Dimers and M-Curves

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Integrable Systems and Algebraic Geometry,
BIMSA, Beijing, June 2024

joint with Nikolai Bobenko and Yuri Suris

Papers

- Dimers and M-Curves [AB-Bobenko-Suris, 2024]
- Dimers and M-Curves. Limit Shapes from Riemann Surfaces [AB-Bobenko, 2024+]

Setup

The Dimer Model

- Measure on dimer configurations (perfect matchings):

$$\mathbb{P}(D) = \frac{1}{Z} \prod_{e \in D} \nu(e).$$

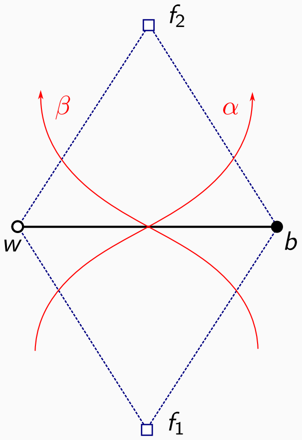
- Physical weights are face weights:

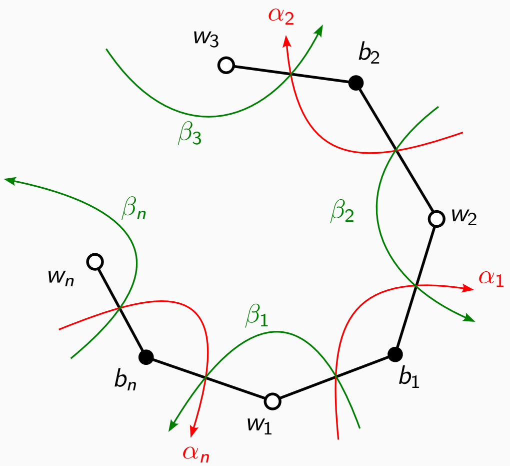
$$W_f = \frac{\nu(e_1)\nu(e_3)\dots\nu(e_{2n-1})}{\nu(e_2)\nu(e_4)\dots\nu(e_{2n})}.$$

- Kasteleyn condition:

$$\text{sign}(W_f) = (-1)^{(n+1)}.$$

Quad Graph and Train Tracks

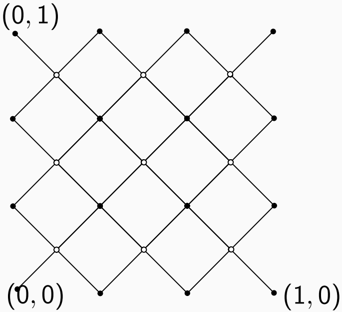




Direct problem

Planar bipartite doubly periodic graph G.

- Doubly periodic weights
- Kasteleyn matrix K



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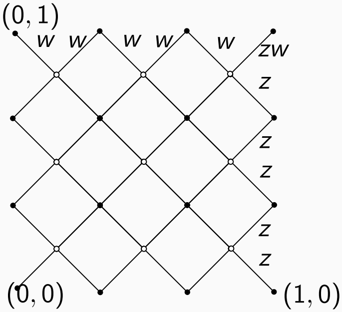
- Spectral curve $\det K(z,w) = 0$

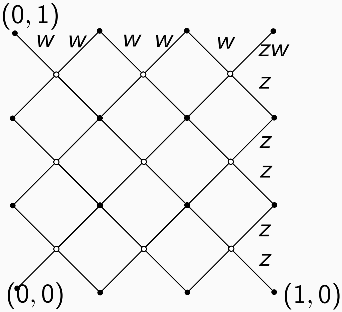


$$\det K(z,w) = 0, \quad K(z,w)\psi(P) = 0, \quad P = (z,w).$$

Direct problem

Planar bipartite doubly periodic graph G.

- Doubly periodic weights
- Kasteleyn matrix K
- Spectral curve $\det K(z,w) = 0$
- Eigenfunction $\psi(P)$, $P = (z,w)$
- Monodromies of $\psi(P)$ are z, w
- Analytic properties of $\psi(P)$ on spectral curve



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Direct and inverse problems

Planar bipartite periodic graph G.

- **Direct problem**:
  Weights $\Rightarrow$ Kasteleyn matrix $\Rightarrow$ Spectral curve $\Rightarrow$ Eigenfunction $\psi(P)$
- **Inverse problem**:
  Spectral curve (Riemann surface) and analytic properties of $\psi(P)$ $\Rightarrow$ Explicit representation for $\psi(P)$ $\Rightarrow$ Weights

More general: Quasiperiodic weights that include all periodic weights

Inverse problem. KP equation. Krichever's scheme 1977

- KP equation

$$3u_{yy} = \partial_x(4u_t - 6uu_x - u_{xxx})$$

- Baker-Akhiezer (BA) function

$$\psi(x,y,t;P) = \frac{\theta(A(P) + U_1x + U_2y + U_3t + D)\theta(D)}{\theta(A(P) + D)\theta(U_1x + U_2y + U_3t + D)} \exp(\xi_1(P)x + \xi_2(P)y + \xi_3(P)t)$$

