Hitchin systems and their quantizations

Lectures by Pavel Etingof
Notes by Henry Liu

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1 Lecture 5

1.1 Quantum Hitchin systems

What is a quantum integrable system, and what does it mean to quantize a classical integrable system? This is not even the most basic question, since classical integrable systems live on symplectic manifolds, and so we should say that it means to quantize a symplectic manifold. This story can be told for smooth manifolds, analytic manifolds, algebraic varieties, etc. over any field.

Let M be a symplectic manifold. Then $\mathcal{O}(M)$ is a Poisson algebra (or sheaf thereof), meaning that there is a Poisson bracket $\{-,-\}$ on elements (or sections) of $\mathcal{O}(M)$. We will pretend M is affine so that there is no need to think about sheaves. In classical mechanics, M will be the phase space, and $\mathcal{O}(M)$ is the algebra of observables. Quantization means that observables are replaced by operators which may no longer commute.

Definition. A *quantization* of $\mathcal{O}(M)$ is a non-commutative algebra A, over $k[[\hbar]]$ or $k[\hbar]$, which is a flat deformation of $A/\hbar A$ and such that $A/\hbar A \cong \mathcal{O}(M)$ and the multiplication $*$ in A satisfies

$$\lim_{\hbar\to 0} \frac{f*g - g*f}{\hbar} = \{f,g\}.$$

Example. Suppose $M = T^*Y$ for some Y. Recall that $\mathcal{O}(T^*Y)$ is locally generated by coordinates x_i and momenta p_i. Then a natural quantization is the algebra $\mathcal{D}(Y)$ of differential operators on Y — more precisely, the sub-algebra $\mathcal{D}_\hbar(Y)$ generated locally by x_i and $\widehat{p}_i \coloneqq \hbar\partial_i$, which satisfy the Heisenberg uncertainty relations

$$[\widehat{p}_i, x_j] = \hbar\delta_{ij}$$

and $[p_i, p_j] = [x_i, x_j] = 0$.

Let $k = \mathbb{C}$ and $n \coloneqq \dim Y$. Recall that a classical integrable system consists of functionally (or, in the algebraic case, algebraically) independent functions $H_1, \ldots, H_n$ on T^*Y such that $\{H_i, H_j\} = 0$. These functions define a map

$$p\colon T^*Y \to \mathbb{A}^n$$

whose pullback $p^*\colon \mathcal{O}(\mathbb{A}^n) = \mathbb{C}[X_1, \dots, X_n] \to \mathcal{O}(T^*Y)$ sends $X_i \mapsto H_i$ and is therefore injective onto a Poisson-commutative subalgebra $\mathbb{C}[H_1, \dots, H_n] \subset T^*Y$.

Theorem. *Any function which Poisson-commutes with all H_i is functionally (algebraically) dependent on them.*

Thus if Y is a smooth algebraic variety, then classical integrable systems on T^*Y correspond to Poisson-commutative subalgebras in $\mathcal{O}(T^*Y)$ of transcendence degree $\dim Y$. This motivates the following definition.

Definition. A *quantization* of such a classical integrable system is a non-commutative algebra A quantizing $\mathcal{O}(T^*Y)$, with an injection

$$\mathbb{C}[X_1, \dots, X_n] \hookrightarrow A, \qquad X_i \mapsto H_i.$$

Theorem (Makar–Limanov)**.** *If $[H, H_i] = 0$ and $H \in \mathcal{D}(Y)$, then H is algebraically dependent on $H_1, \dots, H_n$.*

To construct a quantum integrable system, we therefore require a maximal (up to algebraic extensions) commutative subalgebra in $\mathcal{D}(Y)$. This quantizes a classical integrable system if it converges to it when $\hbar \to 0$ and $\widehat{p}_i \mapsto p_i$.

