Krichever tau-function: old and new

Andrei Marshakov

I.Krichever CAS, Skoltech

Integrable Systems and Algebraic Geometry, BIMSA

June-July 2024

Plan

- Old story:
  - Definition: (**I.M.Krichever, The τ-function of the universal Whitham hierarchy, matrix models and topological field theories**, **arXiv:hep-th/9205110**)
    - Actually $\mathcal{F} = \log \tau$;
    - terminology ...
  - Matrix models & Topological strings;
  - Seiberg-Witten theory and integrable systems;
- Peculiarities:
  - Residue formula;
  - WDVV equations;
- New story:
  - Nekrasov functions, 2d conformal theories and isomonodromic deformations;
  - 2d gravity and Verlinde formula;

Notations:

- Topology of compact oriented Riemann surface: genus g, an example with $g = 3$:

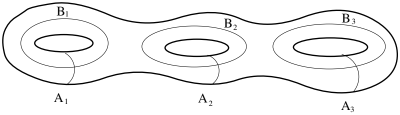

Figure: Riemann surface Σ of genus $g = 3$ with chosen basis of A and B cycles.

- $\dim H_1(\Sigma_g) = 2g$, (symplectic) intersection form $A_\alpha \circ B_\beta = \delta_{\alpha\beta}$.
- Dual basis in $H^1(\Sigma_g)$ of *holomorphic* first kind Abelian differentials $\bar\partial(d\omega_\alpha) = 0$, normalized to the A-cycles

$$\oint_{A_\beta} d\omega_\alpha = \delta_{\alpha\beta}$$

Period matrix

$$\oint_{B_\alpha} d\omega_\beta = T_{\alpha\beta}$$

Riemann Bilinear Relations

- Period matrix is symmetric due to

$$0 = \int_\Sigma d\omega_\beta \wedge d\omega_\gamma = \int_{\partial\Sigma} \omega_\beta d\omega_\gamma =$$

$$= \sum_\alpha \left(\oint_{A_\alpha} d\omega_\beta \oint_{B_\alpha} d\omega_\gamma - \oint_{A_\alpha} d\omega_\gamma \oint_{B_\alpha} d\omega_\beta \right) = T_{\beta\gamma} - T_{\gamma\beta}$$

- Proof from the Stokes theorem on cut Riemann surface:

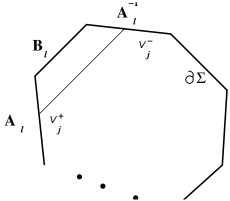

Figure: Cut Riemann surface ($4g$-gon) with the boundary $\partial\Sigma$. The boundary values of Abelian integrals $v^\pm_\alpha = \omega^\pm_\alpha$ on two boundaries of the cut differ by the period integral of the corresponding differential $d\omega_\alpha$ over the dual cycle.

Riemann Bilinear Relations

- Meromorphic: second kind Abelian differentials,

$$d\Omega_k \underset{P\to P_0}{\sim} \frac{d\zeta}{\zeta^{k+1}} + \ldots, \qquad \oint_{A} d\Omega_k = 0, \qquad k \geq 1$$

and third-kind Abelian differentials $d\Omega_\pm$

$$d\Omega_0 = d\log\frac{E(P,P_+)}{E(P,P_-)} \underset{P\to P_\pm}{\sim} \pm\frac{d\zeta_\pm}{\zeta_\pm} + \ldots, \qquad \oint_{A} d\Omega_0 = 0,$$

- E.g. for the first and third kind Abelian differentials

$$0 = \int_\Sigma d\omega_\beta \wedge d\Omega_0 = \sum_\alpha \left(\oint_{A_\alpha} d\omega_\beta \oint_{B_\alpha} d\Omega_0 - \oint_{A_\alpha} d\Omega_0 \oint_{B_\alpha} d\omega_\beta \right) +$$

$$+2\pi i \sum_{P_\pm} \mathrm{res}_{\,P_\pm} \omega_\beta d\Omega_0 = \oint_{B_\beta} d\Omega_0 - 2\pi i \int_{P_+}^{P_-} d\omega_\beta$$

Definition: Krichever data

- Complex curve Σ_g with *pair* of meromorphic differentials (dx and dy), with the *fixed* periods.
- Subfamily of curves $\{\Sigma_g\}$ of dimension

$$(3g-3)-(2g-3)=g$$

- Krichever data: an *integrable system* (back to the Liouville theorem) – on g-dimensional family of Σ_g one can choose g independent functions (Hamiltonians), while the co-ordinates on Jacobian of Σ_g play the role of complexified angle variables.

