Elliptic solitons related to the Lamé functions

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Discrete Krichever-Novikov equation [Adler (1998), Hietarinta (2003)]

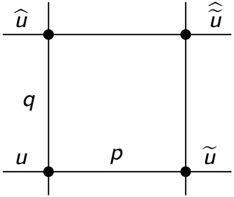
$$p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - r(u\widehat{\widetilde{u}}+\widetilde{u}\widehat{u}) + pqr(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}) = 0$$

$$\begin{aligned} &(p,P) = (\sqrt{k}\,\mathrm{sn}(\alpha;k), \mathrm{sn}'(\alpha;k)), \quad (q,R) = (\sqrt{k}\,\mathrm{sn}(\beta;k), \mathrm{sn}'(\beta;k)) \\ &(r,R) = (\sqrt{k}\,\mathrm{sn}(\gamma;k), \mathrm{sn}'(\gamma;k)), \quad \gamma = \alpha - \beta \end{aligned}$$

points on the elliptic curve: $\Gamma = \{(x,X) : X^2 = x^4 + 1 - (k+1/k)x^2\}$.

$$u \equiv u_{n,m}, \quad \widetilde{u} \equiv u_{n+1,m},$$

$$\widehat{u} \equiv u_{n,m+1}, \quad \widehat{\widetilde{u}} \equiv u_{n+1,m+1}$$

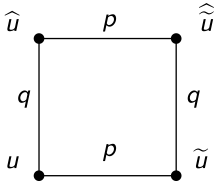


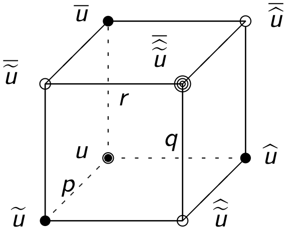
Krichever-Novikov equation [Krichever, Novikov (1981)]

$$v_t = v_{xxx} - \frac{3}{2}\frac{v_{xx}^2}{v_x} + \frac{3}{8v_x} - 6\wp(2v)v_x^3$$

Multidimensional consistency

Consistency Around the Cube (CAC): $Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = 0$





Classification: (MDC+ affine linear +D4+Tetrahedron) [Adler, Bobenko, Suris (2003)]

- Linearity w.r.t. each $\{u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}\}$
- Symmetry: Q invariant under group D_4
- Tetrahedron Condition: $\overline{\widehat{\widetilde{u}}} = f(\widetilde{u}, \widehat{u}, \overline{u}; p, q, r)$

Classification of quad equations [Adler, Bobenko, Suris (2003)]

$$p(u\tilde{u}+\hat{u}\hat{\tilde{u}})-q(u\hat{u}+\tilde{u}\hat{\tilde{u}})-r(u\hat{\tilde{u}}+\tilde{u}\hat{u})+pqr(1+u\tilde{u}\hat{u}\hat{\tilde{u}})=0 \qquad \text{(Q4)}$$

$$(q^2-p^2)(u\hat{\tilde{u}}+\tilde{u}\hat{u})+q(p^2-1)(u\tilde{u}+\hat{u}\hat{\tilde{u}})-p(q^2-1)(u\hat{u}+\tilde{u}\hat{\tilde{u}})$$
$$-\delta^2(p^2-q^2)(p^2-1)(q^2-1)/(4pq)=0 \qquad \text{(Q3(}\delta\text{))}$$

$$p(u-\hat{u})(\tilde{u}-\hat{\tilde{u}})-q(u-\tilde{u})(\hat{u}-\hat{\tilde{u}})$$
$$+pq(p-q)(u+\tilde{u}+\hat{u}+\hat{\tilde{u}})-pq(p-q)(p^2-pq+q^2)=0 \qquad \text{(Q2)}$$

$$p(u-\hat{u})(\tilde{u}-\hat{\tilde{u}})-q(u-\tilde{u})(\hat{u}-\hat{\tilde{u}})+\delta^2 pq(p-q)=0 \qquad \text{(Q1(}\delta\text{))}$$

$$(q^2-p^2)(u\tilde{u}\hat{u}\hat{\tilde{u}}+1)+q(p^2-1)(u\hat{u}+\tilde{u}\hat{\tilde{u}})-p(q^2-1)(u\tilde{u}+\hat{u}\hat{\tilde{u}})=0 \qquad \text{(A2)}$$

$$p(u+\hat{u})(\tilde{u}+\hat{\tilde{u}})-q(u+\tilde{u})(\hat{u}+\hat{\tilde{u}})-\delta^2 pq(p-q)=0 \qquad \text{(A1(}\delta\text{))}$$

$$p(u\tilde{u}+\hat{u}\hat{\tilde{u}})-q(u\hat{u}+\tilde{u}\hat{\tilde{u}})+\delta(p^2-q^2)=0 \qquad \text{(H3(}\delta\text{))}$$

$$(u-\hat{\tilde{u}})(\tilde{u}-\hat{u})+(q-p)(u+\tilde{u}+\hat{u}+\hat{\tilde{u}})+q^2-p^2=0 \qquad \text{(H2)}$$

$$(u-\hat{\tilde{u}})(\tilde{u}-\hat{u})=p-q \qquad \text{(H1)}$$

Q4 solution: J. Atkinson, F. Nijhoff, A constructive approach to the soliton solutions of integrable quadrilateral lattice equations, Commun. Math. Phys., 299 (2010) 283-304.

Elliptic solitons

KdV and 1SS

- KdV:

$$u_t = 6uu_x + u_{xxx}. \qquad \text{(KdV)}$$

- 1SS:

$$u = 2(\ln f)_{xx}, \quad f = 1 + e^{kx+k^3t}.$$

Usual soliton vs Elliptic soliton: Plane wave factor (PWF)

- usual soliton: e^{kx+k^3t} (KdV)
- elliptic soliton: **Lamé function** $\frac{\sigma(x+k)}{\sigma(x)\sigma(k)} e^{-\zeta(k)x-\wp'(k)t}$

E.A. Kuznetsov, A.V. Mikhailov, Stability of stationary waves in nonlinear weakly dispersive media, Sov. Phys. JETP, 40 (1974) 855-859. (Zh. Eksp. Teor. Fiz., 67 (1974) 1717-1727.)
F.W. Nijhoff, J. Atkinson, Elliptic *N*-soliton solutions of ABS lattice equations, Int. Math. Res. Not., 2010 (2010) 3837-3895.

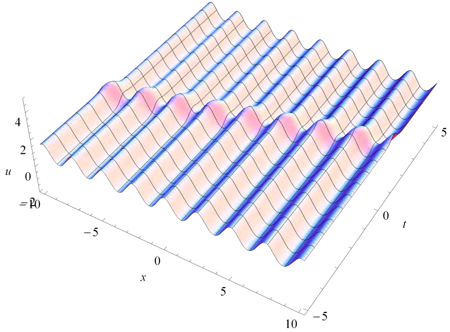


Figure: Elliptic 1-soliton of the KdV

$$u(x,t) = -2\wp(x+w_2) + 2\Big(\ln\big(1+\widetilde{\Psi}_x(k,k)e^{-4\wp'(k_1)t}\big)\Big)_{xx},$$

$$\widetilde{\Psi}_x(a,b) = \frac{\sigma(x+a+b+w_2)}{\sigma(x+w_2)\sigma(a+b)}e^{-(\zeta(a)+\zeta(b))x-\zeta(w_2)(a+b)}$$

Outline

I Introduction

I-1: Lamé function and KdV

KdV:

$$u_t = 6uu_x + u_{xxx} \tag{KdV}$$

Lax Pair:

$$\varphi_{xx} = (\lambda - u)\varphi,$$
$$\varphi_t = 4\varphi_{xxx} + 6u\varphi_x + 3u_x\varphi,$$

Solution to KdV: $u = -2\wp(x)$

Lamé function

$$\varphi_{xx} = (\wp(k) + 2\wp(x))\varphi \tag{Lamé}$$

$$\varphi(x) = \frac{\sigma(x+k)}{\sigma(x)\sigma(k)} e^{-\zeta(k)x}$$

I-2: Weierstrass functions

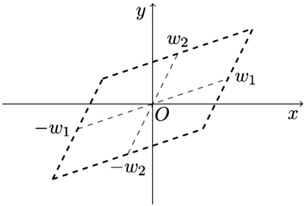


Figure: Fundamental period parallelogram $\mathbb{D}$ with boundary Ω.

Weierstrass functions: $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$, $\wp(z) = -\zeta'(z)$.