This gives rise to a naive quantization procedure: just replace all instances of p_i with $\hbar\partial_i$. This is not a good thing to do in general due to ordering issues: unlike p_i, the partial derivatives ∂_i do not commute with coordinates, so there is ambiguity as to whether, say, $x_i p_i$ should be replaced by $\hbar x_i \partial_i$ or $\hbar \partial_i x_i = \hbar x_i \partial_i + \hbar$. But this naive procedure does work sometimes, e.g. in the Garnier system from last lecture.

Example. Recall the deformed Garnier system, given by the Hamiltonians

$$G_i = \sum_{j \neq i} \frac{-(x_i - x_j)^2 p_i p_j + 2(x_i - x_j)(\lambda_i p_j - \lambda_j p_i) + 2\lambda_i \lambda_j}{t_i - t_j}$$

where x_i and p_j are the standard coordinates and momenta. The naive quantization procedure produces

$$\frac{1}{\hbar^2}\widehat{G}_i \coloneqq \sum_{j \neq i} \frac{-(x_i - x_j)^2 \partial_i \partial_j + (x_i - x_j)(\frac{\lambda_i}{\hbar}\partial_j - \frac{\lambda_j}{\hbar}\partial_i) + 2\frac{\lambda_i}{\hbar}\frac{\lambda_j}{\hbar}}{t_i - t_j}.$$

It is convenient to introduce $\Lambda_i \coloneqq 2\lambda_i/\hbar$, so that this can be rewritten as

$$\sum_{j \neq i} \frac{-(x_i - x_j)^2 \partial_i \partial_j + 2(x_i - x - j)(\Lambda_i \partial_j - \Lambda_j \partial_i) + \frac{1}{2}\Lambda_i \Lambda_j}{t_i - t_j}.$$

In representation theory, there is a slightly different way to write this. Let $\mathfrak{sl}_2 = \langle e, f, h \rangle$, and recall that there is an action of $U(\mathfrak{sl}_2)$ by differential operators on $\mathbb{A}^1$ given by

$$f \mapsto -\partial_x, \qquad h \mapsto 2x\partial_x + \Lambda, \qquad e \mapsto x^2 \partial_x + \Lambda x.$$

The Casimir tensor is the unique element $\Omega \in (\mathfrak{sl}_2 \otimes \mathfrak{sl}_2)^{\mathrm{SL}_2}$ up to scaling, and is given by the formula

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h.$$

Then the numerator of formula for $\widehat{G}_i$ is just $\Omega_{i,j}$:

$$\widehat{G}_i = \sum_{j \neq i} \frac{\Omega_{i,j}}{t_i - t_j}.$$

It is obvious that $[\Omega_{12}, \Omega_{13} + \Omega_{23}] = 0$, in fact, true for any simple $\mathfrak{g}$ if $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$, and therefore $[\widehat{G}_i, \widehat{G}_j] = 0$. The result is a collection of elements in $U(\mathfrak{g})^{\otimes n}$ called the *Gaudin Hamiltonians for* $\mathfrak{g}$.

If we pick representations $V_1, \ldots, V_n$ of $\mathfrak{g}$, then we genuinely get commuting operators

$$\widehat{G}_i \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$$

which also commute with $\mathfrak{g}$ and therefore act on $(V_1 \otimes \cdots \otimes V_n)^{\mathfrak{g}}$. In this way we have produced interesting families of commuting operators.

For $\mathfrak{g}$ of higher rank, this procedure does not directly produce an integrable system, because we are missing higher-order operators; considering only the Casimir is like considering only $\operatorname{tr} \wedge^2 \phi$ in the classical Hitchin system for SL_n. However, the following example shows that the naive procedure does not always work.

