Weight systems related to Lie algebras

Sergei Lando

National Research University Higher School of Economics,
Skolkovo Institute of Science and Technology

June 30, 2024

Publications

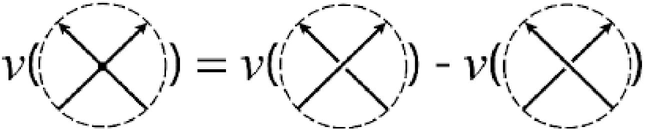
- Zhuoke Yang *New approaches to $\mathfrak{gl}_N$ weight system*, arXiv:2202.12225, Izvestiya Math., no. 6 (2023)
- Zhuoke Yang *On the Lie superalgebra $\mathfrak{gl}(m|n)$ weight system*, arXiv:2207.00327, Journal of Geometry and Physics 2023 Vol. 187
- M. Kazarian, S. Lando, *Weight systems and invariants of graphs and embedded graphs*, Russian Math. Surveys, 2022, vol. 77(5), 131–184
- P. A. Filippova, *Values of the $\mathfrak{sl}_2$ Weight System on Complete Bipartite Graphs*, Funct. Anal. Appl., 54:3 (2020), 208–223
- P. A. Filippova, *Values of the $\mathfrak{sl}_2$ weight system on a family of graphs that are not the intersection graphs of chord diagrams*, Sb. Math., 213:2 (2022), 235–267
- M. Kazarian, P. Zinova, *Algebra of shares, complete bipartite graphs, and the $\mathfrak{sl}_2$-weight system*, Sb. Math. no. 6, (2023)
- P. Zakorko, *Values of the $\mathfrak{sl}_2$ weight system on chord diagrams with complete intersection graphs*, Sb. Math., no. 7 (2023)

Prepublications

- N. Kodaneva, S. Lando, *Polynomial graph invariants induced from the $\mathfrak{gl}$-weight system*, arXiv:2312.17519
- M. Kazarian, N. Kodaneva, S. Lando, *The universal $\mathfrak{gl}$-weight system and the chromatic polynomial*, arXiv:2406.10562
- Maxim Kazarian and Zhuoke Yang, *Universal polynomial $\mathfrak{so}$ weight system*, in preparation
- P. Zakorko, P. Zinova *Duality for the $\mathfrak{sl}_2$ weight system*, in preparation

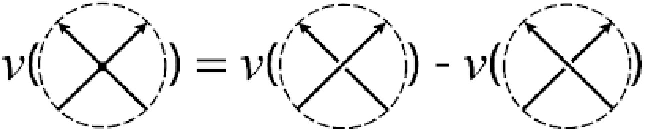
Chord diagrams and weight systems

Any knot invariant v with values in a commutative ring admits an extension to singular knots according to the following *Vassiliev skein relation*:

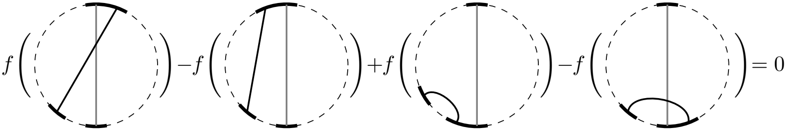


A knot invariant is *of order at most n* if its extension to singular knots with more than n double points vanishes.

Any knot invariant v with values in a commutative ring admits an extension to singular knots according to the following *Vassiliev skein relation*:



A knot invariant is *of order at most n* if its extension to singular knots with more than n double points vanishes.
Each knot invariant of order at most n determines a function on chord diagrams with n chords; this function satisfies *Vassiliev's 4-term relations*:



Weight systems and finite type knot invariants

According to Kontsevich's theorem, each weight system with values in an algebra over a field of characteristic 0 arises from a finite type knot invariant.

Weight systems and finite type knot invariants

According to Kontsevich's theorem, each weight system with values in an algebra over a field of characteristic 0 arises from a finite type knot invariant.