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + O(z^6),$$
$$\zeta(z) = \frac{1}{z} - \frac{g_2}{60}z^3 - \frac{g_3}{140}z^5 + O(z^7),$$
$$\sigma(z) = z - \frac{g_2}{240}z^5 - \frac{g_3}{840}z^7 + O(z^9).$$

Elliptic curve: $y^2 = 4x^3 - g_2 x - g_3$

I-2: Weierstrass functions: Identities:

$$\wp(z)-\wp(u)=-\frac{\sigma(z+u)\sigma(z-u)}{\sigma^2(z)\sigma^2(u)},$$

$$\eta_u(z)=\zeta(z+u)-\zeta(z)-\zeta(u)=\frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)},$$

$$\wp(z)+\wp(u)+\wp(z+u)=\eta_u^2(z)$$

$$\chi_{u,v}(z)=\zeta(u)+\zeta(v)+\zeta(z)-\zeta(u+v+z)=\frac{\sigma(u+v)\sigma(u+z)\sigma(z+v)}{\sigma(u)\sigma(v)\sigma(z)\sigma(z+u+v)}.$$

Frobenius-Stickelberger determinant (elliptic van der Monde):

$$\begin{aligned}&|\mathbf{1},\ \wp(\mathbf{k}),\ \wp'(\mathbf{k}),\ \wp''(\mathbf{k}),\ \cdots,\ \wp^{(n-2)}(\mathbf{k})|\\ =&(-1)^{\frac{(n-1)(n-2)}{2}}\left(\prod_{s=1}^{n-1}s!\right)\frac{\sigma(k_1+\cdots+k_n)\prod_{i<j}\sigma(k_i-k_j)}{\sigma^n(k_1)\sigma^n(k_2),\cdots\sigma^n(k_n)},\end{aligned}$$

where $f(\mathbf{k})=(f(k_1),f(k_2),\cdots,f(k_n))^T$.

$$\prod_{j=1}^{n}\Phi_x(k_j)=\frac{(-1)^{n-1}}{(n-1)!}\Phi_x(k_1+\cdots+k_n)\frac{|\mathbf{1},\ \wp(\mathbf{k}),\ \wp'(\mathbf{k}),\ \cdots,\ \wp^{(n-2)}(\mathbf{k})|}{|\mathbf{1},\ \eta_x(\mathbf{k}),\ \wp(\mathbf{k}),\ \wp'(\mathbf{k}),\ \cdots,\ \wp^{(n-3)}(\mathbf{k})|},$$

where $\Phi_a(b)=\frac{\sigma(a+b)}{\sigma(a)\sigma(b)}$.

I-3: Bilinear

- Hirota's bilinear operator D

$$D_t^m D_x^n f \cdot g = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n f(t,x) g(t',x')|_{t'=t,x'=x}$$

$$e^{\epsilon D_x + \kappa D_y} f(x,y) \cdot g(x,y) = f(x+\epsilon, y+\kappa) g(x-\epsilon, y-\kappa).$$

- Properties with usual PWF: e^{η_j}, $\eta_j = a_j x + b_j t + c_j$

$$D_x^n D_t^m e^{\eta_1} \cdot e^{\eta_2} = (a_1 - a_2)^n (b_1 - b_2)^m e^{\eta_1 + \eta_2},$$

$$D_x^n D_t^m (e^{\eta_1} f) \cdot (e^{\eta_1} g) = e^{2\eta_1} D_x^n D_t^m f \cdot g \ . \qquad \text{(gauge property)}$$

- Example: KdV

$$u_t = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}, \quad u = 2(\ln \tau)_{xx} \qquad \text{(KdV)}$$

$$(D_x^4 - D_x D_t)\tau \cdot \tau = 0$$

D vs Lamé-type PWF: [Xing Li, DJZ (2022)]

- Lamé-type PWF of the KdV:

$$\rho_i(x,t) = \Phi_x(2k_i)e^{\xi_i}, \ \ \xi_i = -2\zeta(k_i)x+\wp'(k_i)t+\xi_i^{(0)}, \quad \Phi_x(y) = \frac{\sigma(x+y)}{\sigma(x)\sigma(y)}$$

- derivatives

$$\rho_{i,x} = -\chi_{k_i,k_i}(x)\rho_i, \quad \rho_{i,xx} = 2\eta_{k_i}(x)\rho_{i,x},$$
$$\rho_{i,xxx} = (6\wp(x) + 2\wp(x+k_i) + 4\wp(k_i))\rho_{i,x},$$

where

$$\eta_x(y) = \zeta(x+y) - \zeta(x) - \zeta(y),$$
$$\chi_{\delta,\varepsilon}(\gamma) = \zeta(\delta) + \zeta(\varepsilon) + \zeta(\gamma) - \zeta(\delta+\varepsilon+\gamma).$$

- Property 1:

$$D_x^2\rho_i \cdot \rho_i = 2(\wp(x) - \wp(x+2k_i))\rho_i^2, , \quad D_x^4\rho_i \cdot \rho_i = 12\wp(x)D_x^2\rho_i \cdot \rho_i,$$

$$D_x^{2n}\ \varrho \cdot \varrho = \frac{\wp^{(2n-1)}(x)}{\wp'(x)}D_x^2\ \varrho \cdot \varrho, \quad \varrho = \Phi_x(a)e^{bx+ct}, \quad a,b,c \in \mathbb{C}.$$

D vs Lamé-type PWF: [Xing Li, DJZ (2022)]

- A general formula: $\varrho_i = \Phi_x(a_i)e^{b_i x + c_i t}, \quad a_i, b_i, c_i \in \mathbb{C}$

$$D_x^n D_t^m \varrho_1 \cdot \varrho_2 = (c_1 - c_2)^m Y_n(G_1, G_2, \cdots, G_n)\varrho_1\varrho_2,$$

where Y_n is the Bell polynomials:

$$Y_n(y_1, y_2, \cdots, y_n) = e^{-y}\partial_x^n e^y,$$

$y := y(x), \; y_i := \partial_x^i y(x),$

$$G_m(x) = \partial_x^{m-1}\alpha_1(x) + (-1)^m \partial_x^{m-1}\alpha_2(x),$$

and $\alpha_i(x) = \zeta(x + a_i) - \zeta(x) + b_i$.

- Property 2 (quasi-gauge):

$$D_x^n D_t^m (\varrho f)\cdot(\varrho g) = \varrho^2 D_x^n D_t^m f \cdot g + \sum_{l=1}^{[\frac{n}{2}]} \begin{pmatrix} n \\ 2l \end{pmatrix} (D_x^{2l}\varrho \cdot \varrho) D_x^{n-2l} D_t^m f \cdot g.$$

I-4: KdV: τ function and vertex operator

$$(4D_xD_t - D_x^4)\tau\cdot\tau = 0, \qquad \text{(Bilinear KdV)}$$

$$\tau_N = \sum_{J\subset S}\left[\left(\prod_{i\in J} c_i\right)\left(\prod_{\substack{i,j\in J\\ i<j}} A_{ij}\right)\exp\left(2\sum_{i\in J}\xi_i\right)\right], \qquad (\tau\text{: NSS})$$

where c_i are arbitrary constants, $A_{ij} = \frac{(k_i-k_j)^2}{(k_i-k_j)^2}$, $\xi_i = k_i x + k_i^3 t$, $S = \{1, 2, \cdots, N\}$, $J \subset S$.

$$\tau_{N+1} = e^{c_{N+1}X(k_{N+1})}\tau_N, \quad \tau_0 = 1,$$

$$X(k) = e^{2\xi(\mathbf{t},k)}\, e^{-2\xi(\widetilde{\partial},\, k^{-1})}, \qquad \text{(vertex operator)}$$

$$\xi(\mathbf{t},k) = \sum_{j=0}^{1} k^{2j+1} t_{2j+1}, \quad \mathbf{t} = (t_1 = x,), \quad \widetilde{\partial} = \left(\partial_1, \frac{\partial_3}{3}\right), \quad \partial_j = \partial_{t_j}.$$

- J. Lepowsky, R.L. Wilson (1978)
- E. Date, M. Kashiwara, T. Miwa (1981)

$$\mathrm{KdV} : A_1^{(1)}; \qquad \mathrm{KP} : gl(\infty); \qquad \mathrm{BKP} : o(\infty)$$

II Bilinear: τ functions, vertex operators

- KdV and KP
- Elliptic N-th root of unity
- Reduction via elliptic dispersion relation
- Degeneration via periods
- Discrete: KdV and KP