Example (Elliptic Calogero–Moser system)**.** Recall that the classical elliptic Calogero–Moser system has Hamiltonian $H_2 = \sum_i p_i^2 - \sum_{j \neq i} \wp(q_i - q_j)$. So the quantized Hamiltonian is

$$\frac{1}{\hbar^2} \widehat{H}_2 \coloneqq \sum_i \partial_i^2 - \frac{1}{\hbar^2} \sum_{j \neq i} \wp(q_i - q_j).$$

The theorem is that $\widehat{H}_2$ indeed defines a quantum integrable system $\mathbb{C}[\widehat{H}_1, \ldots, \widehat{H}_n]$, which is just the centralizer of $\widehat{H}_2$ in the algebra of differential operators in n variables. Trigonometric or rational Calogero–Moser systems can be obtained as limits from this example.

But the integrability of this system does not immediately follow from the integrability of the classical system. For instance,

$$H_3 = \sum_i p_i^3 + \sum_i p_i f_i(q) + g(q)$$

for some functions $f_i(q)$ and $g(q)$. Already in the middle term we have the ambiguity in ordering: in the quantized Hamiltonian $\widehat{H}_3$, do we put $\partial_i f_i(q)$ or $f_i(q)\partial_i$, or something else? Clearly, we need a more systematic approach!

In fact there is no uniform way to quantize an arbitrary integrable system. One needs to go back to the definition of the classical system and see if one can change the way in which it is obtained. This is what we will do.

1.2 Quantum Hitchin systems

Recall our construction of the Hitchin system on $\mathrm{Bun}_G^\circ(X)$. There were two steps.

1. Represent $\mathrm{Bun}_G(X)$ as a double quotient, e.g. $G(X \setminus x)\backslash G(K)/G(\mathcal{O})$.

2. Construct some commuting Hamiltonians on $T^*G(K)$ which are invariant under the left and right actions of $G(K)$, and then descend them to $\mathrm{Bun}_G(X)$ using Hamiltonian reduction by $G(X \setminus x) \times G(\mathcal{O})$.

To retrace our steps, we must discuss a quantized version of Hamiltonian reduction (along a choice of coadjoint orbit).

Classically, a group H acts in a Hamiltonian manner on a symplectic manifold M, with moment map $\mu\colon M \to \mathfrak{h}^*$, and we define the Hamiltonian reduction $\mu^{-1}(0)/H$ which is a symplectic manifold. Note that μ can be viewed as a Poisson homomorphism between Poisson algebras $\mu\colon S(\mathfrak{h}) = \mathcal{O}(\mathfrak{h}^*) \to \mathcal{O}(M)$. In the quantum setting, we therefore must consider a group H acting on an algebra A, and the natural way to quantize μ is to ask for an algebra homomorphism

$$\mu\colon U(\mathfrak{h}) \to A.$$

This is the input data that one must supply. The classical condition that the moment map is H-equivariant and dual to the action map becomes the condition that μ is H-invariant and

$$z \cdot a = [\mu(z), a], \qquad \forall z \in \mathfrak{h}.$$

Finally, classically, we considered the quotient M/H, for which $\mathcal{O}(M/H) = \mathcal{O}(M)^H \subset \mathcal{O}(M)$ is a Poisson sub-algebra, so the following is a natural way to quantize the locus $\mu^{-1}(0)/H \subset M/H$ cut out by $\mu(m) = 0$.

Definition. The *quantum Hamiltonian reduction* of A by H is the algebra

$$A^H/(A\mu(\mathfrak{h}))^H.$$

Note that $A\mu(\mathfrak{h}) \subset A$ is only a left ideal, but one can check that, after taking H-invariants, $(A\mu(\mathfrak{h}))^H \subset A^H$ is a two-sided ideal.

If H is reductive, the operation of quotienting by $A\mu(\mathfrak{h})$ commutes with the operation of taking H-invariants.