- Analytic properties: meromorphic on $\mathcal{R} - P_0$ with essential singularity at P_0

$$\psi(P) = (1 + o(1/k)) \exp\left(kx + k^2y + k^3t\right), k \to \infty, P \to P_0$$

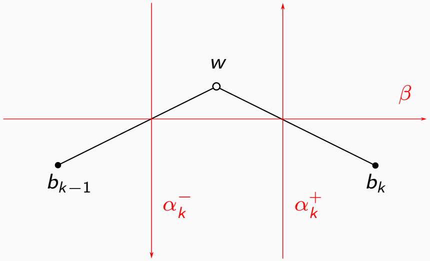
  and pole divisor of degree genus of $\mathcal{R}$

- Explicit solution:

$$u(x,y,z) = 2\partial_x^2 \log \theta(U_1x + U_2y + U_3t + D)$$

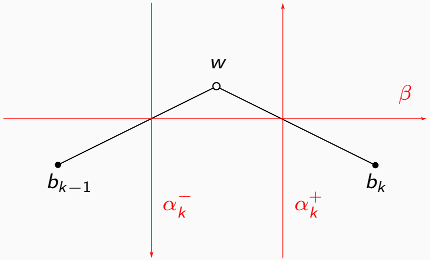
Dimer inverse problem. BA function

- **Data:** Compact Riemann surface $\mathcal{R}$, train track parameters $\alpha_i \in \mathcal{R}$.



Dimer inverse problem. BA function

- **Data:** Compact Riemann surface $\mathcal{R}$, train track parameters $\alpha_i \in \mathcal{R}$.
- Meromorphic function ψ on $\mathcal{R}$ (BA-function). Function $\psi_b : \mathcal{R} \to \mathbb{C}$ on every black vertex b.
- ψ picks up a zero or a pole at $\alpha \in \mathcal{R}$ whenever crossing a train track.



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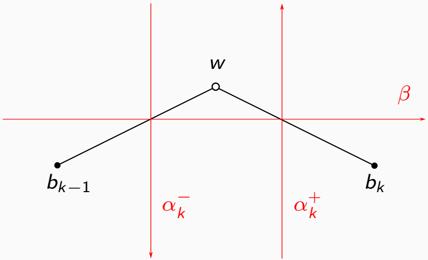
$$\psi_b(P) = \frac{\theta\big(A(P) + \eta(b) + D\big)}{\theta\big(A(P) + \eta(b_0) + D\big)} \prod \frac{E(P, \alpha_k^-)}{E(P, \alpha_k^+)}$$

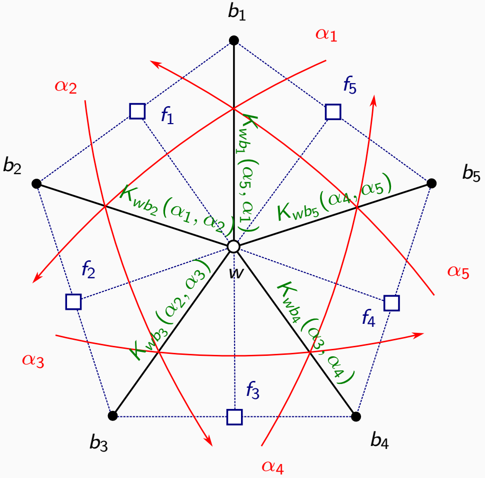
$$\eta(b) - \eta(b_0) = \sum (A(\alpha_k^- - \alpha_k^+))$$



Dimer inverse problem. Fock weights

- Dirac equation

$$\sum_{k=1}^{n} K_{wb_k}(\alpha_{k-1}, \alpha_k)\psi_{b_k}(P) = 0,$$



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- Fock weights

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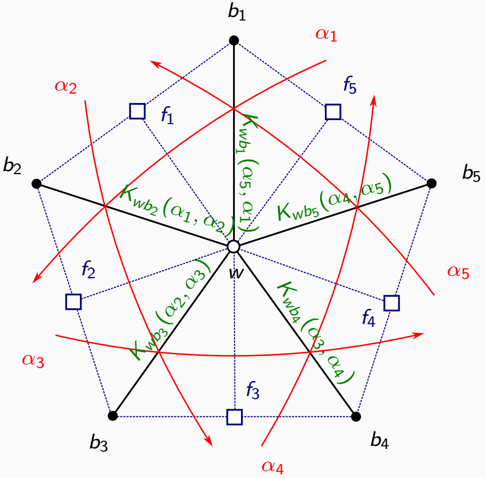
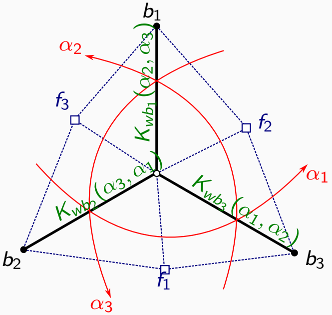