Most of the known weight systems are constructed either from graph invariants, or from Lie algebras,

- graph invariants: easy to construct, easy to compute, not powerful;
- Lie algebras: easy to construct, hard to compute, very powerful.

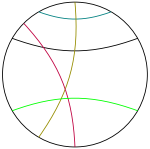
Constructing weight systems from Lie algebras

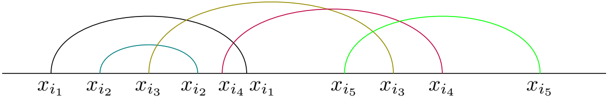
Initial data: finite dimensional Lie algebra $\mathfrak{g}$ with a nondegenerate *invariant* scalar product, $(\mathfrak{g}, (\cdot,\cdot))$: $([x,y],z) = (x,[y,z])\ \forall\ x,y,z$; $d = \dim\ \mathfrak{g}$.

Constructing weight systems from Lie algebras

Initial data: finite dimensional Lie algebra $\mathfrak{g}$ with a nondegenerate *invariant* scalar product, $(\mathfrak{g}, (\cdot,\cdot))$: $([x,y],z) = (x,[y,z])\ \forall\ x,y,z$; $d = \dim\ \mathfrak{g}$.

- Pick an orthonormal basis $x_1,\dots,x_d$ in $\mathfrak{g}$, $(x_i,x_j) = \delta_{ij}$.
- Cut the circle of a chord diagram D at some point and make it into an *arc diagram* A.
- Pick a numbering $\nu : V(A) \to \{1,\dots,d\}$ of the arcs of A.
- Put letters $x_{\nu(a)}$ at the ends of each arc a; the result is a word in $U\mathfrak{g}$.
- Sum over all the numberings $\nu : V(A) \to \{1,\dots,d\}$.





$$D \mapsto \sum_{i_1,i_2,i_3,i_4,i_5=1}^{d} x_{i_1}x_{i_2}x_{i_3}x_{i_2}x_{i_4}x_{i_1}x_{i_5}x_{i_3}x_{i_4}x_{i_5}$$

Constructing weight systems from Lie algebras

Theorem (D. Bar-Natan, M. Kontsevich)

The result is independent of the choice of the orthonormal basis $\{x_i\}$ and the cut point; it belongs to the center of $U\mathfrak{g}$ and satisfies 4-term relations.

Constructing weight systems from Lie algebras

Theorem (D. Bar-Natan, M. Kontsevich)

The result is independent of the choice of the orthonormal basis $\{x_i\}$ *and the cut point; it belongs to the center of* $U\mathfrak{g}$ *and satisfies* 4*-term relations.*

Difficulty: Computations are to be made in a noncommutative algebra.

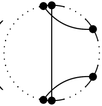
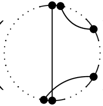
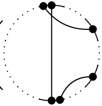
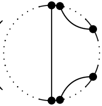
Constructing weight systems from Lie algebras

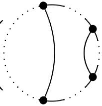
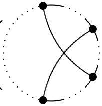
Theorem (D. Bar-Natan, M. Kontsevich)

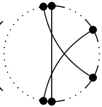
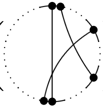
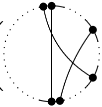
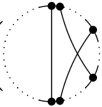
The result is independent of the choice of the orthonormal basis $\{x_i\}$ and the cut point; it belongs to the center of $U\mathfrak{g}$ and satisfies 4-term relations.

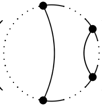
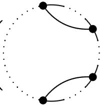
Difficulty: Computations are to be made in a noncommutative algebra.

For $\mathfrak{g} = \mathfrak{sl}(2)$, there is a recurrence relation due to Chmutov and Varchenko (1997).