II-1: KdV: Elliptic 1-soliton solution (1SS) and 2SS

- Bilinear KdV:

$$\bar{v}_t - \frac{3}{4}\bar{v}_x^2 - \frac{1}{4}\bar{v}_{xxx} = 0, \qquad \text{(pKdV)}$$

$$\bar{v} = 2\zeta(x) + \frac{1}{4}g_2 t + 2(\ln\tau)_x, \qquad \text{(transformation)}$$

$$(D_x^4 - 4D_xD_t - 12\wp(x)D_x^2)\tau\cdot\tau = 0. \qquad \text{(bilinear KdV)}$$

alternative ($\tau' = \sigma(x)\tau$): $(D_x^4 - 4D_xD_t - g_2)\tau'\cdot\tau' = 0.$

- 1SS and 2SS:

$$\tau_1 = 1 + \rho_1(x,t) = 1 + \Phi_x(2k_1)e^{\xi_1}, \qquad \Phi_x(y) = \frac{\sigma(x+y)}{\sigma(x)\sigma(y)},$$

$$\tau_2 = 1 + \rho_1(x,t) + \rho_2(x,t) + f^{(2)}(x,t)$$

$$= 1 + \Phi_x(2k_1)e^{\xi_1} + \Phi_x(2k_2)e^{\xi_2} + A_{12}\frac{\sigma(x+2k_1+2k_2)}{\sigma(x)\sigma(2k_1)\sigma(2k_2)}e^{\xi_1+\xi_2},$$

$$\rho_i(x,t) = \Phi_x(2k_i)e^{\xi_i},\ \ \xi_i = -2\zeta(k_i)x + \wp'(k_i)t + \xi_i^{(0)},\ \ A_{12} = \frac{\sigma^2(k_1-k_2)}{\sigma^2(k_1+k_2)}.$$

NSS: Wronskian

- Wronskian $W(\varphi)$, $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_N)^T$:

$$W = |\varphi, \partial_x \varphi, \partial_x^2 \varphi, \cdots, \partial_x^{N-1} \varphi| = |0, 1, 2, \cdots, N-1| = |\widehat{N-1}|$$

- NSS to the bilinear KdV: $\tau = |\widehat{N-1}|$

$$\varphi_{j,xx} = (\wp(k_j) + 2\wp(x))\varphi_j,$$
$$\varphi_{j,t} = \varphi_{j,xxx} - 3\wp(x)\varphi_{j,x} - \frac{3}{2}\wp'(x)\varphi_j,$$

for $j = 1, 2, \cdots, N$ and $k_j \in \mathbb{C}$. A general solution of φ_j:

$$\varphi_j = a_j^+ \varphi_j^+ + a_j^- \varphi_j^-,$$

where $\varphi_j^\pm$ are Lamé functions

$$\varphi_j^\pm = \Phi_x(\pm k_j) e^{\mp \gamma_j}, \quad \gamma_j = \zeta(k_j)x - \frac{1}{2}\wp'(k_j)t + \gamma_j^{(0)},$$

NSS: Hirota's form

- Hirota's form:

$$f = \sum_{\mu=0,1} \frac{\sigma(x + 2\sum_{i=1}^{N} \mu_i k_i)}{\sigma(x)\prod_{j=1}^{N} \sigma^{\mu_j}(2k_j)} \exp\left(\sum_{j=1}^{N} \mu_j \theta_j + \sum_{1\le i<j}^{N} \mu_i \mu_j a_{ij}\right),$$

where

$$\theta_i = -2\gamma_j = -2\zeta(k_i)x + \wp'(k_i)t + \theta_i^{(0)}, \quad e^{a_{ij}} = A_{ij} = \left(\frac{\sigma(k_i - k_j)}{\sigma(k_i + k_j)}\right)^2.$$

- connected with τ:

$$(D_x^4 - 4D_x D_t - 12\wp(x)D_x^2)\tau \cdot \tau = 0,$$

$$(D_x^4 - 4D_x D_t - 12\wp(x)D_x^2)f \cdot f = 0, \quad f = \frac{\widetilde{\tau}}{\widetilde{g}},$$

where $\widetilde{g} = g(x + \sum_{i=1}^{N} k_i)$, $\widetilde{\tau} = \tau(x + \sum_{i=1}^{N} k_i)$,

$$g = (-1)^{\frac{N(N-1)}{2}} \frac{\sigma(x - \sum_{i=1}^{N} k_i)}{\sigma(x)} \cdot \frac{\prod_{1\le i<j\le N} \sigma(k_i - k_j)}{\sigma^N(k_1)\cdots\sigma^N(k_N)} \exp\left(\sum_{i=1}^{N} \gamma_i\right).$$

Vertex operator for τ

- Hirota's form: $f = \tau_N(\bar{\mathbf{t}})$

$$\tau_N(\bar{\mathbf{t}}) = \sum_{J\subset S}\left(\prod_{i\in J} c_i\right)\left(\prod_{\substack{i,j\in J\\ i<j}} A_{ij}\right)\frac{\sigma(t_1 + 2\sum_{i\in J} k_i)}{\sigma(t_1)\prod_{i\in J}\sigma(2k_i)}\exp\left(\sum_{i\in J}\theta_{[e]}(\bar{\mathbf{t}}, k_i)\right),$$

  where c_i are arbitrary constants, $S = \{1, 2, \cdots, N\}$, $J \subset S$.

- vertex operator to generate τ: (cf. Date, Kashiwara, Miwa (1981))

$$X(k) = \Phi_{t_1}(2k)e^{\theta_{[e]}(\bar{\mathbf{t}},k)}e^{\theta(\bar{\tilde{\partial}},k)}$$

$$\tau_N(\bar{\mathbf{t}}) = e^{c_N X(k_N)}\tau_{N-1}(\bar{\mathbf{t}}), \quad \tau_0(\bar{\mathbf{t}}) = 1,$$

- notations

$$\bar{\mathbf{t}} = (t_1 = x, t_3, \cdots, t_{2n+1}, \cdots),$$

$$\bar{\tilde{\partial}} = (\partial_{t_1}, \frac{1}{3}\partial_{t_3}, \cdots, \frac{1}{(2n+1)}\partial_{t_{2n+1}}, \cdots),$$

$$\theta(\bar{\mathbf{t}}, k) = 2\sum_{n=0}^{\infty} k^{2n+1}t_{2n+1}, \quad \theta_{[e]}(\bar{\mathbf{t}}, k) = -2\sum_{n=0}^{\infty}\frac{\zeta^{(2n)}(k)}{(2n)!}t_{2n+1}.$$

Bilinear identity

- Bilinear identity:

$$\oint_{\Omega} \frac{\mathrm{d}q}{2\pi i}\, h(\bar{\mathbf{t}}, q)\, h(\bar{\mathbf{t}}', -q) = 0,$$

where

$$h(\bar{\mathbf{t}}, q) = X(\bar{\mathbf{t}}, q)\tau(\bar{\mathbf{t}}), \quad X(\bar{\mathbf{t}}, q) = \frac{\sigma(t_1 + q)}{\sigma(q)} e^{\frac{1}{2}\theta_{[e]}(\bar{\mathbf{t}}, q)} e^{\frac{1}{2}\theta(\bar{\tilde{\partial}}, q)}.$$

Note that $h(\bar{\mathbf{t}}, q)$ is doubly periodic w.r.t. q.

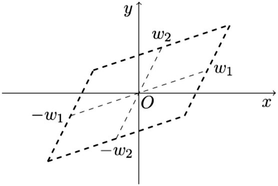


Figure: Fundamental period parallelogram $\mathbb{D}$ with boundary Ω.