Definition: prepotential

- Krichever data: g-parametric family of Riemann surfaces Σ, endowed with a *generating* differential and *connection* ∇_{mod} on moduli space

$$dS \propto ydx, \qquad \nabla_{\mathrm{mod}} dS = \mathrm{holomorphic}$$

  where $y(P) = \int^P dy$, $P \in \Sigma$.
- Prepotential (a particular case of Krichever tau-function)

$$\mathsf{a} = \frac{1}{2\pi i} \oint_{\mathsf{A}} dS, \qquad \mathsf{a}_D = \oint_{\mathsf{B}} dS := \frac{\partial \mathcal{F}}{\partial \mathsf{a}}$$

  where A and B are dual cycles in $H_1(\Sigma)$.
- Defined locally on the moduli (Teichmüller?) space of Σ.
- Integrability from RBI

$$\frac{\partial a^D_\alpha}{\partial a_\beta} = T_{\alpha\beta} = T_{\beta\alpha} = \frac{\partial a^D_\beta}{\partial a_\alpha}$$

Prepotential: proof

- ∇_{mod}: connection via covariantly constant coordinate on Σ, e.g. $x = 0$ (problems at $dx = 0$). Then

$$\nabla_{\mathrm{mod}} dS = (\nabla_{\mathrm{mod}} y) dx$$

  and $\nabla_{\mathrm{mod}} y$ is defined from equation of $\Sigma \subset \mathbb{C}^2$.

- Using this and

$$\delta_{\alpha\beta} = \frac{\partial a_\alpha}{\partial a_\beta} = \frac{1}{2\pi i} \oint_{A_\alpha} \frac{\partial dS}{\partial a_\beta}$$

  one finds that $\frac{\partial dS}{\partial a_\beta} = d\omega_\beta$ is normalized holomorphic differential.

- Then

$$\frac{\partial a^D_\alpha}{\partial a_\beta} = \oint_{B_\alpha} \frac{\partial dS}{\partial a_\beta} = \oint_{B_\alpha} d\omega_\beta = T_{\alpha\beta}$$

Consequence: for the second derivatives

$$\frac{\partial^2 \mathcal{F}}{\partial a_\alpha \partial a_\beta} = T_{\alpha\beta}$$

Prepotential: SW example

- (Σ, dx, dy) given by

$$w + \frac{\Lambda^{2N}}{w} = P_N(z) = z^N + \sum_{k=0}^{N-2} u_k z^k, \qquad dx = \frac{dw}{w}, \quad dy = dz$$

since obviously

$$\oint_{(A,B)} dz = 0, \qquad \oint_{(A,B)} \frac{dw}{w} \in 2\pi i\mathbb{Z}$$

- From $\nabla_{\rm mod} w = 0$ and $\nabla_{\rm mod} z \; P'_N(z) = \sum_{k=0}^{N-2} \delta u_k z^k$

$$\nabla_{\rm mod} dS = \nabla_{\rm mod} z \; \frac{dw}{w} = \sum_{k=0}^{N-2} \delta u_k \frac{z^k}{P'_N(z)} \frac{dw}{w}$$

holomorphic on Σ.

Definition: Krichever tau-function

- To complete definition by the time-variables associated with the second-kind Abelian differentials with singularities at a point P_0

$$t_k = \frac{1}{k}\,\mathrm{res}_{\,P_0}\xi^{-k}dS, \quad k>0$$
$$\frac{\partial \mathcal{F}}{\partial t_k} := \mathrm{res}_{\,P_0}\xi^{k}dS, \quad k>0$$

where ξ is an *inverse* local co-ordinate at P_0: $\xi(P_0)=\infty$.

- The consistency condition for (10) is ensured by

$$\frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_k} = \mathrm{res}_{\,P_0}(\xi^k d\Omega_n)$$

symmetric due to $(\Omega_n)_+ = \xi^n$, for the main, singular at P_0, part.

- Also

$$\frac{\partial^2 \mathcal{F}}{\partial t_n \partial a_\alpha} = \oint_{B_\alpha} d\Omega_n = \mathrm{res}_{\,P_0}\xi^n d\omega_\alpha$$

which again follows from RBI;

Remarks:

- Definition from RBI;
- Can be defined for any set of Abelian differentials $\{dH_I\} = \{d\omega_\alpha, d\Omega_n, d\Omega_0, \ldots\}$ and corresponding flat-coordinates $\{T_I\} = \{a_\alpha, t_n, t_0, \ldots\}$.
- *pq*-duality: $dx \leftrightarrow dy$ generally a nontrivial tiny point:
    - Prepotentials: $\nabla^x_{\rm mod} \leftrightarrow \nabla^y_{\rm mod}$;
    - dKP: a non-trivial relation (e.g. a Fourier-transform for a matrix integral);
    - A nontrivial relation for residue formulas ...
- Starting point for the "topological recursion" ...