$w_{\mathfrak{sl}(2)}$ () $-$ $w_{\mathfrak{sl}(2)}$ () $-$ $w_{\mathfrak{sl}(2)}$ () $+$ $w_{\mathfrak{sl}(2)}$ ()

$=$ $w_{\mathfrak{sl}(2)}$ () $-$ $w_{\mathfrak{sl}(2)}$ () ;

$w_{\mathfrak{sl}(2)}$ () $-$ $w_{\mathfrak{sl}(2)}$ () $-$ $w_{\mathfrak{sl}(2)}$ () $+$ $w_{\mathfrak{sl}(2)}$ ()

$=$ $w_{\mathfrak{sl}(2)}$ () $-$ $w_{\mathfrak{sl}(2)}$ () .

$\mathfrak{sl}_2$-weight system for complete graphs

The value of the $\mathfrak{sl}(2)$-weight system on a chord diagram depends on the intersection graph of the chord diagram rather than on the diagram itself (S. Chmutov and S. L, 2007). The *intersection graph* of a chord diagram is the graph whose vertices are the chords of the diagram, and two vertices are connected by an edge iff the corresponding chords intersect one another.

$\mathfrak{sl}_2$-weight system for complete graphs

The value of the $\mathfrak{sl}(2)$-weight system on a chord diagram depends on the intersection graph of the chord diagram rather than on the diagram itself (S. Chmutov and S. L, 2007). The *intersection graph* of a chord diagram is the graph whose vertices are the chords of the diagram, and two vertices are connected by an edge iff the corresponding chords intersect one another.

The chromatic polynomial for complete graphs on n variables looks very simple: $\chi_{K_n}(c) = c(c-1)\dots(c-n+1) = (c)_n$.
The generating function for it has the continued fraction form

$$\sum_{n=0}^{\infty} \chi_{K_n}(c)t^n = \cfrac{1}{1 - ct + \cfrac{ct^2}{1-(c-2)t+\cfrac{(2c-2)t^2}{1-(c-4)t+\cfrac{(3c-6)t^2}{1-(c-6)t+\dots}}}},$$

where the k th row is $1-(c-2(k-1))t + \left(kc - \frac{k(k-1)}{2}\right) t^2$.

$\mathfrak{sl}_2$-weight system for complete graphs

Theorem (P. Zakorko, 2021, former Lando's conjecture)

We have

$$\begin{aligned}\sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}(K_n)t^n &= 1+ct+c(c-1)t^2+c(c-1)(c-2)t^3 \\ &\quad +c(c^3-6c^2+13c-7)t^4+\dots \\ &= \cfrac{1}{1-ct+\cfrac{ct^2}{1-(c-2)t+\cfrac{(4c-3)t^2}{1-(c-6)t+\cfrac{(9c-18)t^2}{1-(c-12)t+\dots}}}},\end{aligned}$$

where the k th row is $1-(c-k(k-1))t+\left(k^2c-\frac{k^2(k^2-1)}{4}\right)t^2$.

Compare with the chromatic continued fraction: the k th row is $1-(c-2(k-1))t+\left(kc-\frac{k(k-1)}{2}\right)t^2$.

Values of the $\mathfrak{sl}(2)$-weight system on complete bipartite graphs

Theorem (M. Kazarian, P. Zinova)

For the generating functions $G_m(t) = \sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}(K_{m,n})t^n$, *we have*

$$G_m(t) = \frac{c^m + t \cdot \sum_{i=0}^{m-1} s_{i,m} G_i(t)}{1 - \left(c - \frac{m(m+1)}{2}\right) t}$$

with the initial condition

$$G_0(t) = \frac{1}{1 - ct}.$$

There is an explicit formula for the coefficients $s_{i,m}$.

$w_{\mathfrak{sl}(2)}$-duality

If one replaces complete bipartite graphs sequences $K_{m,n}$, $n = 0, 1, 2, \dots$, with the sequences of *joins* (G, n) of a given graph G with discrete graphs on $n = 0, 1, 2, \dots$ vertices, the form of the previous formula remains the same: the generating function for the values of the $\mathfrak{sl}_2$ weight system is

$$\sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}((G, n))t^n = \sum_{m=0}^{|V(G)|} \frac{P_m^G(c)}{1 - \left(c - \frac{m(m+1)}{2}\right) t},$$

for some sequence of polynomials $P_0^G, P_1^G, \dots$.