- Bilinear identity in residue form

$$\operatorname*{Res}_{q=0} \left[h(\bar{\mathbf{t}}, q)\, h(\bar{\mathbf{t}}', -q) \right] = 0$$

Algorithm for calculating residues

- Bilinear identity: Redefining $\tau'(\bar{\mathbf{t}}) = \sigma(t_1)\tau(\bar{\mathbf{t}})$, we have

$$\oint_\Omega \frac{\mathrm{d}q}{2\pi i} \frac{1}{\sigma^2(q)} e^{\frac{1}{2}\theta_{[e]}(\bar{\mathbf{t}}-\bar{\mathbf{t}}',q)} \tau'(\bar{\mathbf{t}}+\bar{\varepsilon}(q))\tau'(\bar{\mathbf{t}}'-\bar{\varepsilon}(q)) = 0.$$

- Introduce $\bar{\mathbf{t}} = \bar{\mathbf{x}} + \bar{\mathbf{y}}$, $\bar{\mathbf{t}}' = \bar{\mathbf{x}} - \bar{\mathbf{y}}$, $\bar{\mathbf{x}} = (x_1, x_3, \cdots)$, $\bar{\mathbf{y}} = (y_1, y_3, \cdots)$,

$$\oint_\Omega \frac{\mathrm{d}q}{2\pi i} \frac{1}{\sigma^2(q)} e^{\theta_{[e]}(\bar{\mathbf{y}},q)} e^{(\bar{\mathbf{y}}+\bar{\varepsilon}(q))\cdot \mathbf{D}_{\bar{\mathbf{x}}}} \tau'(\bar{\mathbf{x}}) \cdot \tau'(\bar{\mathbf{x}}) = 0,$$

$$\operatorname*{Res}_{q=0} \left[\frac{1}{\sigma^2(q)} e^{\theta_{[e]}(\bar{\mathbf{y}},q)} e^{(\bar{\mathbf{y}}+\bar{\varepsilon}(q))\cdot \mathbf{D}_{\bar{\mathbf{x}}}} \tau'(\bar{\mathbf{x}}) \cdot \tau'(\bar{\mathbf{x}}) \right] = 0,$$

where $\mathbf{D}_{\bar{\mathbf{x}}} = (D_{x_1}, D_{x_3}, D_{x_5}, \cdots)$, $\bar{\varepsilon}(q) = (q, \frac{q^3}{3}, \cdots, \frac{q^{2n+1}}{(2n+1)}, \cdots)$.

- Difficulty:

$$\theta_{[e]}(\bar{\mathbf{t}}, k) = -2 \sum_{n=0}^{\infty} \frac{\zeta^{(2n)}(k)}{(2n)!} t_{2n+1} = \sum_{j=-\infty}^{\infty} s_j(\bar{\mathbf{t}}) k^j.$$

Algorithm for calculating residues [Li, DJZ (2022)]

- Algorithm:

$$\operatorname*{Res}_{q=0}\left[(\overline{\mathbf{B}}+\mathbf{D}_{\bar{\mathbf{x}}})^{\overline{\beta}}|_{\le 1}\left(\sum_{n=0}^{||\overline{\beta}||-1}\sum_{j=0}^{n}p_j(\widetilde{\mathbf{D}}_{\bar{\mathbf{x}}})\mu_{n-j}q^{n-2}\right)\tau'(\bar{\mathbf{x}})\cdot\tau'(\bar{\mathbf{x}})\right]=0,$$

$\overline{\beta}=(\beta_1,\beta_3,\cdots,\beta_{2j+1},\cdots)$, $\beta_j\ge 0$, $|\overline{\beta}|=\sum_{j=0}^{\infty}\beta_{2j+1}$,
$||\overline{\beta}||=\sum_{j=0}^{n}(2j+1)\beta_{2j+1}$, $e^{\xi(\mathbf{t},k)}=\sum_{n=0}^{\infty}p_n(\mathbf{t})k^n$, $\frac{1}{\sigma^2(q)}=\sum_{j=0}^{\infty}\mu_j q^{j-2}$,

$$\overline{\mathbf{B}}=-2(\zeta(q),\frac{\zeta''(q)}{2!},\cdots\frac{\zeta^{(2n)}(q)}{(2n)!},\cdots),\quad \widetilde{\mathbf{D}}_{\bar{\mathbf{x}}}=(D_{x_1},0,\frac{1}{3}D_{x_3},0,\frac{1}{5}D_{x_5},\cdots).$$

- Examples: $\overline{\beta}=(3,0,0,\cdots)$, $\overline{\beta}=(2,1,0,\cdots)$ and $\overline{\beta}=(5,0,0,\cdots)$

$$(D_{x_1}^4-4D_{x_1}D_{x_3}-g_2)\tau'\cdot\tau'=0,$$
$$(D_{x_1}^6+4D_{x_1}^3D_{x_3}-32D_{x_3}^2+3g_2D_{x_1}^2-24g_3)\tau'\cdot\tau'=0,$$
$$(D_{x_1}^6+40D_{x_1}^3D_{x_3}+40D_{x_3}^2-216D_{x_1}D_{x_5}+3g_2D_{x_1}^2-24g_3)\tau'\cdot\tau'=0.$$

KP [Li, DJZ (2022)]

- Bilinear

$$4v_t - v_{xxx} - 3(v_x)^2 - 3\partial^{-1}v_{yy} = 0. \qquad \text{(pKP)}$$

$$v = 2\zeta(x) + \frac{g_2}{4}t + 2(\ln\tau)_x,$$

$$(D_x^4 - 4D_xD_t - 12\wp(x)D_x^2 + 3D_y^2)\tau\cdot\tau = 0, \qquad \text{(Bilinear KP)}$$

or

$$(D_x^4 - 4D_xD_t + 3D_y^2 - g_2)\tau'\cdot\tau' = 0. \qquad (\tau' = \sigma(x)\tau)$$

- τ function: $\tau = |\widehat{N-1}|$, $f = \widetilde{\tau}/\widetilde{g}$

$$\tau_N(\mathbf{t}) = \sum_{J\subset S}\left(\prod_{i\in J} c_i\right)\left(\prod_{i<j\in J} A_{ij}\right)\frac{\sigma(t_1 + \sum_{i\in J}(k_i - l_i))}{\sigma(t_1)\prod_{i\in J}\sigma(k_i - l_i)}e^{\sum_{i\in J}(\xi_{[e]}(\mathbf{t},k_i) - \xi_{[e]}(\mathbf{t},l_i))}$$

$$A_{ij} = \frac{\sigma(k_i - k_j)\sigma(l_i - l_j)}{\sigma(k_i - l_j)\sigma(l_i - k_j)}$$

Vertex operator for τ

- τ function:

$$\tau_N(\mathbf{t}) = \sum_{J\subset S}\left(\prod_{i\in J} c_i\right)\left(\prod_{i<j\in J} A_{ij}\right)\frac{\sigma(t_1+\sum_{i\in J}(k_i-l_i))}{\sigma(t_1)\prod_{i\in J}\sigma(k_i-l_i)}e^{\sum_{i\in J}(\xi_{[e]}(\mathbf{t},k_i)-\xi_{[e]}(\mathbf{t},l_i))}$$

- vertex operator to generate τ: (cf. Date, Kashiwara, Miwa (1981))

$$X(k,l) = \Phi_{t_1}(k-l)e^{\xi_{[e]}(\mathbf{t},k)-\xi_{[e]}(\mathbf{t},l)}e^{\xi(\widetilde{\partial},k)-\xi(\widetilde{\partial},l)},$$

$$\tau_N(\mathbf{t}) = e^{c_N X(k_N,l_N)}\tau_{N-1}(\mathbf{t}), \quad \tau_0(\mathbf{t}) = 1,$$

- notations

$$\mathbf{t} = (t_1 = x, t_2, \cdots, t_n, \cdots), \quad \widetilde{\partial} = (\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \cdots, \frac{1}{n}\partial_{t_n}, \cdots),$$

$$\xi(\mathbf{t},k) = \sum_{n=1}^{\infty} k^n t_n, \quad \xi_{[e]}(\mathbf{t},k) = \sum_{n=1}^{\infty}(-1)^n\frac{\zeta^{(n-1)}(k)}{(n-1)!}t_n, \quad \zeta^{(i)}(k) = \partial_k^i\zeta(k).$$

Bilinear identity [Li, DJZ (2022)]

- Bilinear identity

$$\oint_{\Omega} \frac{\mathrm{d}q}{2\pi i}\, h(\mathbf{t}, q)\, h^*(\mathbf{t}', q) = 0, \qquad \text{(bilinear identity)}$$

$$h(\mathbf{t}, q) = X(\mathbf{t}, q)\tau(\mathbf{t}), \quad h^*(\mathbf{t}, q) = X^*(\mathbf{t}, q)\tau(\mathbf{t}),$$

$$X(\mathbf{t}, q) = \frac{\sigma(t_1 + q)}{\sigma(q)} e^{\xi_{[e]}(\mathbf{t},q)} e^{\xi(\widetilde{\partial},q)}, \quad X^*(\mathbf{t}, q) = \frac{\sigma(t_1 - q)}{\sigma(-q)} e^{-\xi_{[e]}(\mathbf{t},q)} e^{-\xi(\widetilde{\partial},q)}.$$

