Hitchin systems and their quantizations

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1 Lecture 4

1.1 G-bundles with parabolic structures

Let G be an affine algebraic group, $H \subset G$ be a closed subgroup, and $\mathcal{E}$ be a G-bundle on X. Pick a point $x \in X$.

Definition. An *H-structure on $\mathcal{E}$ at x* is an H-orbit in E_x.

Note that there are a G/H's worth of choices for H-structures. Note also that H matters only up to conjugation, because right-multiplication by a group element $g \in G$ will transform an H-orbit into a gHg^{-1}-orbit.

Let G now be connected and reductive. A subgroup $P \subset G$ is *parabolic* if it contains a Borel, or, equivalently, if the quotient G/P is a projective variety. For instance, for GL_n, if $n = n_1 + \cdots + n_r$ is a composition, then the parabolic subgroup corresponding to this composition consists of upper block-diagonal matrices with blocks of size $n_1, \ldots, n_r$. The smallest parabolic is the *Borel* B, where all $n_i = 1$.

Definition. Let $\mathrm{Bun}_G(X, t_1, \ldots, t_N, P_1, \ldots, P_N)$ be the moduli stack of G-bundles on X with a P_i-structure at t_i for $i = 1, \ldots, N$.

It is clear that we have a fibration

$$\mathrm{Bun}_G(X, t_1, \ldots, t_N, P_1, \ldots, P_N) \to \mathrm{Bun}_G(X)$$

with fiber $G/P_1 \times \cdots \times G/P_N$.

Example. Since GL_n-bundles are the same as rank-n vector bundles, let $\mathcal{E}$ denote the GL_n-bundle and E denote the associated rank-n vector bundle. Canonically,

$$\mathcal{E}_x = \{\text{bases in } E_x\}$$

is the fiber of $\mathcal{E}$ at a point $x \in X$. In particular, if P is a parabolic subgroup associated to the composition $n = n_1 + \cdots + n_r$, then P is the stabilizer of a partial flag

$$0 \subset V_1 \subset \cdots \subset V_r = V$$

where the quotients V_i/V_{i-1} are vector spaces of dimensions n_i, for $i = 1, \dots, r$. So a P-structure is a set of bases compatible with this flag, i.e. there is an nested sequence of subsets of the basis which are bases of the V_i. Thus, choosing a P-structure is equivalent to fixing a flag in E_x.

Example. The basic example we will consider is $G = \mathrm{GL}_2$ and $P_i = B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. By the discussion above, a B-structure at t_i is the choice of a line $\ell_i \subset E_{t_i}$.

We want to consider parabolic structures because it enables us to consider $g = 0$ and $g = 1$. This is because, without the marked points and extra structure, there are no stable bundles for $g < 2$; all bundles have non-trivial automorphism groups, even if G is adjoint. However, with the extra data, automorphisms must preserve it, so the automorphism group shrinks. In particular, there will be a lot of objects with trivial automorphism group.

In fact, if $N \geq 3$ and G is adjoint, then a generic G-bundle on $\mathbb{P}^1$ with parabolic structures has trivial automorphism group.[1] For example, consider $G = \mathrm{PGL}_2$ and let E be the trivial G-bundle on X. Then $\mathrm{Aut}(E) = \mathrm{PGL}_2$. But we have parabolic structures $\ell_1, \dots, \ell_N$, where $\ell_i \in \mathbb{P}E_{t_i} = \mathbb{P}^1$ for each i. So the set $\mathrm{Bun}_G^{\mathrm{triv}}(X, t_1, \dots, t_N, P_1, \dots, P_N)$ of parabolic structures on trivial bundles is just $[(\mathbb{P}^1)^N / \mathrm{PGL}_2]$. This is still stacky, because when N points in $\mathbb{P}^1$ coincide, there is still a non-trivial automorphism group. But we can consider the smaller open set given by *distinct* points $(y_1, \dots, y_N) \in (\mathbb{P}^1)^N$. It is well-known that PGL_2 acts 3-transitively on $\mathbb{P}^1$, i.e. given three distinct points on $\mathbb{P}^1$, there is a unique element in PGL_2 which sends them to $(0, 1, \infty)$. Hence

$$(\mathbb{P}^1)^{N-3} \subset [(\mathbb{P}^1)^N / \mathrm{PGL}_2]$$

is the open set of N distinct points. It is a variety.

