\mathbb{R},\quad \alpha\beta<\gamma\delta q.$$

Then $M^{\alpha,\beta,\gamma,\delta}_{N;q,t}(X)>0$ for all $X\in\operatorname{Conf}_N(\mathbb{L})$.

2) Degenerate series:

$$\beta<0<\alpha,\quad \alpha^{-1}\in\mathbb{L}_+,\quad \beta^{-1}\in\mathbb{L}_-,\qquad \gamma=\overline{\delta}\in\mathbb{C}\setminus\mathbb{R}.$$

Then $M^{\alpha,\beta,\gamma,\delta}_{N;q,t}(X)>0$ only for configurations X contained in the truncated lattice

$$\mathbb{L}[\beta^{-1}q,\,\alpha^{-1}q]:=\{x\in\mathbb{L}:\beta^{-1}q\le x\le\alpha^{-1}q\}.$$

This happens because the conditions $\alpha^{-1} \in \mathbb{L}_+$ and $\beta^{-1} \in \mathbb{L}_-$ imply that the product $(\alpha x; q)_\infty (\beta x; q)_\infty$ vanishes for all $x \in \mathbb{L}$ outside the lattice interval $[\beta^{-1} q,\, \alpha^{-1} q]$.

2.4 Large-N limit

Let $\mathrm{Conf}_\infty(\mathbb{L})$ denote the set of particle configurations $X \subset \mathbb{L}$ such that:

- $|X| = \infty$.
- X is bounded away from $\pm\infty$.
- If $\tau > 1$, then X is τ-sparse.

Note that the spaces $\mathrm{Conf}_N(\mathbb{L})$ are countable while the space $\mathrm{Conf}_\infty(\mathbb{L})$ has the power of continuum. It is a totally disconnected topological space.

Theorem (Main result). *Let $(\alpha, \beta, \gamma, \delta)$ be in the principal or degenerate series. We still assume $t = q^\tau$ with $\tau \in \{1, 2, 3, \dots\}$. There* *exists a limit*

$$\lim_{N\to\infty} M^{\alpha,\beta,\gamma,\delta}_{N;q,t} = M^{\alpha,\beta,\gamma,\delta}_{\infty;q,t} \in \mathrm{Prob}\big(\,\mathrm{Conf}_\infty(\mathbb{L})\big).$$

In other words, there exists a limit probability measure which determines a particle ensemble on the two-sided q-lattice $\mathbb{L}$, with infinitely many particles accumulating at $0 \notin \mathbb{L}$.

A similar claim holds in fact for any $t \in (0,1)$. Only the description of the configurations becomes more involved.

References: Olshanski [Selecta Math. 2021] and [Comm. Math. Phys. 2021].

2.5 The special case $\tau = 1$ (that is, $q = t$)

In this case one can say more.

Theorem. *If $q = t$, then the limit ensemble on $\mathbb{L}$ given by the measure $M^{\alpha,\beta,\gamma,\delta}_{\infty;q,q}$ is determinantal, meaning that its correlation functions*

ρ_n, $n = 1, 2, \dots$, *are given by a determinantal expression of the form*

$$\rho_n(x_1, \dots, x_n) = \det[K_q^{\alpha,\beta,\gamma,\delta}(x_i, x_j)]_{i,j=1}^n,$$

where $K_q^{\alpha,\beta,\gamma,\delta}(x, y)$ *is a kernel on* $\mathbb{L} \times \mathbb{L}$*, not depending on* n*.*

This kernel admits an explicit expression of the form

$$K_q^{\alpha,\beta,\gamma,\delta}(x, y) = \frac{A(x)B(y) - A(y)B(x)}{x - y}, \quad x, y \in \mathbb{L}.$$

Here $A(x)$ *and* $B(x)$ *are certain functions on* $\mathbb{L}$*, expressed through the* q*-hypergeometric function* ${}_2\phi_1$*.*

Because the correlation functions are explicitly computable, one can say that the model with $q = t$ is exactly solvable.

Theorem. *The limit measure* $M_{\infty;q,q}^{\alpha,\beta,\gamma,\delta}$ *is diffuse, meaning that it has no atoms.*

References: Gorin-Olshanski [J. Funct. Anal. 2016], Cuenca-Gorin-Olshanski [IMRN 2021].

2.6 Degeneration $M_{N;q,t}^{\alpha,\beta,\gamma,\delta} \rightsquigarrow M_{N;2\tau}^{z,z',w,w'}$

Recall that we started with the lattice $\mathbb{Z}$ and then proceeded to the two-sided q-lattice $\mathbb{L}$. Our formulas in these two cases have a similarity.