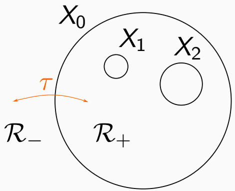
Fay identity. Dirac operator on a triangle



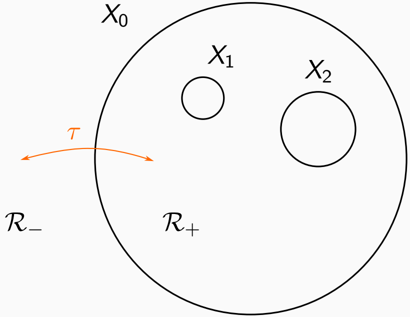
$$\begin{aligned}
&\theta\big(A(\alpha_2)+A(\alpha_3)+D\big)\theta\big(A(P)+A(\alpha_1)+D\big)E(\alpha_2,\alpha_3)E(P,\alpha_1)\\
&+\theta\big(A(\alpha_1)+A(\alpha_3)+D\big)\theta\big(A(P)+A(\alpha_2)+D\big)E(\alpha_3,\alpha_1)E(P,\alpha_2)\\
&+\theta\big(A(\alpha_1)+A(\alpha_2)+D\big)\theta\big(A(P)+A(\alpha_3)+D\big)E(\alpha_1,\alpha_2)E(P,\alpha_3) \quad = \quad 0,
\end{aligned}$$

Reality conditions

- $\mathcal{R}$ M-curve with antiholomorphic involution τ and fixed ovals $X_0, \ldots, X_g$.



M-curve



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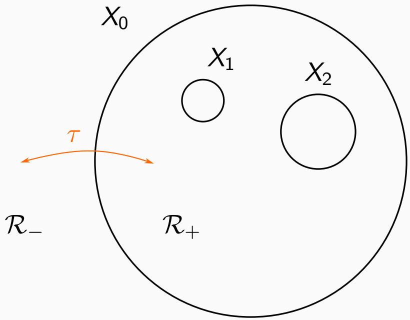
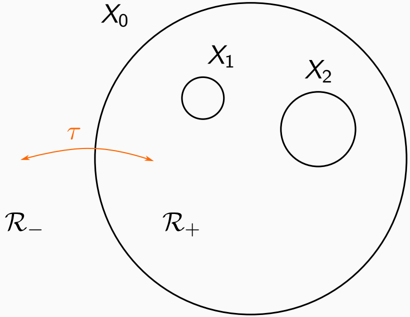


This is a Riemann surface used for computations and not just an illustration

M-curve



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All figures in this talk are not hand drawn but computed

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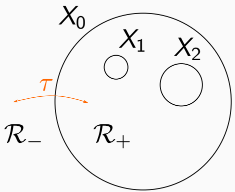

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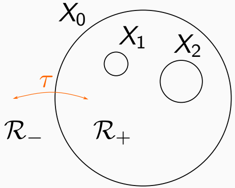

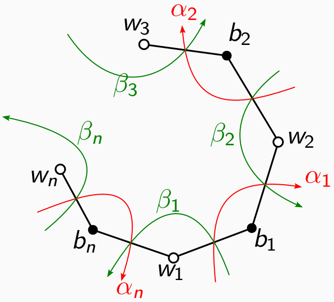

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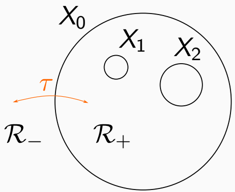
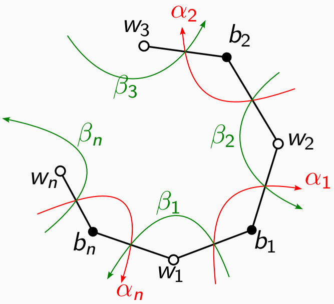
- Ordering condition on α_i, β_i gives

$$\text{sign}(W_f) = (-1)^{(n+1)}.$$

In general notion of minimal graphs [Boutillier, Cimasoni, de Tilière - '20, '21].