$w_{\mathfrak{sl}(2)}$-duality

If one replaces complete bipartite graphs sequences $K_{m,n}$, $n = 0, 1, 2, \dots$, with the sequences of *joins* (G, n) of a given graph G with discrete graphs on $n = 0, 1, 2, \dots$ vertices, the form of the previous formula remains the same: the generating function for the values of the $\mathfrak{sl}_2$ weight system is

$$\sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}((G, n))t^n = \sum_{m=0}^{|V(G)|} \frac{P_m^G(c)}{1 - \left(c - \frac{m(m+1)}{2}\right) t},$$

for some sequence of polynomials $P_0^G, P_1^G, \dots$.

Theorem (P. Zakorko, P. Zinova)

If we replace a graph G with its complement $\overline{G}$, then the polynomials P_k^G remain the same up to a sign: $P_k^{\overline{G}} = (-1)^{|V(G)|-k} P_k^G$.

Here the *complement graph* $\overline{G}$ has the same set of vertices as G, and the complementary set of edges.

Extending $\mathfrak{gl}(N)$-weight system to permutations

Nothing similar to Chmutov–Varchenko recurrence for other Lie algebras!
Kazarian's idea: For the Lie algebra $\mathfrak{gl}(N)$, a recurrence arises if we extend the weight system from chord diagrams to arbitrary permutations.

Extending $\mathfrak{gl}(N)$-weight system to permutations

Nothing similar to Chmutov–Varchenko recurrence for other Lie algebras!
Kazarian's idea: For the Lie algebra $\mathfrak{gl}(N)$, a recurrence arises if we extend the weight system from chord diagrams to arbitrary permutations.

Version of the initial construction: Pick an arbitrary basis $\{x_1, \dots, x_d\}$, not necessarily orthonormal, and write $x_{\nu(a)}$ on the left end of an arc a and the $(\cdot,\cdot)$-dual element $x^*_{\nu(a)}$ on its right end. In the previous example,

$$\sum_{i_1,i_2,i_3,i_4,i_5=1}^{d} x_{i_1}x_{i_2}x_{i_3}x^*_{i_2}x_{i_4}x^*_{i_1}x_{i_5}x^*_{i_3}x^*_{i_4}x^*_{i_5}.$$

The resulting element of the center of the universal enveloping algebra of $\mathfrak{g}$ coincides with the one above.

Main construction for $\mathfrak{gl}(N)$

For $\mathfrak{g} = \mathfrak{gl}(N)$, with the scalar product $(A, B) := \mathrm{Tr}\, AB$, choose the basis consisting of matrix units E_{ij}, $i, j = 1, \dots, N$, with the duality $E_{ij}^* = E_{ji}$.

Definition

For $\sigma \in S_m$, a permutation of m elements, define

$$w_{\mathfrak{gl}(N)} : \sigma \mapsto \sum_{i_1,i_2,\dots,i_m=1}^{N} E_{i_1,i_{\sigma(1)}} E_{i_2,i_{\sigma(2)}} \dots E_{i_m,i_{\sigma(m)}} \in U\mathfrak{gl}(N).$$

Main construction for $\mathfrak{gl}(N)$

For $\mathfrak{g} = \mathfrak{gl}(N)$, with the scalar product $(A, B) := \mathrm{Tr}\, AB$, choose the basis consisting of matrix units E_{ij}, $i, j = 1, \dots, N$, with the duality $E_{ij}^* = E_{ji}$.

Definition

For $\sigma \in S_m$, a permutation of m elements, define

$$w_{\mathfrak{gl}(N)} : \sigma \mapsto \sum_{i_1, i_2, \dots, i_m = 1}^{N} E_{i_1, i_{\sigma(1)}} E_{i_2, i_{\sigma(2)}} \dots E_{i_m, i_{\sigma(m)}} \in U\mathfrak{gl}(N).$$

Theorem

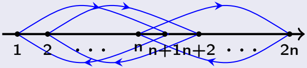
For any permutation σ, $w_{\mathfrak{gl}(N)}(\sigma)$ lies in the center $ZU\mathfrak{gl}(N)$ of $U\mathfrak{gl}(N)$.