- Calculation

$$\operatorname*{Res}_{q=0}\left[(\mathbf{B} + \mathbf{D}_{\mathbf{x}})^{\beta}|_{\leq 1} \left(\sum_{n=0}^{||\beta||-1} \sum_{j=0}^{n} p_j(\widetilde{\mathbf{D}}_{\mathbf{x}})\mu_{n-j} q^{n-2} \right) \tau'(\mathbf{x}) \cdot \tau'(\mathbf{x}) \right] = 0,$$

- Examples:

$$(D_{x_1}^4 + 3D_{x_2}^2 - 4D_{x_1}D_{x_3} - g_2)\tau' \cdot \tau' = 0,$$
$$(D_{x_1}^3 D_{x_2} + 2D_{x_2}D_{x_3} - 3D_{x_1}D_{x_4})\tau' \cdot \tau' = 0,$$
$$(D_{x_1}^6 + 45D_{x_1}^2 D_{x_2}^2 + 20D_{x_1}^3 D_{x_3} + 40D_{x_3}^2 + 90D_{x_2}D_{x_4}$$
$$- 216D_{x_1}D_{x_5} + 3g_2 D_{x_1}^2 - 24g_3)\tau' \cdot \tau' = 0$$

II-2: Elliptic N-th roots of unity [Nijhoff, Sun, DJZ (2023)]

Definition

There exist distinct $\{\omega_j(\delta)\}_{j=0}^{N-1}$, up to the periodicity of the periodic lattice, such that the following equation holds,

$$\prod_{j=0}^{N-1} \Phi_\kappa(\omega_j(\delta)) = \frac{1}{(N-1)!}(\wp^{(N-2)}(-\kappa) - \wp^{(N-2)}(\delta)),$$

where $\omega_0(\delta) = \delta$ and all $\{\omega_j(\delta)\}$ are independent of κ. $\{\omega_j(\delta)\}_{j=0}^{N-1}$ are called elliptic N-th roots of the unity.

These roots also satisfy

$$\sum_{j=0}^{N-1} \omega_j(\delta) = 0$$

and

$$\sum_{j=0}^{N-1} \zeta^{(l)}(\omega_j(\delta)) = 0, \quad (l = 0, 1, \cdots, N-2).$$

Discrete plane wave factors (PWFs):

- discrete PWF/dispersion relation for usual solitons

$$G_N(p,k) := \sum_{j=1}^{N} \alpha_j (p^j - k^j) = \prod_j (p - \omega_j(k)), \quad \alpha_N \equiv 1$$

$$\mathrm{PWF}: \rho = \left(\frac{p+k}{p+\omega_j(k)}\right)^n \left(\frac{q+k}{q+\omega_j(k)}\right)^m$$

- Elliptic Nth root of unity

$$N=2: \quad \Phi_\kappa(\delta)\Phi_\kappa(-\delta) = \wp(\kappa) - \wp(\delta) \ ,$$

$$N=3: \quad \Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta)) = -\tfrac{1}{2}\left(\wp'(\kappa) + \wp'(\delta)\right) \ , \quad \forall$$

$$N=4: \quad \Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta))\Phi_\kappa(\omega_3(\delta)) = \tfrac{1}{6}\left(\wp''(\kappa) - \wp''(\delta)\right) \ ,$$

- Elliptic PWF:

$$\rho = \sum_j \left(\frac{\Phi_p(\omega_j(k))}{\Phi_p(\omega_0(k))}\right)^n \left(\frac{\Phi_q(\omega_j(k))}{\Phi_q(\omega_0(k))}\right)^m$$

II-3: Reduction by dispersion relation

- τ function of KP:

$$\tau_N(\mathbf{t}) = \sum_{J\subset S}\left(\prod_{i\in J} c_i\right)\left(\prod_{i<j\in J} A_{ij}\right)\frac{\sigma(t_1+\sum_{i\in J}(k_i-l_i))}{\sigma(t_1)\prod_{i\in J}\sigma(k_i-l_i)}e^{\sum_{i\in J}(\xi_{[e]}(\mathbf{t},k_i)-\xi_{[e]}(\mathbf{t},l_i))}$$

$$\xi_{[e]}(\mathbf{t},k) = \sum_{n=1}^{\infty}(-1)^n\frac{\zeta^{(n-1)}(k)}{(n-1)!}t_n$$

- Reduction to KdV HIERARCHY: $l_i = -k_i$
- Reduction to Bussinesq:
  ok: from the KP equation to the Boussinesq equation,
  fail: from the KP hierarchy to the Boussinesq hierarchy (because elliptic cube root of unity is not the 6th root of unity.) [It can be zero by redefine $t_2 \to t_2 + 12g_2t_6$.]

$$\begin{aligned}&\wp^{(4)}(\omega_1(\delta)) - \wp^{(4)}(\delta)\\ =\ & 30(\wp'(\omega_1(\delta)) - \wp'(\delta))(\wp'(\omega_1(\delta)) + \wp'(\delta)) + 12g_2(\wp(\omega_1(\delta)) - \wp(\delta)).\end{aligned}$$

II-4: Degeneration by period

- Elliptic curve: $y^2 = R(x) = 4x^3 - g_2 x - g_3$
- Degenerations:

$$\Delta = g_2^3 - 27 g_3^2 = 0.$$

- trigonometric/hyperbolic:

$$g_2 = \frac{4}{3}\alpha^4, \quad g_3 = \frac{8}{27}\alpha^6, \quad \alpha = \frac{\pi}{2w},$$

$$\sigma(q) = \frac{1}{\alpha} e^{\frac{1}{6}(\alpha q)^2} \sin(\alpha q),$$
$$\zeta(q) = \frac{1}{3}\alpha^2 q + \alpha \cot(\alpha q),$$
$$\wp(q) = -\frac{1}{3}\alpha^2 + \alpha^2 \csc^2(\alpha q).$$

- rational: $g_2 = g_3 = 0$,

$$\sigma(q) = q, \quad \zeta(q) = \frac{1}{q}, \quad \wp(q) = \frac{1}{q^2}.$$

II-4: Degeneration by period

$$\operatorname*{Res}_{q=0}\left[(\mathbf{B}+\mathbf{D}_{\mathbf{x}})^{\beta}|_{\leq 1}\left(\sum_{n=0}^{||\beta||-1}\sum_{j=0}^{n} p_j(\widetilde{\mathbf{D}}_{\mathbf{x}})\mu_{n-j}q^{n-2}\right)\tau'(\mathbf{x})\cdot\tau'(\mathbf{x})\right]=0,$$

(bilinear KP)

- trigonometric/hyperbolic:

$$\tau' = e^{\frac{1}{6}(\alpha x_1)^2}\sin(\alpha x_1)\,\tau_N(\mathbf{x}),$$

$$\tau_N(\mathbf{x}) = \sum_{J\subset S}\left(\prod_{i\in J} c_i'\right)\left(\prod_{i<j\in J} A_{ij}'\right)\frac{\sin(\alpha(x_1+\sum_{i\in J}(k_i-l_i)))}{\sin(\alpha x_1)\prod_{i\in J}\sin(\alpha(k_i-l_i))}$$
$$\times\exp\left(\sum_{i\in J}(\xi_{[t]}(\mathbf{x},k_i)-\xi_{[t]}(\mathbf{x},l_i))\right),$$

where

$$\xi_{[t]}(\mathbf{x},k)=\alpha\sum_{n=1}^{\infty}(-1)^n x_n\frac{\partial_k^{n-1}\cot(\alpha k)}{(n-1)!},$$

$$A_{ij}'=\frac{\sin(\alpha(k_i-k_j))\sin(\alpha(l_i-l_j))}{\sin(\alpha(k_i-l_j))\sin(\alpha(l_i-k_j))}.$$

II-4: Degeneration by period

$$\operatorname*{Res}_{q=0}\left[(\mathbf{B}+\mathbf{D_x})^{\beta}|_{\leq 1}\left(\sum_{n=0}^{||\beta||-1}\sum_{j=0}^{n}p_j(\widetilde{\mathbf{D}}_{\mathbf{x}})\mu_{n-j}q^{n-2}\right)\tau'(\mathbf{x})\cdot\tau'(\mathbf{x})\right]=0,$$

(bilinear KP)