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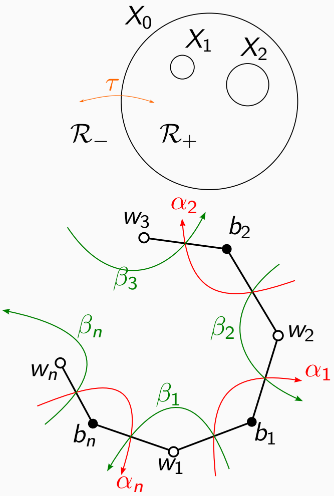
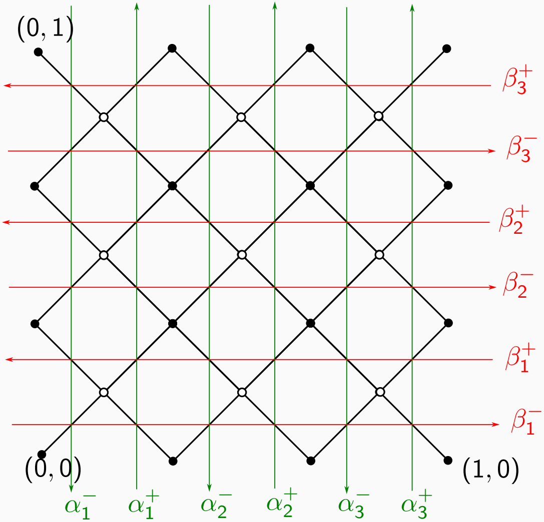
- For $g = 0$ these are isoradial weights. [Kenyon '02, Kenyon-Okounkov '03]

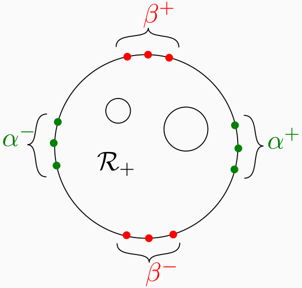


Kasteleyn weights: square grid

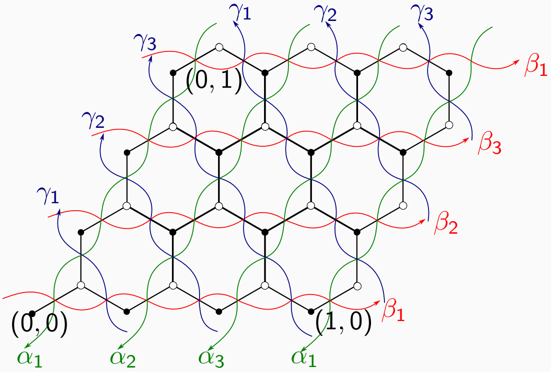


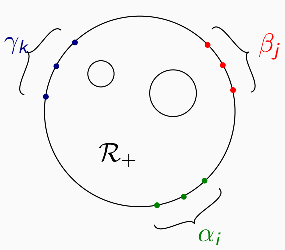
Kasteleyn

$\Updownarrow$

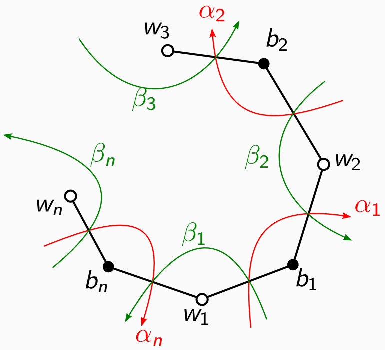


Kasteleyn weights: hex grid



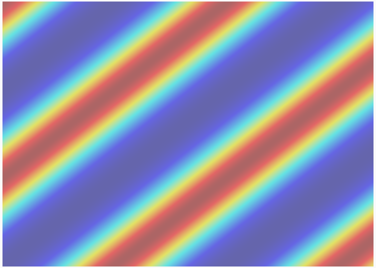


Fock weights. Formulas

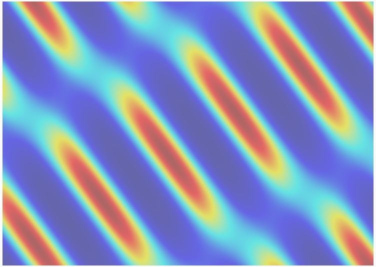


$$W_f = \prod_{i=1}^{n} \frac{\theta[\Delta]\left(\int_{\alpha_i}^{\beta_i} \omega\right)}{\theta[\Delta]\left(\int_{\beta_i}^{\alpha_{i+1}} \omega\right)} \frac{\theta\left(\eta(f_{2i}) + D\right)}{\theta\left(\eta(f_{2i-1}) + D\right)}$$

Quasi-periodic weights



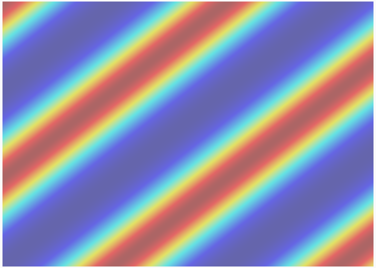
(a) W_f for $g = 1$



(b) W_f for $g = 2$

$g = 0$: isoradial weights, rational functions

Quasi-periodic weights



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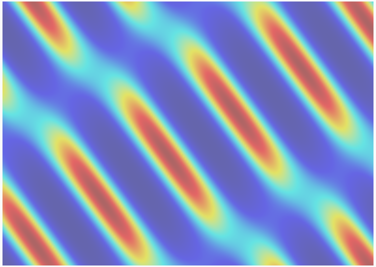