In general,

$$\mathrm{Bun}_G^{\mathrm{triv}}(X, t_1, \dots, t_N, P_1, \dots, P_N) \cong \Big(\prod_i G/P_i \Big) / G$$

where the G-action is diagonal.

1.2 Hitchin system with parabolic structures

Recall that we have constructed the Hitchin system by realizing $\mathrm{Bun}_G(X)$ as a double quotient, and then descending invariant functions upstairs on the loop group. We may do the same when there is parabolic structure. Namely, recall that

$$\mathrm{Bun}_G(X) = G(X \setminus \{t_1, \dots, t_N\}) \backslash \prod_i G(D_{t_i}^\times) / \prod_i G(D_{t_i}),$$

and parabolic structures are local at each of the t_i, so we should modify the right quotient. Let

$$\begin{aligned} \mathrm{ev}\colon G(D) &\to G \\ g(z) &\mapsto g(0) \end{aligned}$$

[1] There is more than one notion of stability for bundles with parabolic structures, but for us it will not matter which notion we use.

be the evaluation function at $z = 0$, and let $\widetilde{P}_i := \mathrm{ev}^{-1}(P_i)$. In other words, it consists of Taylor series whose constant term lies in $P_i \subset G$. Then

$$\mathrm{Bun}_G(X, t_1, \dots, t_N, P_1, \dots, P_N) = G(X \setminus \{t_1, \dots, t_N\}) \backslash \prod_i G(D_{t_i}^\times) / \prod_i \widetilde{P}_i.$$

The discrepancy between $\mathrm{Bun}_G(X, t_1, \dots, t_N, P_1, \dots, P_N)$ and $\mathrm{Bun}_G(X)$ is therefore exactly as stated earlier.

Now take the usual Hamiltonians $H_{i,j,n} := \mathrm{Res}\, P_{t_i}(\phi) z_j^n$ on $T^*G(D_{t_1}^\times) \times \cdots \times T^*G(D_{t_N}^\times)$, where z_j is a local coordinate around t_j, and do the same reduction as before but now with respect to the subgroup $G(X \setminus \{t_1, \dots, t_N\}) \times \prod_i \widetilde{P}_i$. The result is an integrable system on $T^* \mathrm{Bun}_G(X, t_1, \dots, t_N, P_1, \dots, P_N)$. Points in this space are pairs (E, ϕ) where E is a bundle with parabolic structure, and $\phi \in \Omega^1(X \setminus \{t_1, \dots, t_N\}, \mathrm{ad}\, E)$ is a Higgs field *with singularities.* One can check that the condition is:

> ϕ can have at most first-order poles at the points $t_1, \dots, t_N$, and the residue $\mathrm{Res}_{t_i} \phi$ *strictly preserves* the flag F_i (specified by the parabolic structure) at t_i.

Here, "strictly" means that it lies in the unipotent radical of the stabilizer P_i of F_i.

Exercise. Check this. For instance, in the $G = \mathrm{GL}_n$ case, it means that the residue preserves the flag and acts by 0 on the associated graded.