Residue formula: statement

Theorem

$$\frac{\partial^3 \mathcal{F}}{\partial T_I \partial T_J \partial T_K} = \mathrm{res}_{\,dx=0} \left(\frac{dH_I dH_J dH_K}{dxdy} \right)$$

- Idea of proof: to take one mode derivative of a second-derivative formula ...
- Prepotential case:

$$\frac{\partial T_{\alpha\beta}}{\partial a_\gamma} \equiv \partial_\gamma T_{\alpha\beta} = \int_{B_\beta} \partial_\gamma d\omega_\alpha = -\int_{\partial\Sigma} \omega_\beta \partial_\gamma d\omega_\alpha$$

- Further

$$\partial_\gamma T_{\alpha\beta} = -\int_{\partial\Sigma} \omega_\beta \partial_\gamma d\omega_\alpha = \int_{\partial\Sigma} \partial_\gamma \omega_\beta d\omega_\alpha = \sum \mathrm{res}_{\,dx=0} \left(\partial_\gamma \omega_\beta d\omega_\alpha \right)$$

since the expression acquires poles at $dx = 0$.

Residue formula: proof

- Use expansions where $dx = 0$

$$\omega_\beta(x) \underset{x\to x_\alpha}{=} \omega_{\beta a} + c_{\beta a}\sqrt{x-x_a} + \ldots, \qquad d\omega_\beta \underset{x\to x_a}{=} \frac{c_{\beta a}}{2\sqrt{x-x_a}}\,dx + \ldots$$

$$\nabla_{\rm mod}: \quad \partial_\gamma\omega_\beta \equiv \partial_\gamma\omega_\beta\big|_{x=\rm const} \underset{x\to x_a}{=} -\frac{c_{\beta a}}{2\sqrt{x-x_a}}\partial_\gamma x_a + \text{regular}$$

- Then

$$\text{res}\,(\partial_\gamma\omega_\beta d\omega_\alpha) = \sum_a \text{res}\left(\frac{c_{\beta a}\partial_\gamma x_a}{2\sqrt{x-x_a}}d\omega_\alpha\right) = \sum_a \text{res}\left(\frac{d\omega_\beta}{dx}d\omega_\alpha\partial_\gamma x_a\right) =$$

$$= \sum_a \text{res}\left(\frac{d\omega_\alpha d\omega_\beta d\omega_\gamma}{dxdy}\right)$$

where last equality similarly follows from

$$y(x) \underset{x\to x_a}{=} y_a\sqrt{x-x_a} + \ldots, \qquad dy \underset{x\to x_a}{=} \frac{y_a}{2\sqrt{x-x_a}}\,dx + \text{regular}$$

$$d\omega_\gamma = \partial_\gamma dS \underset{x\to x_a}{=} -\frac{y_a\partial_\gamma x_a}{2\sqrt{x-x_a}}\,dx + \text{regular}$$

Landau-Ginzburg topological theories

The topological theories defined by polynomial superpotential (generally of several complex variables)

$$W(\lambda) = \lambda^N + \sum_{k=0}^{N-2} u_k \lambda^k$$

The primaries are given by (dKP equation)

$$\phi_k(\lambda) := \frac{\partial W}{\partial t_k} = \left(\frac{d}{d\lambda} W^{k/N} \right)_+$$

where flat times

$$t_k = \frac{1}{k} \operatorname{res}_{P_0} \xi^{-k} dS = -\frac{N}{k(N-k)} \operatorname{res}_{\infty} \left(W^{1-k/N} d\lambda \right)$$

for $(\Sigma, dx, dy) = (\Sigma_0, dW, d\lambda)$ with $\xi = W(\lambda)^{1/N}$.

Landau-Ginzburg topological theories

The derivatives of the Krichever tau-function are given by

$$\frac{\partial \mathcal{F}}{\partial t_k} = \mathrm{res}_{P_0} \xi^k dS = \frac{N}{N+k} \mathrm{res}_{\infty} \left(W^{1+\frac{k}{N}} d\lambda \right)$$

together with

$$\mathcal{F}_{ik} = \frac{\partial^2 \mathcal{F}}{\partial t_i \partial t_k} = \mathrm{res}_{\infty} \left(W^{k/N} \frac{\partial W}{\partial t_i} \right) = \mathrm{res}_{\infty} \left(W^{k/N} \partial_\lambda W_+^{i/N} \right)$$

and (the Grothendick residue)

$$\mathcal{F}_{ijk} = -\mathrm{res}_{\infty} \frac{\partial_\lambda W_+^{i/N} \partial_\lambda W_+^{j/N} \partial_\lambda W_+^{k/N}}{W'} = \mathop{\mathrm{res}}_{W'=0} \frac{\phi_i(\lambda)\phi_j(\lambda)\phi_k(\lambda)}{W'}$$