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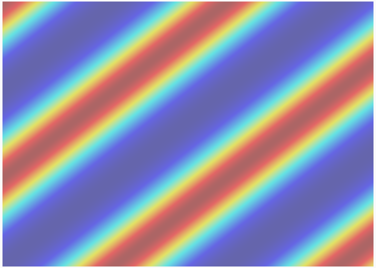

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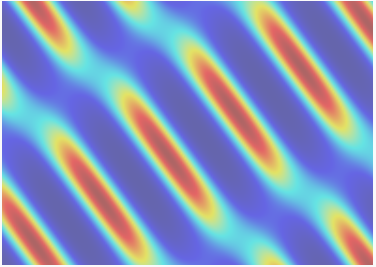

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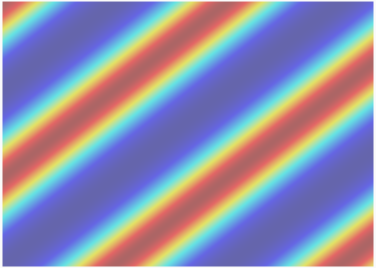
Weights periodic iff $\sum_i(\alpha_i^- - \alpha_i^+), \sum_i(\beta_i^- - \beta_i^+)$ principal divisors.

Quasi-periodic weights



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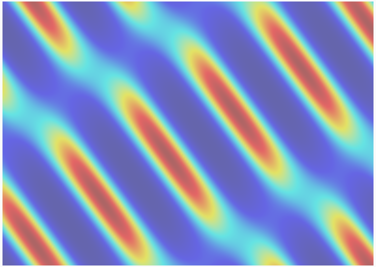

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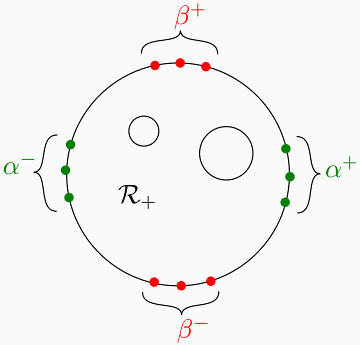
$g > 0$: Train track parameters repeat periodically $\not\Rightarrow$ weights K_{wb}, W_f periodic.

Algebro-geometric description

Amoeba and polygon map

- **Goal**: Describe limiting objects algebro-geometrically.
- **Data** $\mathcal{S}$: M-curve $\mathcal{R}$ with antiholomorphic involution τ. $\{\alpha_i^\pm, \beta_j^\pm\} \in X_0$ with clustering condition.



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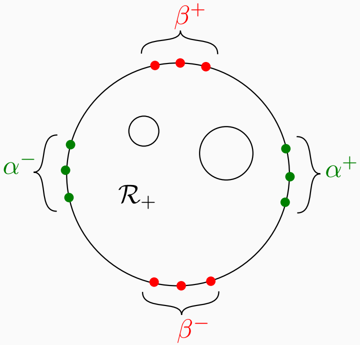
- $\zeta_k(P) = \int^P d\zeta_k = x_k + iy_k$ well defined on $\mathcal{R}_+$.



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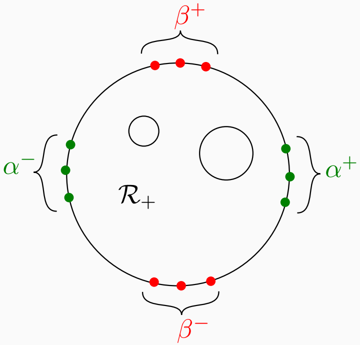
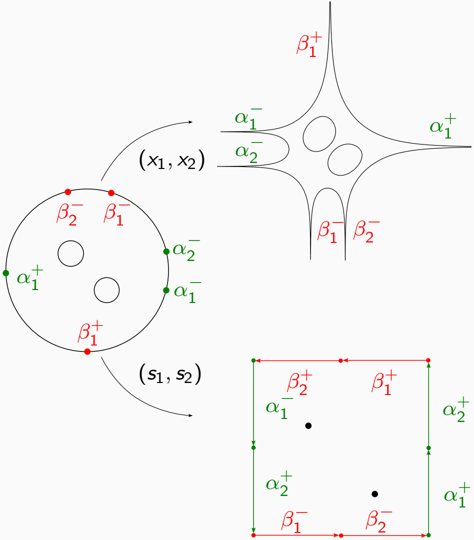
- $\zeta_k(P) = \int^P d\zeta_k = x_k + iy_k$ well defined on $\mathcal{R}_+$.
- **Proposition**: $(x_1, x_2), (y_1, y_2)$ coordinates of $\mathcal{R}_+^\circ$. [Krichever '14]



$(s_1, s_2) = \frac{1}{\pi}(y_2, -y_1)$.