Example. Let's compute the Hitchin system for PGL_2 in genus $g = 0$. For this purpose, we will assume for convenience that $t_1, \dots, t_N \in \mathbb{A}^1 \subset \mathbb{P}^1$ and that the parabolic structure at t_i is given by $y_i \in \mathbb{A}^1$. The Higgs field ϕ is a 1-form with simple poles at t_i, valued in $\mathfrak{sl}_2$. So

$$\phi = \sum_{i=1}^N \frac{A_i}{z - z_i}\, dz, \qquad A_i \in \mathfrak{sl}_2,$$

satisfying the following conditions. First, ϕ must be regular at $\infty \in \mathbb{P}^1$ because there is no marking/puncture there. This is the case if and only if $\sum_{i=1}^N A_i = 0$. Second, the A_i must be nilpotent, i.e. $A_i \begin{pmatrix} y \\ 1 \end{pmatrix} = 0$. This is the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix} = 0,$$

which says $b = -ay$ and $d = -cy$, and the $\mathfrak{sl}_2$ condition says $d = -a$, so $a = cy$. Hence

$$A = \begin{pmatrix} cy & -cy^2 \\ c & -cy \end{pmatrix}.$$

Hence we have

$$A_i = p_i \begin{pmatrix} y_i & -y_i^2 \\ 1 & -y_i \end{pmatrix}.$$

One may check that p_i are the momentum coordinates. In particular, the symplectic form is $\sum dy_i \wedge dp_i$. Let's compute the Hitchin Hamiltonians:

$$H_2 = \frac{1}{2} \operatorname{tr} \phi^2 = \operatorname{tr} \left(\sum_{i,j} \frac{A_i}{z - t_i} \frac{A_j}{z - t_j} \right) (dz)^2.$$

The $i = j$ terms drop out, because A_i is nilpotent and thus $A_i^2 = 0$. The result is

$$H_2 = \sum_{i \neq j} \frac{\operatorname{tr} A_i A_j}{(z - t_i)(z - t_j)} (dz)^2.$$

Using the identity

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \left(\frac{1}{z-a} - \frac{1}{z-b} \right),$$

this can be rewritten as

$$H_2 = \sum_{i \neq j} \frac{\operatorname{tr} A_i A_j}{(t_i - t_j)(z - t_i)} (dz)^2.$$

It remains to compute the trace:

$$\operatorname{tr} A_i A_j = p_i p_j \operatorname{tr} \begin{pmatrix} y_i & -y_i^2 \\ 1 & -y_i \end{pmatrix} \begin{pmatrix} y_j & -y_j^2 \\ 1 & -y_j \end{pmatrix} = -p_i p_j (y_i - y_j)^2.$$

Finally, let's take residues (the result is well-defined up to scaling, which doesn't matter for us):

$$G_i \coloneqq \operatorname{Res}_{t_i} H_2 = \sum_{j \neq i} \frac{p_i p_j (y_i - y_j)^2}{t_j - t_i}.$$

Since $\sum_i A_i = 0$, we have $\sum_i p_i = \sum_i p_i y_i = \sum_i p_i y_i^2 = 0$, so (y, p) belongs to $\mu^{-1}(0) \subset T^*\mathbb{C}^N$. Also, the G_i give only $N - 3$ independent integrals of motion since $\sum_i G_i = \sum_i t_i G_i = \sum_i t_i^2 G_i = 0$, but this is exactly sufficient to get an integrable system on $\mu^{-1}(0)/\operatorname{PGL}_2$, which has dimension $N - 3$.

1.3 Twisted Hitchin system

It turns out that Hitchin systems for bundles with parabolic structures have a twisted generalization, which allows us to produce more general integrable systems. To introduce them, we first explain Hamiltonian reduction along orbits. Let M be a symplectic manifold, with Hamiltonian action by a group H. Let $\mu \colon M \to \mathfrak{h}^*$ be a moment map. Previously, we considered $\mu^{-1}(0)/H$, but more generally, we may consider

$$\mu^{-1}(\mathcal{O})/H$$

for any H-orbit $\mathcal{O} \subset \mathfrak{h}^*$, called a *coadjoint orbit.* If the H-action is nice, this quotient also has a canonical symplectic structure, and we can run the same construction of integrable systems as before: if F_i are H-invariant functions in involution on M, then they descend to functions $\bar{F}_i$ on $\mu^{-1}(\mathcal{O})/M$ which are also in involution.