WDVV equations: definition

LG primaries satisfy the associative algebra (a polynomial ring modulo $W'(\lambda)$)

$$\phi_i(\lambda)\phi_j(\lambda) = \sum_{k=1}^{N-1} C_{ij}^k \phi_k(\lambda) + R_{ij}(\lambda) W'(\lambda)$$

and therefore

$$[\mathsf{C}_i, \mathsf{C}_j] = 0$$

for the matrices $\|\mathsf{C}_i\|_j^k := C_{ij}^k$. In terms of matrices

$$\|\mathsf{F}_i\|_{jk} := \mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial t_i \, \partial t_j \, \partial t_k}$$

it leads to the overdetmined system of the differential equations

$$\mathsf{F}_i \mathsf{F}_j^{-1} \mathsf{F}_k = \mathsf{F}_k \mathsf{F}_j^{-1} \mathsf{F}_i \qquad \text{for all} \;\; i, j, k.$$

for the Krichever tau-function.

WDVV equations: theorem

Theorem

Let $\mathcal{F} = \mathcal{F}(T)$ be the Krichever tau-function, i.e. residue formula $\frac{\partial^3 \mathcal{F}}{\partial T_I \partial T_J \partial T_K} = \mathrm{res}_{\,dx=0} \left(\frac{dH_I dH_J dH_K}{dx dy} \right)$ holds. Then it satisfies the WDVV equations once the matching relation

$$\#\{T\} = \#(dx = 0)$$

is fulfilled.

Remarks:

- The number of critical points $\#(dx = 0)$ is counted modulo possible involution.
- Upon non-degeneracy conditions the proof is obvious.
- Constant "metric" $\eta = \mathsf{F}_1$ is not necessary.

WDVV equations: proof

Idea of proof: finite dimensional ring at $dx = 0$

$$\phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) = \sum_k C_{ij}^k \phi_k(\lambda_\alpha), \qquad \forall\ \lambda_\alpha$$

is solved for

$$C_{ij}^k = \sum_\alpha \phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) \left(\phi_k(\lambda_\alpha)\right)^{-1}$$

upon $\#\{i\} = \#\{\alpha\}$ and $\det_{i\alpha} \|\phi_i(\lambda_\alpha)\| \neq 0$. Modification $(\xi(\lambda_\alpha) \neq 0)$

$$\phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) = \sum_k C_{ij}^k(\xi)\phi_k(\lambda_\alpha) \cdot \xi(\lambda_\alpha), \qquad \forall\ \lambda_\alpha$$

only leads to redefinition

$$\eta_{kn} = \mathcal{F}_{kn1} \longrightarrow \eta_{kn}(\xi) = \sum_a \xi_a \mathcal{F}_{kna}$$

with $\xi_a = \sum_\alpha \xi(\lambda_\alpha) \left(\phi_a(\lambda_\alpha)\right)^{-1}$.

2d minimal gravity

- For each (p,q)-th point take a pair of polynomials

$$X = \lambda^p + \dots, \qquad Y = \lambda^q + \dots$$

  of degrees p and q respectively. Landau-Ginzburg $(p,q) = (N,1)$.
- A dispersionless version of the Lax and Orlov-Shulman operators from KP theory

$$\left[\hat{X}, \hat{Y}\right] = \hbar, \qquad \hat{X} = \partial^p + \dots, \qquad \hat{Y} = \partial^q + \dots$$

- An invariant way: an algebraic equation

$$Y^p - X^q - \sum f_{ij} X^i Y^j = 0$$

  with some $\{f_{ij}\}$. Generally, this is a smooth curve of genus

$$g = \frac{(p-1)(q-1)}{2} = \# \text{ primaries}$$

2d (minimal) gravity

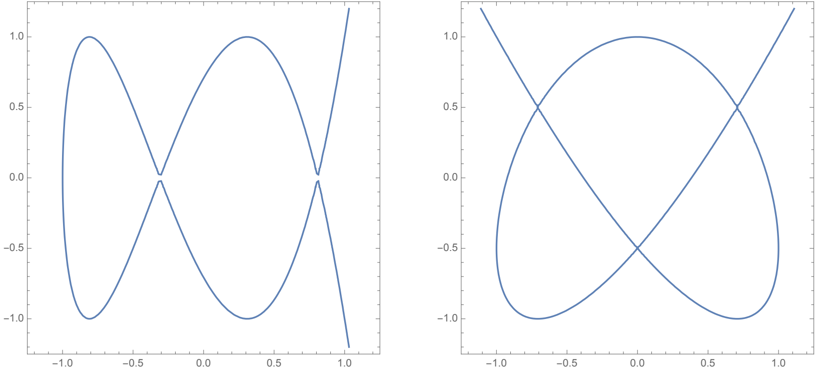

Figure: Degenerate curves of Yang-Lee and Ising models of $g = 2$ and $g = 3$.