Amoeba and polygon map

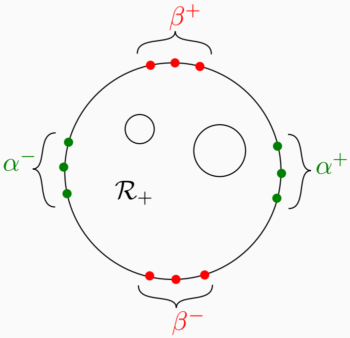


Ronkin function and surface tension

- Ronkin function and surface tension

$$\rho(P) = \rho(x_1, x_2) = -\frac{1}{\pi}\,\mathrm{Im}\int_{\ell} \zeta_2 d\zeta_1 + x_2 s_2$$

$$\sigma(P) = \sigma(s_1, s_2) = \frac{1}{\pi}\,\mathrm{Im}\int_{\ell} \zeta_2 d\zeta_1 - x_1 s_1$$



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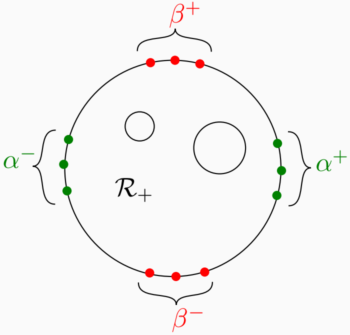
- Legendre dual

$$\nabla\sigma(s_1, s_2) = (x_1, x_2), \quad \nabla\rho(x_1, x_2) = (s_1, s_2).$$



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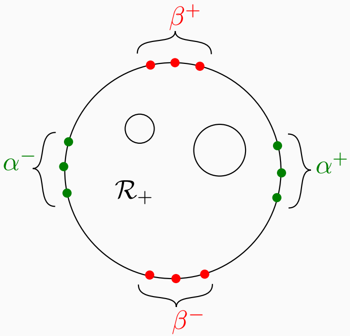
- Legendre dual

$$\nabla\sigma(s_1, s_2) = (x_1, x_2), \quad \nabla\rho(x_1, x_2) = (s_1, s_2).$$

- Definitions agree with algebraic ones in doubly periodic case.

$$F(x_1, x_2) := \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |\mathcal{P}(e^{x_1} z, e^{x_2} w)| \frac{dz}{z} \frac{dw}{w}.$$



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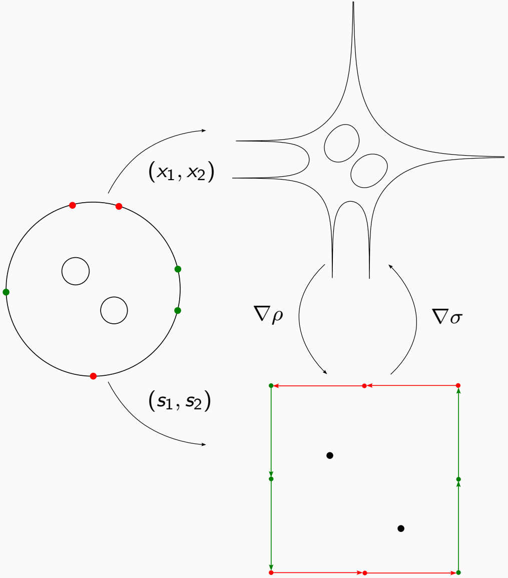
Ronkin function and surface tension



Dictionary

Object	doubly periodic	our setup
Spectral curve	$\det(P(z,w)) = 0$	$\mathcal{R}$
Monodromies	(z, w)	$(\frac{\psi_{(1,0)}}{\psi_{(0,0)}}, \frac{\psi_{(0,1)}}{\psi_{(0,0)}})$
Main differentials	$(\frac{dz}{z}, \frac{dw}{w})$	$(d\zeta_1, d\zeta_2)$
Amoeba map	$(\log\|z\|, \log\|w\|)$	$(\operatorname{Re}\zeta_1, \operatorname{Re}\zeta_2)$

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- In [BB]: purely variational proof, more general boundary conditions.