To replace 0 with a coadjoint orbit $\mathcal{O} \subset \mathfrak{h}^*$, the equation $\mu(m) = 0$ is replaced by $\mu(m) \in \mathcal{O}$. In the quantum setting, we need to find an ideal $I \subset U(\mathfrak{h})$ which quantizes the orbit $\mathcal{O}$, in the sense that $U(\mathfrak{h})/I$ is a quantization of $\mathcal{O}$. Then the quantum Hamiltonian reduction is

$$A^H/(A\mu(I))^H.$$

When $\mathcal{O} = 0$, the ideal I is the augmentation ideal, i.e. the kernel of $U(\mathfrak{h}) \to \mathbb{C}$, so that $A\mu(I) = A\mu(\mathfrak{h})$ and we recover the previous case.

Let us implement this for the Hitchin system (without punctures). We should take the (huge!) algebra $A \coloneqq \mathcal{D}(G(K))$ of differential operators, and the quantum Hitchin system should be obtained from some 2-sided-invariant differential operators on $G(K)$.

This is an infinite-dimensional group and there is actually quite a bit of technical trouble in talking about differential operators on such a group. But this was sorted out by Beilinson and Drinfeld and others, and we will ignore such issues and pretend $L \coloneqq G(K)$ is an ordinary Lie group.

What are 2-sided-invariant differential operators on L? Left-invariant differential operators are well-known to be identified with the universal enveloping algebra $U(\mathfrak{l})$ of the Lie algebra of L. Then 2-sided-invariant differential operators are therefore identified with

$$U(\mathfrak{l})^L = Z(U(\mathfrak{l})),$$

the center of $U(\mathfrak{l})$. Explicitly, $\mathfrak{l} = \mathfrak{g}((t))$, but for G semisimple, unfortunately the center of $U(\mathfrak{l})$ is trivial. (Technically, since $\mathfrak{l}$ is infinite-dimensional, one should take a completion of the universal enveloping algebra, but, even so, the center is still trivial.)

We can explain what went wrong. Classically, recall that $H_2 = \frac{1}{2}\operatorname{tr}\phi^2$, and the Higgs field ϕ has the form $\phi(z)\,dz$ with $\phi \in \mathfrak{g}((t))$. Writing $\phi =: \sum \phi_n z^{-n}$,

$$\begin{aligned} H_2 &= \frac{1}{2}\sum_n z^n \sum_m \phi_m\phi_{n-m} \\ &= \frac{1}{2}\sum_n z^n \sum_{m,i} \phi_m^i\phi_{n-m}^i \end{aligned}$$

where, in the second equality, we picked an orthonormal basis a_i of $\mathfrak{g}$ and wrote $\phi_m =: \sum_i \phi_m^i a_i$. In particular, we want $[\phi_p^j, H_2] = 0$, but we have infinite sums and *normal-ordering* is required after quantization to make it meaningful on highest-weight representations. But then this commutation relation will fail.

Exercise. Check this.

In fact, we were doomed to fail, because Beilinson and Drinfeld showed that every globally-defined differential operator on $\operatorname{Bun}_G(X)$ is a scalar. However, we can recall something from physics to save the day: differential operators on a manifold are *not* the most natural quantization of functions on the cotangent bundle. Recall from quantum mechanics that classical observables on $M = T^*Y$ should quantize to operators on $L^2(Y)$. But to define $L^2(Y)$, one must fix a measure on Y, and there is no natural choice for the measure in general. The solution is to take $L^2(Y, \Omega^{1/2})$, where Ω is the bundle of densities, and the L^2-norm is given by

$$\| f(y)|dy|^{1/2}\|^2 \coloneqq \int |f(y)|^2\,|dy|.$$

Thus, from this point of view, the most natural quantization of our functions on the cotangent bundle is *twisted* differential operators $\mathcal{D}(Y, K_Y^{1/2})$.

Definition. Let Y be a smooth variety and L be a line bundle on Y. An *L-twisted differential operator* on Y is a differential operator acting on sections of L. Let $\mathcal{D}(Y, L^{\otimes n})$ be the space of $L^{\otimes n}$-twisted differential operators. It is generated by functions and vector fields, with relations containing n as a parameter. Namely, if one locally picks a connection on L with curvature denoted ω, the only relation that changes is that, usually, $[\nabla_v, \nabla_u] = \nabla_{[v,u]}$, but now we ask for

$$[\nabla_v, \nabla_u] = \nabla_{[v,u]} + n\omega(v, u).$$

So it makes sense to take any $n \in \mathbb{C}$.