Solution to dKP

- On rational curve

$$S = \sum_{k=1}^{p+q} t_k H_k = \sum_{k=1}^{p+q} t_k X^{k/p}(\lambda)_+, \qquad k \bmod p,$$

$$dS \underset{\xi\to\infty}{=} \sum \left(k t_k \xi^{k-1} d\xi + \frac{\partial \mathcal{F}}{\partial t_k} \frac{d\xi}{\xi^{k+1}} \right)$$

- Dependence of $X(\lambda) = \lambda^p + \sum_{k=0}^{p-2} X_k \lambda^k$ over $\{t\}$ from $dS|_{dX=0} = 0$, a system "hodograph" equations $\frac{dS}{d\lambda} = 0$ at $p-1$ roots of $X'(\lambda) = 0$.
- Any hamiltonian

$$H_k(\lambda) = \frac{\partial S}{\partial t_k} = \xi^k(\lambda)_+$$

is a polynomial of variable $\lambda = H_1$: dispersionless Hirota equations: all second derivatives $\{\frac{\partial^2 \mathcal{F}}{\partial t_k \partial t_n}\}$ are expressed in terms of $\{\frac{\partial^2 \mathcal{F}}{\partial t_k \partial t_1}\}$. E.g.

$$\frac{\partial^2 \mathcal{F}}{\partial t_3 \partial t_3} = 3 \left(\frac{\partial^2 \mathcal{F}}{\partial t_1^2} \right)^3$$

Ising $(p,q) = (3,4)$

$$X = \lambda^3 + X_1\lambda + X_0$$
$$Y = \lambda^4 + Y_2\lambda^2 + Y_1\lambda + Y_0$$

Flat times $\{t_1, t_2, 0, 0, t_5, 0, t_7 = \mathrm{const}\}$:

$$t_1 = -\frac{2}{3}X_0^2 + \frac{4}{27}X_1^3 + \frac{5}{9}t_5X_1^2$$
$$t_2 = -\frac{2}{3}X_0X_1 - \frac{5}{3}t_5X_0$$

Solving for X_0

$$t_1 = -\frac{6t_2^2}{(2X_1 + 5t_5)^2} + \frac{4}{27}X_1^3 + \frac{5}{9}t_5X_1^2 \underset{t_2=0}{=} \frac{4}{27}X_1^3 + \frac{5}{9}t_5X_1^2$$

the Boulatov-Kazakov equation.

$(p,q) = (2, 2K+1))$ series

- $p = 2$ KdV reduction

$$X = \lambda^2 + 2u, \quad \xi = \sqrt{X} = \sqrt{\lambda^2 + 2u}$$

$$S = \sum_{k=0}^{K+1} t_{2k+1} X^{k+1/2}(\lambda)_+$$

- Dependence $u = u(t)$ from $dS\big|_{dX=0} = 0$ gives

$$P(u) \equiv \tfrac{1}{2} \left.\frac{dS}{d\lambda}\right|_{\lambda=0} = \sum_{k=0}^{K+1} \frac{(2k+1)!!}{k!} t_{2k+1} u^k = 0$$

- Explicit formula $\mathcal{F} = \frac{1}{2}\sum_{k,l} t_k t_l \ \mathrm{res}_{P_0}(\xi^k dH_l)$ for the tau-function

$$\mathcal{F} = \tfrac{1}{2} \sum_{k,l=0}^{K+1} t_{2k+1} t_{2l+1} \frac{(2k+1)!!(2l+1)!!}{k!l!(k+l+1)} u^{k+l+1} = \tfrac{1}{2} \int_0^u P^2(v)dv$$

$(p, q) = (2, 2K + 1)$ series

In order to compare with the world-sheet gravity: resonances and analytic terms ...

- Resonances: absent for $(2K + 1)$-reduction;
- Residue formula (contributions from infinity?);
- $p - q$ or $X - Y$ duality;
- Verlinde formula (with A.Artemev and P.Gavrylenko).

Only $\mu \neq 0$: Chebyshev background ...