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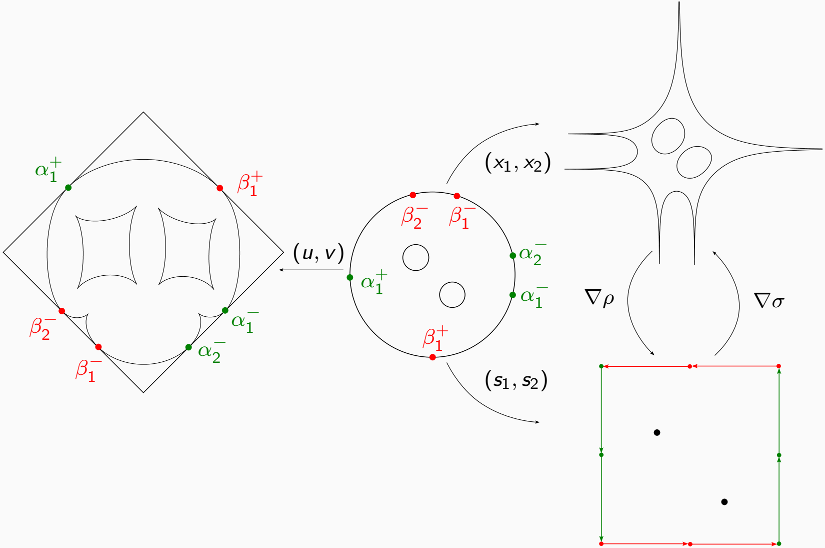
All Maps



α_1^+
β_1^+
β_2^-
α_1^-
β_1^-
α_2^-
(u, v)
β_2^-
β_1^-
α_2^-
α_1^+
α_1^-
β_1^+
(x_1, x_2)
(s_1, s_2)
$\nabla\rho$
$\nabla\sigma$

Arctic curve, isoradial case ($g = 0$)

$d\zeta_i(z) = f_i(z)dz$.

$$\begin{aligned} f_1(z) &= \sum_i \frac{1}{z-\alpha_i^-} - \frac{1}{z-\alpha_i^+}, \\ f_2(z) &= \sum_i \frac{1}{z-\beta_i^-} - \frac{1}{z-\beta_i^+}, \\ f_3(z) &= \sum_i \frac{1}{z-\alpha_i^-} + \frac{1}{z-\alpha_i^+} - \frac{1}{z-\beta_i^-} - \frac{1}{z-\beta_i^+} \end{aligned}$$

imply

$$u = \frac{W(f_1, f_3)}{W(f_1, f_2)}, \quad v = \frac{W(f_1, f_3)}{W(f_1, f_2)},$$

where W is the Wronskian

$$W(f_i, f_j) = \begin{vmatrix} f_i & f_j \\ f_i' & f_j' \end{vmatrix}.$$

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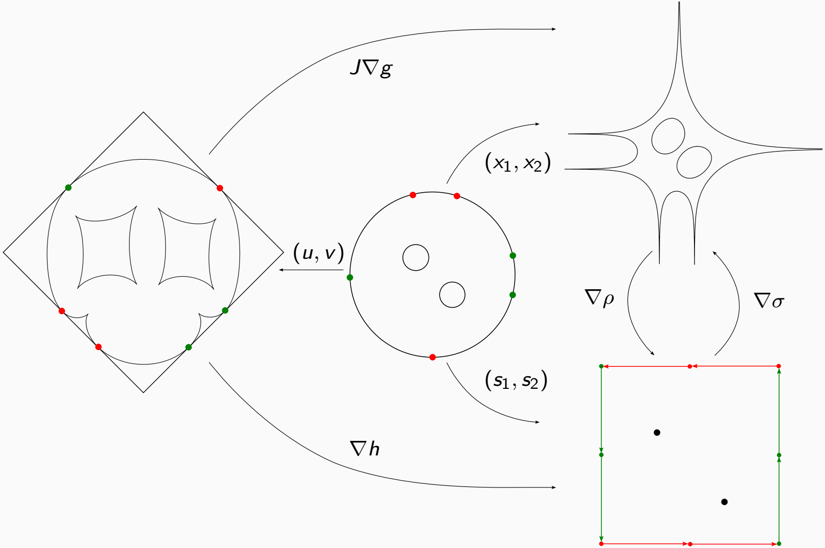
All Maps



Computation

- All these formulas can be efficiently computed via Schottky uniformization,
- Schottky group is a free group G generated by inversions in circles X_i,
- differentials are given by Poincarè theta series

$$d\zeta_1(z) := \sum_{g\in G}\sum_i \left(\frac{1}{z-g(\alpha_i^-)} - \frac{1}{z-g(\alpha_i^+)}\right) dz,$$

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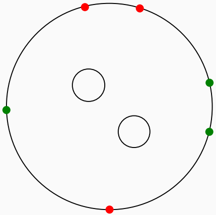
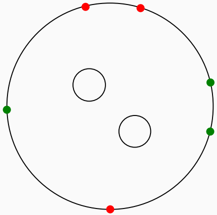
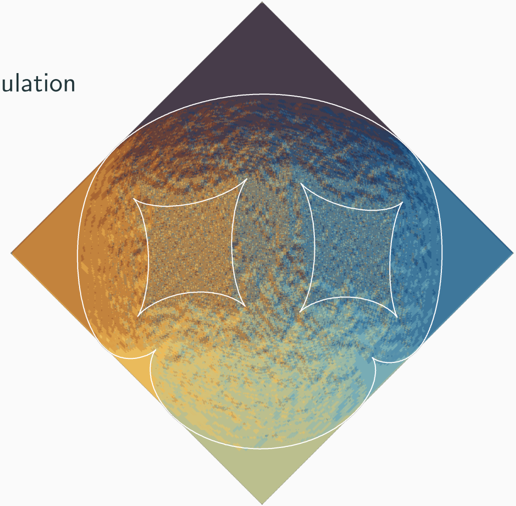


Computation

- Pictures shown are actual computations, not just illustrations. github.com/nikolaibobenko/FockDimerSimulation
- Theoretical predictions match simulations on practical scales.