In the setting of Hitchin systems, recall the group

$$G(X \setminus \{t_1, \ldots, t_N\}) \times \prod_i G(D_{t_i}),$$

acting on $\prod_i G(D_{t_i}^\times)$, and let ker denote the kernel of the evaluation map $\prod G(D_{t_i}) \to G^N$ at $(t_1, \ldots, t_N)$. We reduce first by ker, after which there is a residual action of G^N, and then for a coadjoint orbit $\mathcal{O} \subset (\mathfrak{g}^*)^N$, we can descend the Hitchin Hamiltonians to $\mu^{-1}(\mathcal{O})/G^N$. Parabolic structures will arise from specific choices of $\mathcal{O}$.

As before, points in $\mu^{-1}(\mathcal{O})/G^N$ are Higgs pairs (E, ϕ) where ϕ must satisfy some conditions. To illustrate, take $G = \mathrm{PGL}_2$. At t_i, take the coadjoint orbit in $\mathfrak{sl}_2^* \cong \mathfrak{sl}_2$ given by $\mathrm{diag}(\lambda_i, -\lambda_i)$ for generic λ_i. (The previous Hitchin system with parabolic structure corresponds to the coadjoint orbit of a nilpotent element $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$.) Then ϕ has simple poles at each t_i with

$$\mathrm{Res}_{t_i} \phi|_{\ell_i} = \lambda_i \cdot \mathrm{id}\,.$$

As before, writing $\phi = \sum_i \frac{A_i}{z - t_i}\, dz$,

$$A_i \begin{pmatrix} y_i \\ 1 \end{pmatrix} = \lambda_i \begin{pmatrix} y_i \\ 1 \end{pmatrix}.$$

Solving, we obtain

$$A_i = \begin{pmatrix} -\lambda_i + p_i y_i & 2\lambda_i y_i - p_i y_i^2 \\ p_i & \lambda_i - p_i y_i \end{pmatrix}.$$

The trace becomes

$$\mathrm{tr}(A_i A_j) = -(y_i - y_j)^2 p_i p_j + 2(\lambda_i p_j - \lambda_j p_i)(y_i - y_j) + 2\lambda_i \lambda_j.$$

The resulting Hamiltonians

$$G_i(\lambda_1, \ldots, \lambda_N) = \sum_{j \neq i} \frac{-(y_i - y_j)^2 p_i p_j + 2(\lambda_i p_j - \lambda_j p_i)(y_i - y_j) + 2\lambda_i \lambda_j}{t_i - t_j}$$

define the *deformed* or *twisted* Garnier system. The ordinary Garnier system is a particular limit of this, when $\lambda_i \to 0$.

Example (Genus 1)**.** Let X be an elliptic curve with zero denoted $0 \in X$, and consider a generic bundle of degree 0 and rank n. Line bundles of degree-0 are all of the form

$$L_q = \mathcal{O}(q) \otimes \mathcal{O}(0)^{-1}$$

for a point $q \in X$, with a meromorphic section given by $\frac{\theta(z-q)}{\theta(z)}$. Atiyah showed that generic rank-n bundles all have the form

$$E = L_{q_1} \oplus \cdots \oplus L_{q_n},$$

say with $q_i \neq q_j$. Consider $G = \mathrm{PGL}_n$, put one puncture at 0, and perform the twisted reduction procedure for the orbit

$$\mathcal{O} := \langle \mathrm{diag}(c, c, \ldots, c, (-n+1)c) \rangle \subset \mathfrak{sl}_2.$$