Chebyshev curves

- At only cosmological constant $\mu \neq 0$

$$T_p(Y) = T_q(X)$$

parameterized by $z \in \mathbb{P}^1$

$$X = T_p(z), \qquad Y = T_q(z)$$

Degenerate at

$$U_{p-1}(Y) = 0, \qquad U_{q-1}(X) = 0$$

- For $(p,q) = (2, 2K+1)$ a degenerate hyperelliptic curve with nodal singularities at

$$U_1(Y) = Y = 0, \qquad U_{2K}(X) = 0$$

$2K$ pairwise glued points

$$z_n^{\pm} = \pm\cos\frac{\pi(2n-1)}{2(2K+1)}, \qquad n = 1, 2, \dots, K$$

$$X_n = T_2(z_n^{\pm}) = \cos\frac{\pi(2n-1)}{2K+1}, \quad Y_n = T_{2K+1}(z_n^{\pm}) = \pm\cos\pi\left(n - \frac{1}{2}\right) = 0$$

Chebyshev curves

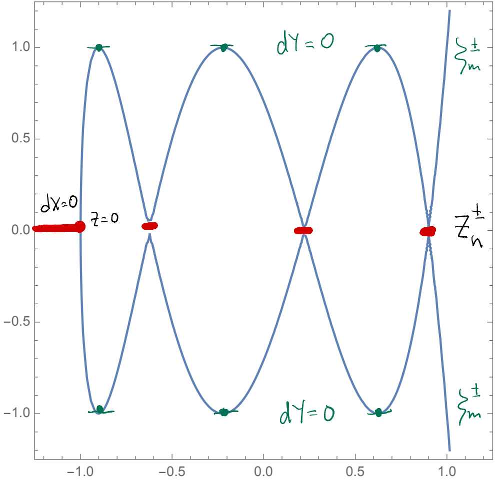

Figure: Chebyshev curve for $(2, 2K+1)$-series, with (degenerate) cuts $\{z_n^\pm\}$marked in red, critical points $\{\zeta_m^\pm\}$ where $dY = 0$ – in green, and the point $z = 0$ where $dX = 0$.

Ground ring and tachyons

- Ground ring of minimal $(2, 2K+1)$ gravity - isomorphic to

$$U_k(x)U_l(x) = U_{k+l}(x) + U_{k+l-2}(x) + \ldots + U_{|k-l|}(x), \quad k, l = 0, 1, \ldots$$

modulo $U_{2K}(x) = 0$.

- KP hamiltonians $H_{2n+1}(z) = T_{2n+1}(z) \sim Y(z)_+^{(2n+1)/(2K+1)}$.
- Non-faithful "tachyonic" module:

$$\mathcal{T}_n = \frac{dH_{2K+1-2n}}{dz} = U_{2(K-n)}(z), \qquad n = 1, \ldots, K$$

ground ring acts as

$$\mathcal{T}_n(z) = U_{n-1}(X)\mathcal{T}_1(z) = U_{n-1}(T_2(z)), \qquad n = 1, \ldots, 2K$$

- The tachyonic operators $\mathcal{T}_n \sim \mathcal{T}_{2K-n}$ are identified up to a sign, due to "reflection relations"

$$U_{2K+l}(x) + U_{2K-l}(x) = 0$$

Proof

Indeed, $\mathcal{T}_1 \sim U_{2K-2}(z)$

$$\mathcal{T}_n \sim U_{n-1}(T_2(z))U_{2K-2}(z) = \frac{1}{z}U_{2n-1}(z)U_{2K-2}(z) \underset{\text{ring } U}{=}$$

$$= \frac{1}{z}\left(U_{2K+2n-3}(z) + U_{2K+2n-1}(z) + \ldots + U_{2K-2n+1}(z) + U_{2K-2n-1}(z)\right) \underset{\text{reflection}}{=}$$

$$= \frac{1}{z}\left(U_{2K-2n+1}(z) + U_{2K-2n-1}(z)\right) \underset{\text{ring } U}{=} U_{2(K-n)}(z)$$

Identification $\mathcal{T}_n \sim \mathcal{T}_{2K-n}$ due to

$$U_{2K-n}(X) = U_{2K-n}(T_2(z)) = \frac{1}{z}U_{4K-2n-1}(z) \underset{\text{reflection}}{=} -\frac{1}{z}U_{2n-1}(z) =$$

$$= -U_{n-1}(T_2(z)) = -U_{n-1}(X)$$

Residue formula

Residue formula for on a Chebyshev curve

$$\frac{\partial^3 \mathcal{F}}{\partial t_{\tilde{i}} \partial t_{\tilde{j}} \partial t_{\tilde{k}}} = -\operatorname{res}_{dY=0} \frac{dH_{\tilde{i}} dH_{\tilde{j}} dH_{\tilde{k}}}{dX dY} = -\frac{1}{2K+1} \operatorname{res}_{U_{2K}(z)=0} \frac{U_{2\tilde{i}} U_{2\tilde{j}} U_{2\tilde{k}}(z) dz}{2z U_{2K}(z)} =$$