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- Minimize $\int_{[-1,1]^2} \sigma(\nabla h)$.
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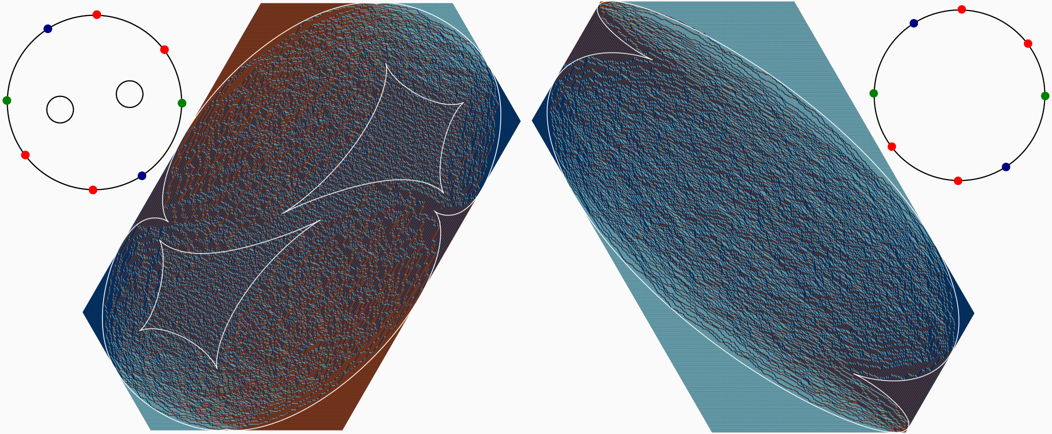
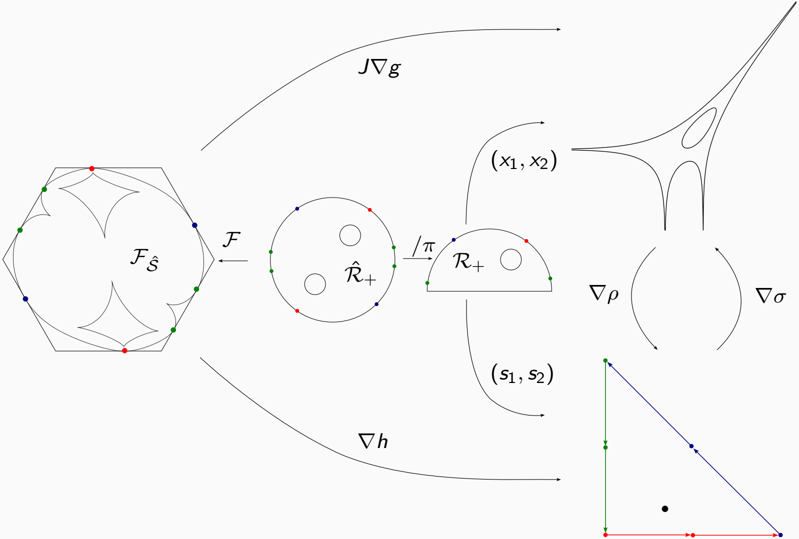
- $g = 0$: [Astala, Duse, Prause, Zhong, '20].
- In general follows from existence of extension of g to gas bubbles and frozen regions.

Hexagonal case

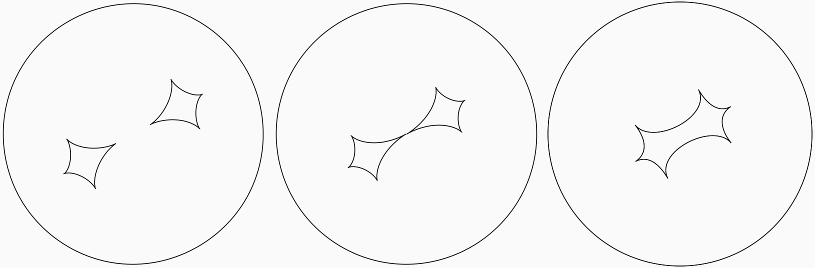


Regular hexagonal lattice

Hexagonal case. All maps



Three arctic curves for the hexagon



Talk "Limit shapes from Riemann Surfaces" by Nikolai Bobenko at BIMSA conference "Representation Theory, Integrable systems and Related Topics" July 8-12, 2024