Motivated by this, we replace $\mathcal{D}(\mathrm{Bun}_G(X))$ with $\mathcal{D}(\mathrm{Bun}_G(X), K^{1/2})$. In fact $K^{1/2}$ can be obtained by descending a line bundle on the pre-quotient $G(K)$. Note that non-trivial line bundles exist on loop groups, in contrast to on ordinary Lie groups, because H^2 of a loop group is H^3 of the original group by transgression. There exists a Kac–Moody group $\widehat{G}$ such that

$$1 \to \mathbb{C}^\times \to \widehat{G} \to G((t)) \to 1;$$

this is just a central extension. The Lie algebra of $\widehat{G}$ is the affine Kac–Moody algebra

$$\widehat{\mathfrak{g}} \coloneqq \mathfrak{g}((t)) \oplus \mathbb{C}K$$

with commutator given by $[a(t), b(t)] \coloneqq [a, b](t) + \mathrm{Res}_{t=0}(a(t)\, db(t))K$. The form on $\mathfrak{g}$ is normalized so that long roots have squared length 2.

Theorem. *Let $\mathcal{E}$ be the $G((t))^2$-equivariant principal $\mathbb{C}^\times$-bundle on $G((t))$ given by $\widehat{G}$. Then $\mathcal{K}_{\mathrm{Bun}_G(X)}$ is obtained by reduction of $\mathcal{E}^{-2h^\vee}$ where $h^\vee$ is the dual Coxeter of G.*

So we need to work with two-sided-invariant elements in $\mathcal{D}(G((t)), \mathcal{E}^{-h^\vee})$, which is the center of the quotient

$$\widehat{U}(\widehat{\mathfrak{g}})/\langle K = -h^\vee\rangle$$

where $\widehat{U}(\widehat{\mathfrak{g}})$ denotes a suitable completion of the universal enveloping algebra $U(\widehat{\mathfrak{g}})$.

Theorem (Feigin–Frenkel, 1991)**.** *The center of $\widehat{U}(\widehat{\mathfrak{g}})/\langle K = k\rangle$ is non-trivial if and only if $k = -h^\vee$, and all classical Hitchin Hamiltonians $P_i(\phi)$ lift to this center and can therefore be quantized.*

In other words, there is a so-called *quantum anomaly*: K must be a specific non-zero value in order for the center to be non-trivial. The value of K is called the *level*, and this special value $-h^\vee$ is called the *critical level*. Representation theory of $\widehat{\mathfrak{g}}$ behaves very differently at the critical level due to the presence of this non-trivial center.

Theorem (Beilinson–Drinfeld)**.** *The two-sided-invariant differential operators on $G((t))$ acting on $\mathcal{E}^{-h^\vee}$ descend to differential operators on $\mathrm{Bun}_G(X)$. This map is surjective onto the algebra*

$$\mathcal{D}(\mathrm{Bun}_G(X), K^{1/2}).$$

This algebra is commutative, and is a polynomial algebra with $(g-1)\dim G = \dim \mathrm{Bun}_G(X)$ generators that quantize the Hitchin system.

The commutativity may be surprising. An example to keep in mind is globally-defined differential operators on an elliptic curve: since they are globally-defined, they have constant and hence commutative coefficients.

Example. Quantum Hitchin systems include, as special cases, the Gaudin system, elliptic Calogero–Moser, affine Toda, etc. (One has to quantize the twisted Hitchin system to get elliptic Calogero–Moser.) The previous issue with ordering ambiguity is solved systematically by this Beilinson–Drinfeld theorem, although it is rather hard to compute the higher quantum Hamiltonians explicitly.