$$= -\frac{1}{2K+1} \sum_{m=1}^{K} \frac{U_{2\tilde{i}} U_{2\tilde{j}} U_{2\tilde{k}}(\zeta_m)}{\zeta_m U'_{2K}(\zeta_m)}$$

for $\tilde{n} = 2(K-n)+1$ etc, and

$$U_{2K}(z) = 0, \qquad z = \zeta_m^{\pm} = \pm \cos \frac{\pi m}{2K+1}, \qquad m = 1, 2, \ldots, K$$

$$X_m = T_2(\zeta_m^{\pm}) = \cos \frac{2\pi m}{2K+1}, \qquad Y_m^{\pm} = \pm T_{2K+1}(\zeta_m^{\pm}) = \pm \cos \pi m = \mp(-1)^m$$

Residue formula

Due to $T_{n+1}(x) = xU_n(x) - U_{n-1}(x)$ and

$$\zeta_m^{\pm} U_{2K-1}(\zeta_m^{\pm}) = (-1)^m \cos \frac{\pi m}{2K+1}, \qquad m = 1, 2, \ldots, K$$

at $U_{2K}(z) = 0$ further

$$\frac{\partial^3 \mathcal{F}}{\partial t_{\tilde{i}} \partial t_{\tilde{j}} \partial t_{\tilde{k}}} = -\frac{1}{2K+1} \sum_{m=1}^{K} \frac{U_{2\tilde{i}} U_{2\tilde{j}} U_{2\tilde{k}}(\zeta_m)}{\zeta_m U'_{2K}(\zeta_m)} =$$

$$= -\frac{1}{(2K+1)^2} \sum_{m=1}^{K} \frac{U_{2\tilde{i}} U_{2\tilde{j}} U_{2\tilde{k}}(\zeta_m)(1-\zeta_m^2)}{\zeta_m U_{2K-1}(\zeta_m)} =$$

$$= \frac{2}{(2K+1)^2} \sum_{m=1}^{K} \frac{\sin \frac{2\pi mi}{2K+1} \sin \frac{2\pi mj}{2K+1} \sin \frac{2\pi mk}{2K+1}}{\sin \frac{2\pi m}{2K+1}} = \frac{1}{2(2K+1)} (-1)^{1+i+j+k} N_{ijk}$$

with the Verlinde expression at the r.h.s.

Verlinde formula: basics

- S-matrix: $\chi_a(-1/\tau) = \sum_b \mathcal{S}_a^b \chi_b(\tau)$, unitarity $\mathcal{S}^\dagger \mathcal{S} = 1$
- Verlinde formula: relation with fusion algebra:

$$\mathcal{N}_{ab}^c = \sum_m \frac{\mathcal{S}_a^m \mathcal{S}_b^m (\mathcal{S}^\dagger)_m^c}{\mathcal{S}_1^m}$$

- Minimal (p,q)-model $\mathcal{S}^2 = 1$:

$$\mathcal{S}_{rs,\rho\sigma} = 2\sqrt{\frac{2}{pq}}(-1)^{1+s\rho+r\sigma} \sin \pi \frac{p}{q} r\rho \ \sin \pi \frac{q}{p} s\sigma$$

- $(p,q) = (2K+1, 2)$ series $s = \sigma = 1$:

$$\mathcal{S}_{r,\rho} = \frac{2}{\sqrt{2K+1}}(-1)^{1+\rho+r+K} \sin \frac{2\pi r\rho}{2K+1}$$

Verlinde formula

Verlinde formula for $(p,q) = (2K+1, 2)$

$$\frac{4}{2K+1}(-1)^{1+i+j+k}\sum_{m=1}^{K}\frac{\sin\frac{2\pi mi}{2K+1}\sin\frac{2\pi mj}{2K+1}\sin\frac{2\pi mk}{2K+1}}{\sin\frac{2\pi m}{2K+1}} = N_{ijk}$$

with $\{N_{ijk}\} \in \{0,1\}$. (One more?) nontrivial proof:

$$N_{ijk} = (-1)^{1+i+j+k}\frac{4}{2K+1}\sum_{m=1}^{K}\frac{\sin\frac{2\pi mi}{2K+1}\sin\frac{2\pi mj}{2K+1}\sin\frac{2\pi mk}{2K+1}}{\sin\frac{2\pi m}{2K+1}} =$$

$$= (-1)^{i+j+k}\,\mathrm{res}_{\,U_{2K}(z)=0}\frac{U_{2\tilde{i}}U_{2\tilde{j}}U_{2\tilde{k}}(z)dz}{zU_{2K}(z)} =$$

$$= (-1)^{1+i+j+k}\left(\mathrm{res}_{\,z=0} + \mathrm{res}_{\,z=\infty}\right)\frac{U_{2\tilde{i}}U_{2\tilde{j}}U_{2\tilde{k}}(z)dz}{zU_{2K}(z)}$$

by moving the contour to $dX = 0$ or $z = 0$ and $z = \infty$.