As $c \to 0$, this orbit degenerates into a rank-1 nilpotent matrix, corresponding to the parabolic subgroup P with blocks of size $(n-1) \times (n-1)$ and 1×1, i.e. $G/P = \mathbb{P}^{n-1}$. Since $\mathrm{Aut}(E) = (\mathbb{C}^\times)^{n-1}$, acting on $\mathbb{P}^{n-1}$, we consider the free orbit of the vector whose entries are all non-zero — without loss of generality, $(1, 1, \ldots, 1)$. We think of the components ϕ_{ij} of ϕ as sections of $L_{q_i} \otimes L_{q_j}^{-1}$, and they should have a first-order pole at $z = 0$ whose residue acts on the vector $(1, \ldots, 1)$ with eigenvalue $(1-n)c$. Hence, for $i \neq j$

$$\phi_{ij} = a_{ij} \frac{\theta(z - q_i + q_j)}{\theta(z)\theta(q_i - q_j)},$$

and $\phi_{ii} = p_i$ are the momenta. What is the condition for the matrix $A = (a_{ij})$? It must satisfy

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = (1-n)c \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

and have the diagonal form as above. This means that all off-diagonal entries a_{ij} are equal to some constant C, and so

$$\phi_{ij} = C \frac{\theta(z - q_i + q_j)}{\theta(z)\theta(q_i - q_j)}.$$

This ϕ is Krichever's Lax matrix for the *elliptic Calogero–Moser (CM) system.* The resulting trace can be computed to be

$$\operatorname{tr} \phi^2 = \sum p_i^2 + C \sum_{j \neq i} \frac{\theta(z - q_i + q_j)\theta(z - q_j + q_i)}{\theta(z)^2 \theta(q_i - q_j)^2}.$$

One can check that the second summand is $C(\wp(z) - \wp(q_i - q_j))$, where $\wp(z)$ is the Weierstrass function, because this expression has zeros and poles in the correct places. Coefficients of non-zero powers of z may be constant and are not so interesting, but the constant coefficient is

$$H_2 = \sum p_i^2 - c^2 \sum_{i \neq j} \wp(q_i - q_j),$$

which is exactly the second Hamiltonian in the elliptic CM system. Higher traces $\operatorname{tr} \wedge^i \phi$ yield the higher Hamiltonians, establishing the complete integrability of the elliptic CM system.

1.4 Problem session

Problem 1. Let $A\colon \mathbb{C}^\times \to \mathrm{Mat}_{n \times n}(\mathbb{C})$ be a meromorphic function with $\det A \not\equiv 0$. Let $q \in \mathbb{C}^\times$ with $|q| < 1$ and consider the q-difference equation

$$f(qz) = A(z) f(z) \tag{1}$$

for $f\colon \mathbb{C}^\times \to \mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that (1) has a meromorphic solution $f(z)$ with $\det f \not\equiv 0$. (Hint: use the GAGA theorem.)

Problem 2. The Garnier system is the Hitchin system for parabolic PGL_2-bundles on $\mathbb{P}^1$ with marked points $t_1, \ldots, t_N \in \mathbb{C}$, parabolic structures $y_1, \ldots, y_N$, and Higgs field

$$\Phi = \sum_{i=1}^{N} \frac{\begin{pmatrix} p_i y_i & -p_i y_i^2 \\ p_i & -p_i y_i \end{pmatrix} dz}{z - t_i}.$$

Find the spectral curve and compute its genus. Compute it explicitly for $N = 4$. (Hint: let a, b be coprime polynomials of X of degree n_a and n_b, and with simple roots. What is the genus of the normalization of $y^2 = a(X)/b(X)$?)

Problem 3. Show that for $N = 4$ the Garnier system is equivalent to the elliptic CM flow for 2 particles. What is a geometric reason for it? Solve the Hamilton equation of the flow.

Problem 4. Calculate the first integral H_3 for the elliptic CM system with Hamiltonian $H = \sum_{i=1}^{N} p_i^2 - 2\sum_{j>i} \wp(q_i - q_j)$ such that

$$H_3 = \sum_{i=1}^{N} p_i^3 + (\text{lower degree terms in } p_i).$$

Problem 5. Let $L = \partial_z^2 - 2\sum_{i=1}^{N} \wp(z - q_i)$. Show that L commutes with a third-order differential operator if and only if $(q_1, \ldots, q_N)$ is a critical point of the Calogero–Moser potential $\sum_{j>i} \wp(q_i - q_j)$.