Moreover, for each fixed N, the ensemble on $\mathbb{L}$ can be degenerated to the ensemble on $\mathbb{Z}$.

Namely, assume for simplicity $\zeta_\pm = \pm 1$, so that $\mathbb{L} = -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$.
There is a natural bijection $\mathbb{Z} \leftrightarrow q^{\mathbb{Z}}$: $\ell \leftrightarrow q^\ell = x$.
Likewise, we have a natural bijection

$$\operatorname{Conf}_N(\mathbb{Z}) \leftrightarrow \operatorname{Conf}_N(\mathbb{L}_+), \quad \mathscr{L} \leftrightarrow X$$
$$\mathscr{L} = (\ell_1 > \cdots > \ell_N) \leftrightarrow (q^{\ell_1} < \cdots < q^{\ell_N}) = X.$$

Fix a quadruple (z, z', w, w') from the principal series and set

$$\alpha = q^{w+1}, \quad \beta = q^{w'+1}, \quad \gamma = q^{-z}, \quad \delta = q^{-z'}.$$

We consider the limit regime in which $q \nearrow 1$. Then the q-lattice $\mathbb{L} \subset \mathbb{R}$ becomes more and more dense.

Theorem. *In this limit regime, the random configurations on* $\mathbb{L}$ *governed by* $M_{N;q,t}^{\alpha,\beta,\gamma,\delta}$ *tend to concentrate near the point* 1*. In particular, they tend to run away from the negative part* $\mathbb{L}_-$ *of the lattice.*

More precisely, for any fixed small $\varepsilon > 0$

$$\lim_{q \nearrow 1} \sum_{X \subset (1-\varepsilon, 1+\varepsilon)} M_{N;q,t}^{\alpha,\beta,\gamma,\delta}(X) = 1.$$

Next, for any $\mathscr{L} \in \mathrm{Conf}_N(\mathbb{Z})$ *we have*

$$\lim_{q \nearrow 1} M_{N;q,t}^{\alpha,\beta,\gamma,\delta}(q^{\mathscr{L}}) = M_{N,2\tau}^{z,z',w,w'}(\mathscr{L}).$$

Remark. The point is that, in this limit regime, the negative part of the q-lattice becomes negligible. However, in the framework of our approach one cannot build a (q,t)-version of discrete beta-ensembles solely on $\mathbb{L}_+$. The two-sided lattice $\mathbb{L}$ seems to be absolutely necessary.

It would be interesting to find a representation-theoretic interpretation of our construction, at least for the case $q=t$. A natural suggestion would be to work with representations of the quantized algebras $\mathcal{U}_q(\mathfrak{gl}(N),\mathbb{C})$. However, I don't see how to reconcile this algebra with the two-sided lattice $\mathbb{L}$.

Reference: Olshanski [Comm.Math.Phys. 2021].

3 Big q-Jacobi symmetric functions

3.1 Big q-Jacobi symmetric polynomials

We focus on the degenerate series of parameters $(\alpha,\beta,\gamma,\delta)$:

$$\beta<0<\alpha, \qquad \gamma=\overline{\delta}\in\mathbb{C}\setminus\mathbb{R},$$

and deal with the truncated q-lattice

$$\mathbb{L}^{\alpha,\beta}:=\mathbb{L}[\beta^{-1}q,\ \alpha^{-1}q]=\beta^{-1}q^{\mathbb{Z}_{\geq 1}}\cup\alpha^{-1}q^{\mathbb{Z}_{\geq 1}}.$$

Let $\mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta})$ denote the set of N-particle configurations on this lattice. As above, for $\tau > 1$ we additionally suppose that the configurations are τ-sparse.

Let

$$\mathrm{Sym}(N) := \mathbb{R}[x_1, \dots, x_N]^{S_N}$$

denote the $\mathbb{R}$-algebra of symmetric polynomials with N variables.

There is a natural embedding

$$\begin{aligned} \mathrm{Sym}(N) &\xrightarrow{\iota} \text{bounded functions on } \mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta}) \\ f &\mapsto f(X), \quad X = (x_1, \dots, x_N) \in \mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta}). \end{aligned}$$

Recall that for each N we have defined a hypergeometric probability measure $M_{N;q,t}^{\alpha,\beta,\gamma,\delta}$ on $\mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta})$. Using the embedding ι we can realize $\mathrm{Sym}(N)$ as a dense subspace of the Hilbert space $\ell^2\big(\mathrm{Conf}_N(\mathbb{L}^{\alpha.\beta}), M_{N;q,t}^{\alpha,\beta,\gamma,\delta}\big)$. Let $(-,-)$ be the induced scalar product in $\mathrm{Sym}(N)$.