Consider the quantization of the classical $\frac{1}{2}\operatorname{tr}\phi^2$. In the quantum setting, it corresponds (upstairs on the loop group) to

$$T_n \coloneqq \frac{1}{2}\sum_{m,i} {:}\phi^i_m \phi^i_{n-m}{:}$$

where :: denotes *normal-ordering*, i.e. put the term with bigger index (m or $n-m$) first.

Theorem (Sugawara construction)**.**

$$\begin{aligned}
[\phi_p, T_n] &= p(K+h^\vee)\phi_{n+p} \\
[T_n, T_m] &= (K+h^\vee)(n-m)T_{m+n} + \frac{n^3-n}{12}K(K+h^\vee)\dim\mathfrak{g}\cdot\delta_{n-m}.
\end{aligned}$$

Hence, when $K \neq -h^\vee$, the operators

$$L_n \coloneqq \frac{T_n}{K+h^\vee}$$

form a Virasoro algebra with central charge $c = \frac{K\dim\mathfrak{g}}{K+h^\vee}$.

Beilinson–Drinfeld used the Feigin–Frenkel theorem as input, and with a lot of work, they were able to construct the quantum Hitchin system and prove its quantum integrability.

Remark. Analytic Langlands is the spectral theory of the quantum Hitchin system acting on $L^2(\mathrm{Bun}_G(X))$.

1.3 Problem session

Problem 1. Recall the Gaudin system in N variables

$$G_i = \sum_{j\neq i} \frac{-(x_i-x_j)^2\partial_i\partial_j + (\Lambda_i x_j - \Lambda_j x_i)(\partial_i - \partial_j)}{t_i - t_j}.$$

For $N=4$ this reduces to a second order differential operator L in 1 variable with 4 singularities. Compute this operator after sending $(t_1,t_2,t_3,t_4)\mapsto(0,1,\infty,t)$. The answer for $\Lambda_i = -1$ should be the *Lamé operator* (with parameter $-1/2$)

$$L = \partial x(x-1)(x-t)\partial + x.$$

(Hint: you can get the general shape of L by using that it has 4 regular singularities.)

Problem 2. Let E be the elliptic curve $y^2 = x(x-1)(x-t)$. Let us lift L to E. Show that the lift $\tilde{L}$ is the *Darboux operator*

$$\tilde{L} = \partial_z^2 - \sum_{i=0}^{3} \Lambda_i(\Lambda_i + 1)\wp(z+\epsilon;\tau)$$

for $\epsilon_1 = 0$, $\epsilon_2 = 1/2$, $\epsilon_3 = \tau/2$, and $\epsilon_4 = (1+\tau)/2$.

Problem 3. An *oper* for SL_2 on a smooth curve X is a differential operator

$$L = \partial^2 + u \colon K_X^{-1/2} \to K_X^{3/2}.$$

Show that this notion is well-defined and compute how L transforms under coordinate changes in X. (Hint: you should see the object called "Schwarzian derivative".)

Problem 4. Let X be a smooth irreducible projective curve of genus $g > 1$ and fix a square root $K_X^{1/2}$. Show that X admits a unique vector bundle E_X such that

$$0 \to K_X^{1/2} \to E_X \to K_X^{-1/2} \to 0$$

is a non-split short exact sequence. Show that SL_2 opers on X are equivalent to connections on E_X. Show that opers on X exist and form a torsor over $H^0(X, K_X^{\otimes 2})$ (i.e. an affine space of dimension $3g-3$). Compute opers on a genus-2 curve using the hyperelliptic realization.

Problem 5. Show that the coadjoint representation of the Virasoro algebra $\mathbb{C}((t))\partial_z \oplus \mathbb{C}c$ is isomorphic to the space of "opers" $\alpha\partial^2 + u \colon K^{-1/2} \to K^{3/2}$ on $D^\times$, for $\alpha \in \mathbb{C}$, as a module over $\mathrm{Aut}(D)$.