Verlinde formula

Substituting $z = \frac{1}{2}\left(w + \frac{1}{w}\right)$

$$(-1)^{1+i+j+k} N_{ijk} = \left(\text{res}_{\,z=0} + \text{res}_{\,z=\infty}\right) \frac{U_{2\tilde{i}} U_{2\tilde{j}} U_{2\tilde{k}}(z)dz}{zU_{2K}(z)} \underset{z=\frac{1}{2}\left(w+\frac{1}{w}\right)}{=}$$

$$= \left(\text{res}_{\,w=i} + \text{res}_{\,w=0}\right) \frac{\left(w^{2\tilde{i}+1} - w^{-2\tilde{i}-1}\right)(i \to j)(j \to k)}{\left(w^2 - w^{-2}\right)\left(w^{2K+1} - w^{-2K-1}\right)} \frac{dw}{w} =$$

$$= (-1)^{1+i+j+k} + \text{res}_{\,w=0} \frac{\left(w^{2\tilde{i}+1} - w^{-2\tilde{i}-1}\right)(i \to j)(j \to k)}{\left(w^2 - w^{-2}\right)\left(w^{2K+1} - w^{-2K-1}\right)} \frac{dw}{w}$$

The last residue gives

$$N_{ijk} = 1 - \sum_{l=0}^{K-2} \delta_{i+j+k+2l,2K} - \sum_{l=0}^{\left[\frac{K-3}{2}\right]} \left(\delta_{i+j+2l+1,k} + \delta_{i+k+2l+1,j} + \delta_{k+j+2l+1,i}\right) =$$

$$= \sum_{l=0}^{\min(i,j)-1} \delta_{|i-j|+2l+1,k} + \sum_{l=0}^{\left[\frac{K-2}{2}\right]} \delta_{i+j+k,2(K+l+1)}$$

Non-algebraic generalization

"Continuous" theory (Collier, Eberhardt, Mühlmann and Rodriguez)

$$\mathcal{N}(p_1, p_2, p_3) = 2b \sum_{m=1}^{\infty} (-1)^m \frac{\sin 2\pi mbp_1 \sin 2\pi mbp_2 \sin 2\pi mbp_3}{\sin \pi mb^2}$$

($b^2 = \frac{p}{q}$ in minimal theory). Here

$$(-1)^m \sin \pi mb^2 = \sin(\pi mb^2 + \pi m) = \sin 2\pi mbp_0 = \mathcal{S}_0^m$$

where $p_0 = (1/b + b)/2$ corresponds to $h_0 = 0$ or "unity" operator. Zamolodchikov's (?) formula comes from residue formula for a non-algebraic curve

$$x(z) = \cos \pi b^{-1} z, \qquad y(z) = \cos \pi bz$$

with

$$\left(\frac{p}{q} = \right) b^2 \in \mathbb{R}$$

Non-algebraic residue formula

Indeed

$$\mathcal{N}(p_1,p_2,p_3) = \sum_{dx=0} \frac{dH_{p_1} dH_{p_2} dH_{p_3}}{dx\ dy} = \sum_{x'(z)=0} \frac{\phi(p_1 z)\phi(p_2 z)\phi(p_3 z)}{x''(z)y'(z)}$$

since $x'(z) \sim \sin \pi b^{-1} z = 0$ at $z_m = bm$, $m \in \mathbb{Z}$, where

$$y'(z_m) \sim b \sin \pi b z_m = b \sin \pi b^2 m, \qquad x''(z_m) \sim b^{-2} \cos \pi b^{-1} z_m = b^{-2}(-1)^m$$

and the rest comes from identification $\phi(pz) = \sin 2\pi pz$

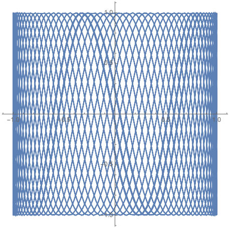

Figure: Non-algebraic curve gives rise to an infinite sum over $dx = 0$.

Many other developments

- Instanton partition functions;
- 2d conformal field theories;
- "Relativistic" (qt)-deformations;
- "Topological vertices", cluster algebras, double-loop algebras;
- Isomonodromic deformations;
- ...