Theorem (Stokman). *Let λ range over the set $\mathbb{Y}(N)$ of partitions of* *length $\leq N$ and let $P_{\lambda|N}(X;q,t)$ denote the N-variate Macdonald polynomial indexed by λ.*

There exists a basis $\{\varphi_{\lambda|N}(X) : \lambda \in \mathbb{Y}(N)\}$ in $\mathrm{Sym}(N)$ such that

$$\varphi_{\lambda|N}(X) = P_{\lambda|N}(X) + \text{lower degree terms}$$

and

$$(\varphi_{\lambda|N}, \varphi_{\mu|N}) = 0, \quad \lambda \neq \mu.$$

References: Stokman [SIAM J. Math. Anal. 1997], Stokman-Koornwinder [Canad. J. Math. 1997].

The polynomials $\varphi_{\lambda|N}$ are called the N-variate symmetric big q-Jacobi polynomials. In the simplest case $N = 1$ these are the classic univariate big q-Jacobi polynomials discovered by Andrews and Askey [In: Lecture Notes in Math. vol. 1171, 1984].

3.2 ‘Almost-stable" expansion on Macdonald polynomials

Theorem. *The expansion of N-variate big q-Jacobi polynomials in the basis of Macdonald polynomials has the form*

$$\varphi_{\lambda|N} = \sum_{\mu:\, \mu\subseteq\lambda} \frac{(t^N;q,t)_\lambda}{(t^N;q,t)_\mu}\, \pi(\lambda,\mu;q,t;\alpha,\beta,\gamma,\delta)\, P_{\mu|N},$$

where

$$(t^N;q,t)_\lambda := \prod_{i=1}^{l(\lambda)} (t^{N+1-i};q)_{\lambda_i} = \prod_{(i,j)\in\lambda} (1-q^{\lambda_i+j-1}t^{N+1-i}),$$

$$(t^N;q,t)_\mu := \prod_{i=1}^{l(\mu)} (t^{N+1-i};q)_{\mu_i} = \prod_{(i,j)\in\mu} (1-q^{\mu_i+j-1}t^{N+1-i})$$

are certain products of q-Pochhammer factors and $\pi(\lambda,\mu;q,t;\alpha,\beta,\gamma,\delta)$ are certain coefficients, which do not depend on N and admit an explicit expression.

We call this expansion almost stable, because the dependence on N is localized in the fraction

$$\frac{(t^N;q,t)_\lambda}{(t^N;q,t)_\mu}$$

The proof (Olshanski [Comm. Math. Phys. 2021]) relies on results of Rains [Transf. Groups 2005] and the theory of interpolation Macdonald polynomials due to Okounkov, Knop, and Sahi.

3.3 Big q-Jacobi symmetric functions

Let

$$\mathbb{Y} = \bigcup_N \mathbb{Y}(N)$$

denote the set of all partitions (=Young diagrams) and let

$$\mathrm{Sym} := \varprojlim \mathrm{Sym}(N)$$

denote the $\mathbb{R}$-algebra of symmetric functions.

Recall the embedding

$$\mathrm{Sym}(N) \xrightarrow{\iota} \text{bounded functions on } \mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta}),$$
$$f \mapsto f(X), \quad X = (x_1, \ldots, x_N) \in \mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta}).$$

Likewise, we have a natural embedding

$$\mathrm{Sym} \xrightarrow{\iota} \text{bounded functions on } \mathrm{Conf}_\infty(\mathbb{L}^{\alpha,\beta}),$$
$$F \mapsto F(X), \quad X = \{x_i\} \in \mathrm{Conf}_\infty(\mathbb{L}^{\alpha,\beta}),$$

where $\mathrm{Conf}_\infty(\mathbb{L}^{\alpha,\beta})$ is the space of ∞-particle τ-sparse configurations on the truncated q-lattice $\mathbb{L}^{\alpha,\beta}$.

We can regard Sym as an algebra of bounded functions on the (totally disconnected topological) space $\mathrm{Conf}_\infty(\mathbb{L}^{\alpha,\beta})$.

Note that for fixed partitions $\lambda, \mu \in \mathbb{Y}$,

$$\lim_{N\to\infty} (t^N; q, t)_\lambda = 1, \qquad \lim_{N\to\infty} (t^N; q, t)_\mu = 1.$$

It follows that, as $N \to \infty$, the N-variate big q-Jacobi polynomials converge, in a natural sense, to certain symmetric functions

$$\Phi_\lambda = \Phi(-; q, t; \alpha, \beta, \gamma, \delta) \\ = \sum_{\mu\colon \mu \subseteq \lambda} \pi(\lambda, \mu; q, t; \alpha, \beta, \gamma, \delta)\, P_\mu(-; q, t),$$

where the $P_\mu(-; q, t)$, $\mu \in \mathbb{Y}$, are the Macdonald symmetric functions.