Graph of a permutation

Definition (digraph of the permutation)

A permutation can be represented as an oriented graph. The m vertices of the graph correspond to the permuted elements. They are placed on the horizontal line, and numbered from left to right in the increasing order. The arc arrows show the action of the permutation (so that each vertex is incident with exactly one incoming and one outgoing arc edge). The digraph $G(\sigma)$ of a permutation $\sigma \in S_m$ consists of these m vertices and m oriented edges, for example:

$$G((1\ n+1)(2\ n+2)\cdots(n\ 2n)) =$$

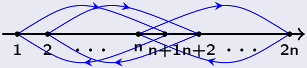


Graph of a permutation

Definition (digraph of the permutation)

A permutation can be represented as an oriented graph. The m vertices of the graph correspond to the permuted elements. They are placed on the horizontal line, and numbered from left to right in the increasing order. The arc arrows show the action of the permutation (so that each vertex is incident with exactly one incoming and one outgoing arc edge). The digraph $G(\sigma)$ of a permutation $\sigma \in S_m$ consists of these m vertices and m oriented edges, for example:

$$G((1\ n+1)(2\ n+2)\cdots(n\ 2n)) =$$



Chord diagrams are permutations of special kind, involutions without fixed points. For them, the initial definition coincides with the one above.

The center $ZU\mathfrak{gl}(N)$

Define *Casimir elements* $C_m \in U\mathfrak{gl}(N)$, $m = 1, 2, \dots$:

$$C_m = w_{\mathfrak{gl}(N)}((1, 2, \dots, m)) = \sum_{i_1, i_2, \dots, i_m = 1}^{N} E_{i_1, i_2} E_{i_2, i_3} \dots E_{i_m, i_1};$$

associated to the standard cycles $1 \mapsto 2 \mapsto 3 \mapsto \cdots \mapsto m \mapsto 1$.

The center $ZU\mathfrak{gl}(N)$

Define *Casimir elements* $C_m \in U\mathfrak{gl}(N)$, $m = 1, 2, \dots$:

$$C_m = w_{\mathfrak{gl}(N)}((1, 2, \dots, m)) = \sum_{i_1, i_2, \dots, i_m = 1}^{N} E_{i_1, i_2} E_{i_2, i_3} \dots E_{i_m, i_1};$$

associated to the standard cycles $1 \mapsto 2 \mapsto 3 \mapsto \cdots \mapsto m \mapsto 1$.

Theorem

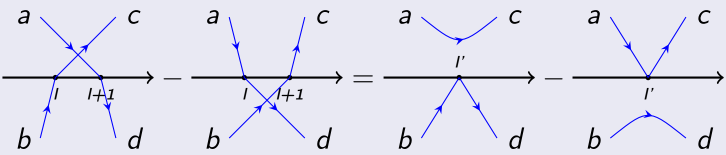
The center $ZU\mathfrak{gl}(N)$ of the universal enveloping algebra $U\mathfrak{gl}(N)$ of $\mathfrak{gl}(N)$ is identified with the polynomial ring $\mathbb{C}[C_1, \dots, C_N]$.

Recurrence relation

Theorem (Zhuoke Yang)

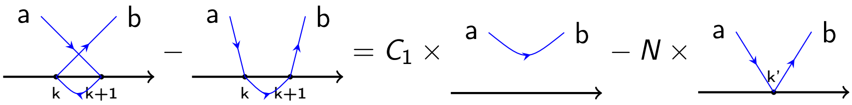
The $w_{\mathfrak{gl}(N)}$ invariant of permutations possesses the following properties:

- *for the empty permutation, the value of $w_{\mathfrak{gl}(N)}$ is equal to* 1*;*
- *$w_{\mathfrak{gl}(N)}$ is multiplicative with respect to concatenation of permutations;*
- (***Recurrence Rule***) *For the graph of an arbitrary permutation σ in S_m, and for any two neighboring elements $l, l+1$, of the permuted set $\{1, 2, \ldots, m\}$, we have for the values of the $w_{\mathfrak{gl}(N)}$ weight system*



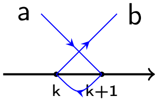
Recurrence rule for the special case

For the special case $\sigma(k+1) = k$, the recurrence looks like follows:

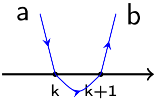


Recurrence rule for the special case

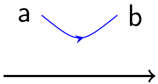
For the special case $\sigma(k+1) = k$, the recurrence looks like follows:



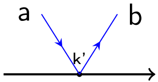
$-$



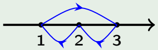
$= C_1 \times$



$- N \times$

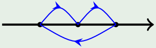


Example



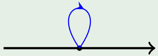
(1 3 2)

$-$



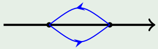
(1 2 3)

$= C_1 \times$



(1)

$- N \times$



(1 2)

$$
\begin{aligned}
w_{\mathfrak{gl}(N)}((1\ 3\ 2)) &= w_{\mathfrak{gl}(N)}((1\ 2\ 3)) + C_1 \cdot w_{\mathfrak{gl}(N)}((1)) - N \cdot w_{\mathfrak{gl}(N)}((1\ 2)) \\
&= C_3 + C_1^2 - NC_2
\end{aligned}
$$

Universal $\mathfrak{gl}$-weight system

Corollary

The $\mathfrak{gl}(N)$-weight systems, for $N = 1, 2, \dots,$ are combined into a universal $\mathfrak{gl}$-weight system $w_{\mathfrak{gl}}$ taking values in the ring of polynomials in infinitely many variables $\mathbb{C}[N; C_1, C_2, \dots]$.
After substituting a given value of N and an expression of higher Casimirs $C_{N+1}, C_{N+2}, \dots$ in terms of the lower ones $C_1, C_2, \dots, C_N$, this weight system specifies into the $\mathfrak{gl}(N)$-weight system.

Inducing graph invariants from the universal $\mathfrak{gl}$-weight system

It is easy to show that no substitution for $N, C_1, C_2, \dots$ makes $w_{\mathfrak{gl}}$ into the chromatic polynomial of the intersection graph of a chord diagram: the corresponding system of equations for K_1, K_2, K_3, K_4, K_5 has no solutions.

Theorem

Under the substitution $C_k = xN^{k-1}$, $k = 1, 2, 3, \dots$, the value of $w_{\mathfrak{gl}}$ on a chord diagram becomes a polynomial in N whose leading term is the chromatic polynomial of the intersection graph of the chord diagram.

Theorem

The assertion remains true if one replaces chord diagram with an arbitrary positive permutation.

A permutation is *positive* if each of its disjoint cycles is strictly increasing, with the exception of the last element.

Inducing graph invariants from the universal $\mathfrak{gl}$-weight system

Theorem

There is a substitution for N and C_k, $k = 1, 2, 3, \ldots$, which makes the value of $w_{\mathfrak{gl}}$ on a chord diagram into the interlace polynomial of its intersection graph.

An extension for Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{so}$, $\mathfrak{sp}$

There is a similar construction of weight systems from Lie superalgebras endowed with nondegenerate invariant scalar product (A. Vaintrob, 1994).

An extension for Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{so}$, $\mathfrak{sp}$

There is a similar construction of weight systems from Lie superalgebras endowed with nondegenerate invariant scalar product (A. Vaintrob, 1994).

Theorem (Zhuoke Yang)

There is an extension of the Lie superalgebra $\mathfrak{gl}(m|n)$ weight system to permutations similar to that for the Lie algebra $\mathfrak{gl}(N)$. The corresponding universal weight system, which works for all values of m and n together, coincides with the result of substitution $N = m - n$ into the universal weight system $w_{\mathfrak{gl}}$.