- rational:

$$\tau' = x_1\,\tau_N(\mathbf{x}),$$

$$\tau_N(\mathbf{x}) = \sum_{J\subset S}\left(\prod_{i\in J}c_i\right)\left(\prod_{i<j\in J}A_{ij}\right)\frac{x_1+\sum_{i\in J}(k_i-l_i)}{x_1\prod_{i\in J}(k_i-l_i)}$$
$$\times\exp\left(\sum_{i\in J}(\xi_{[r]}(\mathbf{x},k_i)-\xi_{[r]}(\mathbf{x},l_i))\right).$$

where

$$\xi_{[r]}(\mathbf{x},k) = -\sum_{n=1}^{\infty}\frac{1}{k^n}x_n,\quad A_{ij}=\frac{(k_i-k_j)(l_i-l_j)}{(k_i-l_j)(l_i-k_j)}$$

II-5: Discrete KdV and KP [Xing Li, ZDJ (2023)]

- Bilinearisation

$$(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})=p^2-q^2, \quad p^2=\wp(\delta)-e_0, \quad q^2=\wp(\varepsilon)-e_0$$

$$u=\zeta(\xi+N\gamma)-N\zeta(\gamma)-n\zeta(\delta)-m\zeta(\varepsilon)-h\zeta(\gamma)-\zeta(\xi_0)+\frac{g}{f}$$

where $\xi=n\delta+m\varepsilon+h\gamma$,

$$\mathcal{H}_1\equiv\chi_{\delta,-\varepsilon}(\widehat{\xi}+N\gamma)\widetilde{f}\widehat{f}+\widetilde{f}\widehat{g}-\widetilde{g}\widehat{f}-\Phi_\delta(-\varepsilon)f\widehat{\widetilde{f}}=0,$$

$$\mathcal{H}_2\equiv\chi_{\delta,\varepsilon}(\xi+N\gamma)f\widehat{\widetilde{f}}+\widehat{\widetilde{f}}g-\widehat{\widetilde{g}}f-\Phi_\delta(\varepsilon)\widetilde{f}\widehat{f}=0,$$

$$\chi_{u,v}(z)=\zeta(u)+\zeta(v)+\zeta(z)-\zeta(u+v+z)$$

- NSS:

$$f=\sigma(\xi)|\widehat{N-1}|, \qquad g=\sigma(\xi)|\widehat{N-2},N|,$$

$$\phi=(\phi_1,\cdots,\phi_N)^T,\ \phi_i=\rho^-_{n,m,h}(k_i)\Phi_\xi(k_i)+\rho^-_{n,m,h}(l_i)\Phi_\xi(l_i),$$

$$\rho^\pm_{n,m,h}(z)=\left(\frac{\sigma(\delta\pm z)}{\sigma(\delta)\sigma(\pm z)}\right)^n\left(\frac{\sigma(\varepsilon\pm z)}{\sigma(\varepsilon)\sigma(\pm z)}\right)^m\left(\frac{\sigma(\gamma\pm z)}{\sigma(\gamma)\sigma(\pm z)}\right)^h\rho^\pm_{0,0,0},$$

lpKdV: τ function and vertex operator

- τ function

$$\tau_N = \sum_{J \subset S} \frac{\sigma(\xi + 2\sum_{i\in J} k_i)}{\sigma(\xi)\prod_{i\in J}\sigma(2k_i)} \left(\prod_{\substack{i,j\in J \\ i<j}} A_{ij} \right) \prod_{i\in J} \rho_{n,m,h}(k_i),$$

$$A_{ij} = \left(\frac{\sigma(k_i - k_j)}{\sigma(k_i + k_j)} \right)^2$$

$$\rho_{n,m,h}(k_i) = \left(\frac{\sigma(k_i - \delta)}{\sigma(k_i + \delta)} \right)^n \left(\frac{\sigma(k_i - \varepsilon)}{\sigma(k_i + \varepsilon)} \right)^m \left(\frac{\sigma(k_i - \gamma)}{\sigma(k_i + \gamma)} \right)^h \rho_{0,0,0}(k_i).$$

- vertex operator: same as for the continuous KdV

$$\rho = e^{\theta_{[e]}(\bar{\mathbf{t}},k)}, \quad \theta_{[e]}(\bar{\mathbf{t}}, k) = -2\sum_{n=0}^{\infty} \frac{\zeta^{(2n)}(k)}{(2n)!} t_{2n+1}$$

after redefining $c_i = \left(\frac{\sigma(k_i-\gamma)}{\sigma(k_i+\gamma)} \right)^h \rho_{0,0,0}(k_i)$ and introducing Miwa's coordinates

$$t_{2j+1} = \frac{\delta^{2j+1} n + \varepsilon^{2j+1} m}{2j+1}.$$

lpKP

- lpKP

$$(\widehat{\overline{w}} - \widehat{\widetilde{w}})(\overline{w} - \widehat{w}) = (\widetilde{\overline{w}} - \widehat{\widetilde{w}})(\overline{w} - \widetilde{w}), \qquad \text{(lpKP)}$$

$$w = \zeta(\xi + N\gamma) - N\zeta(\gamma) - n\zeta(\delta) - m\zeta(\varepsilon) - h\zeta(\gamma) - \zeta(\xi_0) + \frac{g}{f},$$

$$\mathcal{H}_1 \equiv \chi_{\varepsilon,-\gamma}(\xi + (N+1)\gamma)\overline{f}\widehat{f} + \overline{g}\widehat{f} - \widehat{g}\overline{f} - \Phi_\varepsilon(-\gamma)f\widehat{\overline{f}} = 0,$$

$$\mathcal{H}_2 \equiv \chi_{\delta,-\gamma}(\xi + (N+1)\gamma)\widetilde{f}\overline{f} + \overline{g}\widetilde{f} - \widetilde{g}\overline{f} - \Phi_\delta(-\gamma)f\widetilde{\overline{f}} = 0.$$

- Casoratian solution
- τ function
- vertex operator: same as for the continuous KP in light of Miwa's coordinates
- discrete AKP reduction (cf. Jing Wang, DJZ, K Maruno (2024))

III Direct linearisation (DL) approach

- Elliptic DL scheme
- Marchenko equation

III-1: Elliptic scheme of DL approach

- DL for the KdV [Fokas, Ablowitz (1981)]

$$\varphi(x,t;k) + ie^{i(kx+k^3t)} \int_L \frac{\varphi(x,t;l)}{l+k} d\lambda(l) = e^{i(kx+k^3t)},$$

$u = -\partial_x \int_L \varphi(x,t;l) d\lambda(l)$, Lax pair is needed.

- DL+: infinite matrix [Nijhoff, Qiuspel, Caple, et al. (1980s)]

$$\mathbf{u}_k + \rho_k \iint_D d\lambda(l,l') \mathbf{u}_l \sigma_{l'} \Omega_{k,l'} = \rho_k \mathbf{c}_k,$$

$$\mathbf{c}_k = (\cdots, k^{-1}, 1, k, \cdots)^T, \quad \Omega_{k,l'} = \frac{1}{k+l'},$$

$$\mathbf{U} = \iint_D d\lambda(k,k') \mathbf{u}_k \, {}^t\mathbf{c} \, \sigma_{k'}$$

$$\text{Dis PWF} : \rho_k = \prod_{j=1} (p_j + k)^{n_j}, \quad \sigma_{k'} = \prod_{j=1} (p_j - k')^{-n_j}$$

$$\text{Cont PWF} : \rho_k = e^{\sum_{j=1}^{\infty} k^j t_j}, \quad \sigma_{k'} = e^{-\sum_{j=1}^{\infty} k'^j t_j}$$

no need to check Lax pair

Elliptic scheme of DLA: [Nijhoff, Sun, DJZ (2023)]

- Scheme:

$$\mathbf{u}_\kappa + \rho_\kappa \iint_D d\mu(\ell,\ell')\sigma_{\ell'}\mathbf{u}_\ell \Phi_\xi(\kappa+\ell') = \rho_\kappa \Phi_\xi(\boldsymbol{\Lambda})\mathbf{c}_\kappa$$

$$\mathbf{U}_\xi \ := \ \iint_D d\mu(\ell,\ell')\mathbf{u}_\ell(\xi)\,{}^t\mathbf{c}_{\ell'}\sigma_{l'}\Phi_\xi({}^t\boldsymbol{\Lambda})$$

$$\Phi_\xi(x) := \frac{\sigma(x+\xi)}{\sigma(x)\sigma(\xi)}, \quad \boldsymbol{\Lambda}\mathbf{c}_\kappa = \kappa\mathbf{c}_\kappa, \quad \xi = \xi_0 - n\delta - m\varepsilon - l\nu$$

$$\rho_\kappa(n,m,l) = (\Phi_\delta(\kappa))^n(\Phi_\varepsilon(\kappa))^m(\Phi_\nu(\kappa))^l\rho_\kappa(0,0,0)$$

$$\sigma_{\kappa'}(n,m,l) = (\Phi_\delta(-\kappa'))^{-n}(\Phi_\varepsilon(-\kappa'))^{-m}(\Phi_\nu(-\kappa'))^{-l}\sigma_{\kappa'}(0,0,0)$$