We call the functions $\Phi(-; q, t; \alpha, \beta, \gamma, \delta)$ the big q-Jacobi symmetric functions. We regard them as bounded functions on $\mathrm{Conf}_\infty(\mathbb{L}^{\alpha,\beta})$.

By Stokman's theorem, the measures $M^{\alpha,\beta,\gamma,\delta}_{N;q,t}$ are the orthogonality measures for the N-variate big q-orthogonal polynomials $\varphi_{\lambda|N}$. The next result is its analog in the context of symmetric functions.

Theorem. *The limit measure on infinite configurations in the truncated lattice,*

$$M^{\alpha,\beta,\gamma,\delta}_{\infty;\,q,t} = \lim_{N \to \infty} M^{\alpha,\beta,\gamma,\delta}_{N;\,q,t} \in \mathrm{Prob}(\mathrm{Conf}_\infty(\mathbb{L}^{\alpha,\delta})),$$

is the orthogonality measure for the symmetric functions $\Phi(-;q,t;\alpha,\beta,\gamma,\delta)$*.*

That is, the big q*-Jacobi symmetric functions form an orthogonal basis in the Hilbert space*

$$L^2\big(\operatorname{Conf}_\infty(\mathbb{L}^{\alpha,\beta}),\, M^{\alpha,\beta,\gamma,\delta}_{\infty;\,q,t}\big).$$

This theorem is related to the general idea of constructing analogs of various systems of classical orthogonal polynomials in the algebra Sym of symmetric functions.

Other results in this direction: Cuenca-Olshanski [Mosc.Math.J., 2020]. There we show that part of the q-Askey scheme can be transferred into Sym.

In a different form, the idea of lifting N-variate analogs of orthogonal polynomials to the algebra Sym is present in earlier papers by Rains [Transf. Groups, 2005], Sergeev-Veselov [Adv. Math. 2009], Desrosiers-Hallnäs [SIGMA, 2012].

4 Stochastic links connecting N-particle ensembles with varying $N = 1, 2, 3, \dots$

4.1 Sketch of abstract formalism

By a stochastic link $\Lambda : \Omega \dashrightarrow \Omega'$ between two spaces Ω and Ω' we mean a Markov kernel $\Lambda(x, dy)$ on $\Omega \times \Omega'$. This means that for any fixed $x \in \Omega$, $\Lambda(x, -)$ is a probability measure on Ω'.

In particular, if both spaces are discrete, then Λ is simply a stochastic matrix of format $\Omega \times \Omega'$: its entries $\Lambda(x, y)$ are non-negative and all row sums equal 1.

A stochastic link $\Lambda : \Omega \dashrightarrow \Omega'$ can be regarded as a kind of generalized map; the difference with conventional maps is that the image of a point is not a single point but a probability distrubution. Like an ordinary map, Λ gives rise to a map of probability measures:

$$\mathrm{Prob}(\Omega) \to \mathrm{Prob}(\Omega'), \qquad M \mapsto M\Lambda.$$

Namely, if $M \in \mathrm{Prob}(\Omega)$, then

$$(M\Lambda)(dy) = \int_{x\in\Omega} M(dx)\Lambda(x,dy).$$

In the case of discrete spaces, this is simply the operation of multiplying a row vector by a matrix .

Let us consider the category whose objects are (sufficiently good) spaces and morphisms are stochastic links. Assume now that we are given an infinite chain of spaces connected by stochastic links $\Lambda^N_{N-1} : \Omega_N \dashrightarrow \Omega_{N-1}$,

$$\Omega_1 \overset{\Lambda^2_1}{\dashleftarrow} \Omega_2 \overset{\Lambda^3_2}{\dashleftarrow} \dots \overset{\Lambda^{N-1}_{N-2}}{\dashleftarrow} \Omega_{N-1} \overset{\Lambda^N_{N-1}}{\dashleftarrow} \Omega_N \overset{\Lambda^{N+1}_N}{\dashleftarrow} \dots$$

Under suitable assumptions, one can prove the existence of a kind of projective limit

$$\Omega_\infty = \varprojlim(\Omega_N, \Lambda^N_{N-1}).$$

Definition. Assume that each Ω_N is equipped with a probability measure $M_N \in \mathrm{Prob}(\Omega_N)$. We say that $\{M_N\}$ is a coherent family of measures if

$$M_N \Lambda^N_{N-1} = M_{N-1}, \quad \forall N \geq 2.$$

Theorem. *There is a 1-1 correspondence between coherent families* $\{M_N\}$ *of measures and probability measures* $M_\infty \in \mathrm{Prob}(\Omega_\infty)$*.*

Thus, any coherent family on a chain $\{\Omega_N, \Lambda^N_{N-1}\}$ gives rise to a probability measure $M_\infty \in \mathrm{Prob}(\Omega_\infty)$.