An extension for Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{so}$, $\mathfrak{sp}$

There is a similar construction of weight systems from Lie superalgebras endowed with nondegenerate invariant scalar product (A. Vaintrob, 1994).

Theorem (Zhuoke Yang)

There is an extension of the Lie superalgebra $\mathfrak{gl}(m|n)$ weight system to permutations similar to that for the Lie algebra $\mathfrak{gl}(N)$. The corresponding universal weight system, which works for all values of m and n together, coincides with the result of substitution $N = m - n$ into the universal weight system $w_{\mathfrak{gl}}$.

For the other classical series of Lie algebras and Lie superalgebras, the corresponding construction is elaborated by M.Kazarian and Zhoke Yang.

Real-normalized differentials and chord diagrams and permutations

I. Krichever, in cooperation with S. Grushevsky, applied effectively real-normalized differentials to the study of geometry of moduli spaces of complex curves. A meromorphic differential ω on a complex curve X is said to be *real-normalized* if all its periods are real, that is $\int_\gamma \omega$ is real, for an arbitrary closed curve $\gamma : S^1 \to X$ not passing through the poles of ω.

Real-normalized differentials and chord diagrams and permutations

I. Krichever, in cooperation with S. Grushevsky, applied effectively real-normalized differentials to the study of geometry of moduli spaces of complex curves. A meromorphic differential ω on a complex curve X is said to be *real-normalized* if all its periods are real, that is $\int_\gamma \omega$ is real, for an arbitrary closed curve $\gamma : S^1 \to X$ not passing through the poles of ω.

Any meromorphic differential ω on X determines a line field V_ω on $X \setminus \{$ poles of $\omega\}$: at each point q the line $V_\omega(x)$ looks the direction where the imaginary part of $\int_q \omega$ increases, the real part being constant. *Separatirces* of the line field V_ω are its integral trajectories passing through the zeroes of ω. For a given point $A \in X$ not belonging to the separatrices, the function $q \mapsto \int_A^q \omega$ determines a mapping from X cut along the separatrices to $\mathbb{C}$. If ω is real normalized, with a single pole of order 2, then the image is $\mathbb{C}$ cut along several vertical half-lines.

Real-normalized differentials and chord diagrams and permutations

For (X, ω), ω real normalized, with a single pole of order 2, and ω *in general position*, the vertical cut half-lines split into pairs starting at the same height, and determine thus a chord (or arc) diagram. Such a cut diagram determines the pair (X, ω) uniquely: X is reconstructed by gluing the opposite sides of the cuts belonging to the same pair, and ω is the image of dz. Under *isoperiodic deformations*, the diagram is subject to *second Vassiliev moves*. This construction has been applied to the study of the isoperiodic foliation in the space of real normalized differentials by I. Krichever, S. L., and A. Skripchenko (2021).

Real-normalized differentials and chord diagrams and permutations

For (X, ω), ω real normalized, with a single pole of order 2, and ω *in general position*, the vertical cut half-lines split into pairs starting at the same height, and determine thus a chord (or arc) diagram. Such a cut diagram determines the pair (X, ω) uniquely: X is reconstructed by gluing the opposite sides of the cuts belonging to the same pair, and ω is the image of dz. Under *isoperiodic deformations*, the diagram is subject to *second Vassiliev moves*. This construction has been applied to the study of the isoperiodic foliation in the space of real normalized differentials by I. Krichever, S. L., and A. Skripchenko (2021).

For a more general real normalized differential, the corresponding cut diagram determines a chord diagram no longer. Instead, it determines a diagram of a permutation.

Open problems

Open problems

- The $\mathfrak{sl}(2)$-weight system depends on the intersection graph of a chord diagram rather than on the diagram itself. Whether the $\mathfrak{sl}(2)$-weight system can be induced from a polynomial graph invariant satisfying 4-term relations for graphs?