- lattice KP: $u(\xi) := (\mathbf{U}_\xi)_{0,0}$

$$\begin{aligned}&[\zeta(\delta)-\zeta(\varepsilon)+\zeta(\xi-\delta)-\zeta(\xi-\varepsilon)]\,\widehat{\widetilde{u}}(\xi-\delta-\varepsilon)\\ &\quad+[\zeta(\nu)-\zeta(\delta)+\zeta(\xi-\varepsilon-\nu)-\zeta(\xi-\delta-\varepsilon)]\,\widehat{u}(\xi-\varepsilon)\\ &\qquad+\left(\widehat{\widetilde{u}}(\xi-\delta-\varepsilon)-\widehat{\overline{u}}(\xi-\varepsilon-\nu)\right)\widehat{u}(\xi-\varepsilon)\ +\ \text{cycl.}\ =0\ ,\end{aligned}$$

Elliptic scheme of DLA:

- lattice mKP/SKP: variables:

$$v_\alpha(\xi) = 1 - \left([\zeta(\xi) + \zeta(\alpha) + \zeta(\mathbf{\Lambda}) - \zeta(\xi + \alpha + \mathbf{\Lambda})]^{-1} \mathbf{U}_\xi \right)_{0,0}$$

$$w_\alpha(\xi) = 1 - \left(\mathbf{U}_\xi [\zeta(\xi) + \zeta(\alpha) + \zeta({}^t\mathbf{\Lambda}) - \zeta(\xi + \alpha + {}^t\mathbf{\Lambda})]^{-1} \right)_{0,0}$$

$$s_{\alpha,\beta}(\xi) = \Big([\zeta(\xi) + \zeta(\alpha) + \zeta(\mathbf{\Lambda}) - \zeta(\xi + \alpha + \mathbf{\Lambda})]^{-1} \cdot \mathbf{U}_\xi \\ \cdot [\zeta(\xi) + \zeta(\beta) + \zeta({}^t\mathbf{\Lambda}) - \zeta(\xi + \beta + {}^t\mathbf{\Lambda})]^{-1} \Big)_{0,0}$$

- lattice mKP:

$$[\zeta(\delta) - \zeta(\xi - \varepsilon) - \zeta(\alpha) + \zeta(\xi + \alpha - \delta - \varepsilon)] \frac{\widehat{v}_\alpha(\xi - \varepsilon)}{\widehat{\widetilde{v}}_\alpha(\xi - \delta - \varepsilon)} \\ - [\zeta(\varepsilon) - \zeta(\xi - \delta) - \zeta(\alpha) + \zeta(\xi + \alpha - \delta - \varepsilon)] \frac{\widetilde{v}_\alpha(\xi - \delta)}{\widehat{\widetilde{v}}_\alpha(\xi - \delta - \varepsilon)} \\ + \text{ cycl. } = 0$$

- lattice SKP:

$$
\begin{aligned}
&\frac{1-\chi^{(1)}_{\alpha,-\delta}(\xi-\nu)\overline{s}_{\alpha,\beta}(\xi-\nu)-\chi^{(1)}_{\beta,\delta}(\xi-\delta-\nu)\widetilde{\overline{s}}_{\alpha,\beta}(\xi-\delta-\nu)}{1-\chi^{(1)}_{\alpha,-\varepsilon}(\xi-\nu)\overline{s}_{\alpha,\beta}(\xi-\nu)-\chi^{(1)}_{\beta,\varepsilon}(\xi-\varepsilon-\nu)\widehat{\overline{s}}_{\alpha,\beta}(\xi-\varepsilon-\nu)} \\
=\;&\frac{1-\chi^{(1)}_{\alpha,-\delta}(\xi-\varepsilon)\widehat{s}_{\alpha,\beta}(\xi-\varepsilon)-\chi^{(1)}_{\beta,\delta}(\xi-\delta-\varepsilon)\widehat{\widetilde{s}}_{\alpha,\beta}(\xi-\delta-\varepsilon)}{1-\chi^{(1)}_{\alpha,-\nu}(\xi-\varepsilon)\widehat{s}_{\alpha,\beta}(\xi-\varepsilon)-\chi^{(1)}_{\beta,\nu}(\xi-\varepsilon-\nu)\widehat{\overline{s}}_{\alpha,\beta}(\xi-\varepsilon-\nu)} \\
&\times\frac{1-\chi^{(1)}_{\alpha,-\nu}(\xi-\delta)\widetilde{s}_{\alpha,\beta}(\xi-\delta)-\chi^{(1)}_{\beta,\nu}(\xi-\delta-\nu)\widetilde{\overline{s}}_{\alpha,\beta}(\xi-\delta-\nu)}{1-\chi^{(1)}_{\alpha,-\varepsilon}(\xi-\delta)\widetilde{s}_{\alpha,\beta}(\xi-\delta)-\chi^{(1)}_{\beta,\varepsilon}(\xi-\delta-\varepsilon)\widehat{\widetilde{s}}_{\alpha,\beta}(\xi-\delta-\varepsilon)}
\end{aligned}
$$

where

$$\chi^{(1)}_{\delta,\varepsilon}(\gamma)=\zeta(\delta)+\zeta(\varepsilon)+\zeta(\gamma)-\zeta(\delta+\varepsilon+\gamma)$$

DBSQ:

- elements

$$u_{0,0} := (\mathbf{U}_\xi)_{0,0} \ , \quad u_{1,0} := (\mathbf{\Lambda}_\xi \mathbf{U}_\xi)_{0,0} \ , \quad u_{0,1} := \left(\mathbf{U}_\xi\, {}^t\mathbf{\Lambda}_\xi\right)_{0,0}$$

- DBSQ

$$\begin{aligned}
&p_\xi \widetilde{u_{0,0}} + \widetilde{u_{0,1}} = p_\xi u_{0,0} - u_{1,0} - \widetilde{u_{0,0}} u_{0,0} \ , \\
&q_\xi \widehat{u_{0,0}} + \widehat{u_{0,1}} = q_\xi u_{0,0} - u_{1,0} - \widehat{u_{0,0}} u_{0,0} \ , \\
&\frac{1}{2} \frac{\wp'(\delta) - \wp'(\varepsilon)}{p_\xi - q_\xi + \widehat{u_{0,0}} - \widetilde{u_{0,0}}} = \frac{1}{2} \frac{\wp'(\delta) - \wp'(\varepsilon)}{p_\xi - q_\xi} + \widehat{\widetilde{u_{1,0}}} + u_{0,1} + u_{0,0} \widehat{\widetilde{u_{0,0}}} \\
&\qquad\qquad\qquad\qquad + (p_{\widehat{\xi}} + q_\xi)(\widehat{\widetilde{u_{0,0}}} - u_{0,0}) \ ,
\end{aligned}$$

- deformation

$$\begin{aligned}
&\widetilde{w} - u\widetilde{u} + v = 0 \ , \quad \widehat{w} - u\widehat{u} + v = 0 \ , \\
&\frac{1}{2} \frac{\wp'(\delta) - \wp'(\varepsilon)}{\widehat{u} - \widetilde{u}} = w - u\widehat{\widetilde{u}} + \widehat{\widetilde{v}}
\end{aligned}$$

- transformation:

$$u_{0,0} = x_0 - u \ , \quad u_{1,0} = y_0 - v - x_0 u_{0,0} \ , \quad u_{0,1} = z_0 - w - x_0 u_{0,0} \ ,$$

where

$$\begin{aligned} x_0 &= \zeta(\xi) + n\zeta(\delta) + m\zeta(\varepsilon) - \zeta(\xi_0) \ , \quad \xi = \xi_0 - n\delta - m\varepsilon, \\ y_0 &= \frac{1}{2}x_0^2 - \frac{1}{2}\wp(\xi) + \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)) \ , \\ z_0 &= \frac{1}{2}x_0^2 - \frac{1}{2}\wp(\xi) - \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)) \end{aligned}$$

- 1-component form (9-point)

$$\begin{aligned} &\frac{\frac{1}{2}(\wp'(\delta) - \wp'(\varepsilon))}{\widehat{\widetilde{u}} - \widetilde{\widetilde{u}}} - \frac{\frac{1}{2}(\wp'(\delta) - \wp'(\varepsilon))}{\widehat{\widehat{u}} - \widehat{\widetilde{u}}} \\ = \ &(\widehat{\widehat{\widetilde{u}}} - \widehat{\widetilde{\widetilde{u}}})(\widehat{u} - \widehat{\widehat{\widetilde{\widetilde{u}}}}) - (\widehat{u} - \widetilde{u})(u - \widehat{\widetilde{\widetilde{u}}}) \end{aligned}$$

- There are other DBSQs. [Nijhoff, Sun, DJZ (2023)]

III-2: Marchenko equation

- DLA for the KdV [Fokas, Ablowitz (1981)]

$$\varphi(x,t;k) + ie^{i(kx+k^3t)} \int_L \frac{\varphi(x,t;l)}{l+k} d\lambda(l) = e^{i(kx+k^3t)},$$

$u = -\partial_x \int_L \varphi(x,t;l) d\lambda(l)$, Lax pair is needed.