In this sense $(\Omega_\infty, M_\infty)$ serves as a large-N limit of the probability spaces $(\Omega_N, M_N))$.

4.2 Stochastic links $\Lambda^N_{N-1} : \mathrm{Conf}_N(\mathbb{L}) \dashrightarrow \mathrm{Conf}_{N-1}(\mathbb{L})$

Return to our setting: we take as Ω_N the set $\mathrm{Conf}_N(\mathbb{L})$ of τ-sparse N-particle configurations on $\mathbb{L}$. We are going to define stochastic links between these sets.

Theorem. *For each $N \geq 2$ there exists a stochastic matrix $\Lambda^N_{N-1}(X,Y)$ of format $\mathrm{Conf}_N(\mathbb{L}) \times \mathrm{Conf}_{N-1}(\mathbb{L})$, which is consistent with the*

Macdonald polynomials in the sense that

$$\sum_{Y\in\mathrm{Conf}_{N-1}(\mathbb{L})} \Lambda^N_{N-1}(X,Y)\frac{P_{\lambda|N-1}(Y;q,t)}{(t^{N-1};q,t)_\lambda} = \frac{P_{\lambda|N}(X;q,t)}{(t^N;q,t)_\lambda}$$

for any $X \in \mathrm{Conf}_N(\mathbb{L})$ *and any* $\lambda \in \mathbb{Y}(N-1)$.

Furthermore, the entries $\Lambda^N_{N-1}(X,Y)$ *are nonzero iff the configurations* X *and* Y *interlace in a certain sense.*

Such a matrix is unique.

4.3 Identification of the projective limit space

Theorem. *The projective limit of the sequence of sets*

$$\mathrm{Conf}_1(\mathbb{L}) \dashleftarrow \mathrm{Conf}_2(\mathbb{L}) \dashleftarrow \mathrm{Conf}_3(\mathbb{L}) \dashleftarrow \dots$$

connected by the stochastic links Λ^N_{N-1} *can be identified, in a natural way, with the space* $\mathrm{Conf}_\infty(\mathbb{L})$ *of infinite* τ*-sparse particle configurations on* $\mathbb{L}$.

Reference: Olshanski [Selecta Math. 2021].

The theorem says that $\mathrm{Conf}_\infty(\mathbb{L})$ is the universal object with the property that there are stochastic links

$$\Lambda^\infty_N : \mathrm{Conf}_\infty(\mathbb{L}) \dashrightarrow \mathrm{Conf}_N(\mathbb{L}), \qquad \forall N \ge 1,$$

such that

$$\Lambda^\infty_N \Lambda^N_{N-1} = \Lambda^\infty_{N-1}, \quad \forall N \ge 2.$$

4.4 The coherency relation

Fix a quadruple $(\alpha, \beta, \gamma, \delta)$ from the principal or degenerate series and consider the corresponding hypergeometric measures $M^{\alpha,\beta,\gamma,\delta}_{N;\,q,t} \in \mathrm{Prob}\,(\mathrm{Conf}_N(\mathbb{L}))$.

Theorem. *These measures satisfy the coherency relation:*

$$M^{\alpha,\beta,\gamma,\delta}_{N;\,q,t} \Lambda^N_{N-1} = M^{\alpha,\beta,\gamma,\delta}_{N-1;\,q,t}, \quad N \ge 2.$$

When written explicitly, this becomes a nontrivial combinatorial summation formula: for each $Y \in \mathrm{Conf}_{N-1}(\mathbb{L})$,

$$\sum_{X \in \mathrm{Conf}_N(\mathbb{L})} M^{\alpha,\beta,\gamma,\delta}_{N;\,q,t}(X)\Lambda^N_{N-1}(X,Y) = M^{\alpha,\beta,\gamma,\delta}_{N-1;\,q,t}(Y).$$

This formula is proved first for the degenerate series, by using the big q-Jacobi polynomials. Next, the result is extended to the principal series by analytic continuation.

This result, combined with the abstract formalism, leads to the main theorem: the existence of the large-N limit measure $M^{\alpha,\beta,\gamma,\delta}_{\infty;\,q,t}$.