Open problems

- The $\mathfrak{sl}(2)$-weight system depends on the intersection graph of a chord diagram rather than on the diagram itself. Whether the $\mathfrak{sl}(2)$-weight system can be induced from a polynomial graph invariant satisfying 4-term relations for graphs?
  A partial answer (D. Fomichev, M. Karev, arXiv:2406.15084): The value of the $\mathfrak{sl}(2)$-weight system at $c = 3/4$ admits a natural extension to graphs.

Open problems

- The $\mathfrak{sl}(2)$-weight system depends on the intersection graph of a chord diagram rather than on the diagram itself. Whether the $\mathfrak{sl}(2)$-weight system can be induced from a polynomial graph invariant satisfying 4-term relations for graphs?
  A partial answer (D. Fomichev, M. Karev, arXiv:2406.15084): The value of the $\mathfrak{sl}(2)$-weight system at $c = 3/4$ admits a natural extension to graphs.
- The chromatic polynomial of the intersection graph of a chord diagram is the leading term in N of the universal $\mathfrak{gl}$-weight system under the substitution $C_k = xN^{k-1}$, $k = 1, 2, 3, \dots$. What is the combinatorial meaning of the coefficient of the next term in N? of the other terms?

Open problems

- The $\mathfrak{sl}(2)$-weight system depends on the intersection graph of a chord diagram rather than on the diagram itself. Whether the $\mathfrak{sl}(2)$-weight system can be induced from a polynomial graph invariant satisfying 4-term relations for graphs?
  A partial answer (D. Fomichev, M. Karev, arXiv:2406.15084): The value of the $\mathfrak{sl}(2)$-weight system at $c = 3/4$ admits a natural extension to graphs.
- The chromatic polynomial of the intersection graph of a chord diagram is the leading term in N of the universal $\mathfrak{gl}$-weight system under the substitution $C_k = xN^{k-1}$, $k = 1, 2, 3, \dots$. What is the combinatorial meaning of the coefficient of the next term in N? of the other terms?
- What is the combinatorial meaning of the chromatic substitution for permutations?

Open problems

- The $\mathfrak{sl}(2)$-weight system depends on the intersection graph of a chord diagram rather than on the diagram itself. Whether the $\mathfrak{sl}(2)$-weight system can be induced from a polynomial graph invariant satisfying 4-term relations for graphs?
  A partial answer (D. Fomichev, M. Karev, arXiv:2406.15084): The value of the $\mathfrak{sl}(2)$-weight system at $c = 3/4$ admits a natural extension to graphs.
- The chromatic polynomial of the intersection graph of a chord diagram is the leading term in N of the universal $\mathfrak{gl}$-weight system under the substitution $C_k = xN^{k-1}$, $k = 1, 2, 3, \ldots$. What is the combinatorial meaning of the coefficient of the next term in N? of the other terms?
- What is the combinatorial meaning of the chromatic substitution for permutations?
- Same questions about interlace polynomial.

Open problems

Open problems

- Chord diagrams are orientable maps with a single vertex. Permutations are orientable hypermaps with a single vertex. How can one extend the construction of $\mathfrak{gl}$-weight system to arbitrary hypermaps?

Open problems

- Chord diagrams are orientable maps with a single vertex. Permutations are orientable hypermaps with a single vertex. How can one extend the construction of $\mathfrak{gl}$-weight system to arbitrary hypermaps?
- Stratification of the moduli spaces of meromorphic differentials by strata corresponding to permutations suggests that we must consider permutations (and chord diagrams as a special case) as metrized rather than just combinatorial objects. What is the correct way to impose Vassiliev's 4-term relations and construct corresponding invariants in continuous case?

Open problems

- Chord diagrams are orientable maps with a single vertex. Permutations are orientable hypermaps with a single vertex. How can one extend the construction of $\mathfrak{gl}$-weight system to arbitrary hypermaps?
- Stratification of the moduli spaces of meromorphic differentials by strata corresponding to permutations suggests that we must consider permutations (and chord diagrams as a special case) as metrized rather than just combinatorial objects. What is the correct way to impose Vassiliev's 4-term relations and construct corresponding invariants in continuous case?
- ...

Thank you
for your attention