- DLA to GLM:

$$\psi(x,t;k) = \varphi(x,t;k)e^{-i(kx+k^3t)},$$

$$K(x,y,t) = -\frac{1}{2}\int_L \psi(x,t;k)e^{i(ky+k^3t)} d\lambda(k),$$

$$F(x,t) = \int_L e^{i(ky+k^3t)} d\lambda(k),$$

$$K(x,y,t) + F(x+y,t) + \int_x^{+\infty} K(x,\xi;t)F(y+\xi,t)d\xi = 0,$$

$$u = 2\partial_x K(x,x;t).$$

Fokas-Ablowitz's DLA (elliptic)

- KP:

$$u_t + u_{xxx} + 6uu_x + 3\partial_x^{-1} u_{yy} = 0$$

- Elliptic DLA for the KP

$$\psi(x,y,t;k) + \rho_k(y,t) \iint_D \psi(x,y,t;l)\sigma_{l'}(y,t)\Psi_x(k,l')d\lambda(l,l') \\ = \Psi_x(k)\rho_k(y,t),$$

$$u(x,y,t) = -2\wp(x) - 2\partial_x \iint_D \psi(x,y,t;l)\sigma_{l'}(x,y,t)\Phi_x(l')d\lambda(l,l').$$

- notations:

$$\rho_k(x,y,t) = \exp\Big(\wp(k)y - 2\wp'(k)t + \rho^{(0)}(k)\Big),$$

$$\Psi_x(k) = \frac{\sigma(x+k)}{\sigma(x)\sigma(k)} e^{-\zeta(k)x} = \Phi_x(k)e^{-\zeta(k)x}.$$

- Lax pair

$$P\,\psi(x,y,t) = 0, \quad P = \partial_y - \partial_x^2 - u(x,y,t),$$

$$M\,\psi(x,y,t) = 0, \quad M = \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x + 3\partial_x^{-1}u_y.$$

Marchenko equation

- Elliptic DLA for the KP: deformation

$$\psi(x,y,t;k)+\rho_k(y,t)\iint_D \psi(x,y,t;l)\sigma_{l'}(y,t)\Psi_x(k,l')d\lambda(l,l') = \Psi_x(k)\rho_k(y,t),$$

- real valued: $w_1 > 0$, w_2 purely imaginary

$$\psi(x,y,t;k)+\rho_k(y,t)\iint_D \psi(x,y,t;l)\sigma_{l'}(y,t)\widetilde{\Psi}_x(k,l')d\lambda(l,l') = \widetilde{\Psi}_x(k)\rho_k(y,t),$$

$$\widetilde{\Psi}_x(a) = \frac{\sigma(x+a+w_2)}{\sigma(x+w_2)\sigma(a)}e^{-\zeta(a)x-\zeta(w_2)a},$$

$$\widetilde{\Psi}_x(a,b) = \frac{\sigma(x+a+b+w_2)}{\sigma(x+w_2)\sigma(a+b)}e^{-(\zeta(a)+\zeta(b))x-\zeta(w_2)(a+b)},$$

- nonsingular, $\widetilde{\Psi}_x(k,l)$ exponentially decays when $x \to +\infty$,

$$\int_x^{+\infty} \widetilde{\Psi}_\xi(k)\widetilde{\Psi}_\xi(l')d\xi = \widetilde{\Psi}_x(k,l).$$

Marchenko equation

- Elliptic DLA for the KP

$$\psi(x,y,t;k)+\rho_k(y,t)\iint_D \psi(x,y,t;l)\sigma_{l'}(y,t)\widetilde{\Psi}_x(k,l')d\lambda(l,l') = \widetilde{\Psi}_x(k)\rho_k(y,t),$$

- Marchenko equation

$$K(x,s,y,t)+F(x,s,y,t)+\int_x^{+\infty} K(x,\xi,y,t)F(\xi,s,y,t)d\xi=0,$$

$$K(x,s,y,t) = -\iint_D \psi(x,y,t;k)\widetilde{\Psi}_s(k')\sigma_{k'}(y,t)d\lambda(k,k'),$$

$$F(x,s,y,t) = \iint_D \widetilde{\Psi}_x(k)\rho_k(y,t)\widetilde{\Psi}_s(k')\sigma_{k'}(y,t)d\lambda(k,k'),$$

- solution

$$u(x,y,t) = -2\wp(x+w_2)+2\frac{\partial}{\partial x}K(x,x,y,t).$$

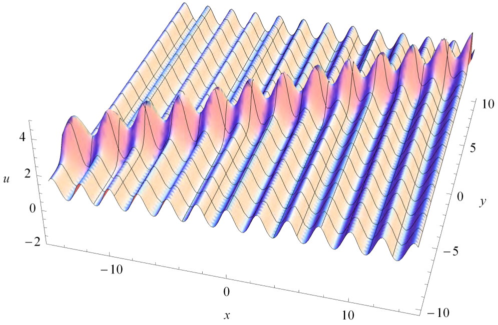


Figure: Elliptic 1-soliton of KP

IV Related problems

- Discrete Krichever-Novikov equation:

$$p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-r(u\widehat{\widetilde{u}}+\widetilde{u}\widehat{u})+pqr(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})=0$$

$$(p,P)=(\sqrt{k}\,\mathrm{sn}(\alpha;k),\mathrm{sn}'(\alpha;k)),\quad (q,R)=(\sqrt{k}\,\mathrm{sn}(\beta;k),\mathrm{sn}'(\beta;k))$$
$$(r,R)=(\sqrt{k}\,\mathrm{sn}(\gamma;k),\mathrm{sn}'(\gamma;k)),\quad \gamma=\alpha-\beta$$

- Algebras for the vertex operators?

$$X(k)=\Phi_{t_1}(2k)e^{\theta_{[e]}(\bar{\mathbf{t}},k)}e^{\theta(\bar{\widetilde{\partial}},k)}$$
$$\tau_N(\bar{\mathbf{t}})=e^{c_N X(k_N)}\tau_{N-1}(\bar{\mathbf{t}}),\quad \tau_0(\bar{\mathbf{t}})=1,$$

- High order Boussinesq?

$$\begin{aligned}&\wp^{(4)}(\omega_1(\delta))-\wp^{(4)}(\delta)\\ =\;&30(\wp'(\omega_1(\delta))-\wp'(\delta))(\wp'(\omega_1(\delta))+\wp'(\delta))+12g_2(\wp(\omega_1(\delta))-\wp(\delta)).\end{aligned}$$

- New formulation of KP?

New formulation of KP?

Saburo Kakei, Solutions to the KP hierarchy with an elliptic background [2310.11679]

$$L_0 = V_0 \partial_x^{-1} V_0^{-1},$$

$$V_0 = 1 + \sum_{n=0}^{\infty} v_{0,n}(x) \partial_x^{-1},$$

$$v_{0,1}(x) = -\zeta(x), \quad v_{0,2}(x) = \frac{1}{2}[\zeta^2(x) - \wp(x)], \quad \cdots .$$

$$L_0 = \partial_x - \wp(x)\partial_x^{-1} + \frac{1}{2}\wp'(x)\partial_x^{-2} + \cdots ,$$

$$L_0^2 = \partial_x^2 - 2\wp(x) - \frac{g_2}{10}\partial_x^{-2} - 6\wp(x)\wp'(x)\partial_x^{-3} + \cdots ,$$

Thank You!

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