Hitchin systems and their quantizations

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June 27, 2024

1 Lecture 3

1.1 Classical Hitchin system for general G

Last time we discussed the classical Hitchin system, in the case that $G = \mathrm{SL}_n$ and X is a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. We considered the moduli space $\mathrm{Bun}_G^\circ(X)$ of stable G-bundles on X, or equivalently, rank-n vector bundles with trivial determinant. Its cotangent bundle $\mathcal{M}_g^\circ \coloneqq T^* \mathrm{Bun}_G^\circ(X)$ consists of Higgs pairs (E, ϕ) where E is a stable G-bundle and $\phi \in \Omega^1(X, \operatorname{ad} E)$ is a Higgs field. The Hitchin base is

$$\mathcal{B} \coloneqq \bigoplus_{i=1}^{n-1} H^0(X, \mathcal{K}_X^{\otimes(i+1)}),$$

with $\dim \mathcal{B} = \dim \mathrm{Bun}_G(X) = (n^2 - 1)(g - 1) =: d$, and we constructed the Hitchin map

$$\begin{aligned} p\colon T^* \mathrm{Bun}_G^\circ(X) &\to \mathcal{B} \\ (E, \phi) &\mapsto (\operatorname{tr} \wedge^2 \phi, -\operatorname{tr} \wedge^3 \phi, \ldots, (-1)^n \operatorname{tr} \wedge^n \phi). \end{aligned}$$

Theorem (Hitchin)**.** *p defines an integrable system.*

This means, in part, that coordinate functions on $\mathcal{B}$ pulled back by p Poisson commute on $T^* \mathrm{Bun}_G^\circ(X)$. Explicitly, if we choose a basis $b_1, \ldots, b_d \in \mathcal{B}$ and write $p(E, \phi) = \sum_{j=1}^d H_j(E, \phi) b_j$, then the coordinate functions H_j satisfy $\{H_i, H_j\} = 0$.

Today we will generalize this to an arbitrary semisimple G. To do this, we will need something about the Lie algebra $\mathfrak{g}$ of G. Namely, recall Chevalley's theorem:

$$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[P_1, \ldots, P_r],$$

where $r \coloneqq \operatorname{rank} G$ and the P_i are homogeneous polynomials. Let $d_i \coloneqq \deg P_i$.

Example. $\mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} = \mathbb{C}[\operatorname{tr} \wedge^2 A, \operatorname{tr} \wedge^3 A, \ldots, \operatorname{tr} \wedge^n A]$.

Let $P \in \mathbb{C}[\mathfrak{g}]^G$ be homogeneous of degree m (semisimple Lie algebras have no invariant linear functions, so $m \geq 2$), and let (E, ϕ) be a Higgs pair. A conjugation-invariant function like P may be evaluated fiber-wise on Higgs fields $\phi \in \Omega^1(X, \operatorname{ad} E)$ to produce elements

$$P(\phi) \in H^0(X, \mathcal{K}_X^{\otimes m}).$$

The Hitchin base should therefore be $\mathcal{B} \coloneqq \bigoplus_{i=1}^r H^0(X, \mathcal{K}_X^{\otimes d_i})$.

Remark. It is important to consider *conjugation-invariant* functions, because fibers of $\operatorname{ad} E$ are isomorphic to $\mathfrak{g}$ *non-canonically.* So to evaluate a function on fibers of $\operatorname{ad} E$ in a well-defined way, it had better be conjugation-invariant.

Last time, we computed using Riemann–Roch that $\dim H^0(X, \mathcal{K}_X^{\otimes m}) = (2m-1)(g-1)$. Recall that $2d_i - 1$ are the degrees of generators of the cohomology ring of G. Hence

$$\dim \mathcal{B} = \sum_i (2d_i - 1)(g-1) = (g-1)\dim G = \dim \operatorname{Bun}_G^\circ(X).$$

So, again, there is some hope to define an integrable system as before, using the map

$$\begin{aligned} p \colon T^* \operatorname{Bun}_G^\circ(X) &\to \mathcal{B} \\ (E, \phi) &\mapsto (P_1(\phi), \dots, P_r(\phi)). \end{aligned}$$

Theorem (Hitchin)**.** *p is an integrable system.*

Hitchin proved this for classical groups G, and then other people completed the proof for the exceptional cases. The proof has two parts: we must show that 1. the coordinate functions H_i are in involution, and that 2. they are functionally independent. Functional independence is equivalent to p being a dominant map, meaning that the image contains some open dense subset.

1.2 Proof for Hitchin's theorem, step 1

This step is for arbitrary G.

We will show that $\{H_i, H_j\} = 0$. For this, we must first review Marsden–Weinstein symplectic reduction. Let Y be a manifold (or variety), and H be a Lie group (or algebraic group) acting on Y on the right. In this case, H acts by Hamiltonian automorphisms on T^*Y, and so there is a moment map

$$\mu \colon T^*Y \to \mathfrak{h}^*$$

where $\mathfrak{h}$ is the Lie algebra of H. This is defined to be dual to the action map

$$a \colon H \to \operatorname{Vect}(Y) = \Gamma(Y, T_Y),$$

meaning that

$$\mu(x,p)(b) \coloneqq \langle p, a(b)_x \rangle, \qquad \forall (x,p) \in T^*Y,\ b \in \mathfrak{h}.$$

Theorem (Marsden–Weinstein symplectic reduction)**.** *The quotient $\mu^{-1}(0)/H$ has a natural symplectic structure. Furthermore, if H acts freely on Y, then there is a natural isomorphism of symplectic manifolds*

$$\mu^{-1}(0)/H \cong T^*(Y/H).$$

This can be used to construct integrable systems as follows. Suppose $\dim Y/H = n$, and $F_1, \dots, F_n$ are H-invariant functions on T^*Y which Poisson-commute. Then they descend to functions $\bar{F}_i$ on the quotient $T^*(Y/H)$, by first restricting F_i to $\mu^{-1}(0) \subset T^*Y$ and then descending. It is easy to check that $\{\bar{F}_i, \bar{F}_j\} = 0$. There is already the right number of them to form an integrable system; if in addition they are functionally independent, then $\bar{F}_1, \ldots, \bar{F}_n$ form an integrable system on $T^*(Y/H)$.

Note that there are too few functions to form an integrable system on T^*Y, so it was necessary to descend to the quotient.

In general, especially when there is no effective way of writing the functions explicitly, it is difficult to check whether a set of functions are in involution. But in this method of constructing integrable systems, sometimes the functions F_i are in involution on T^*Y for silly reasons, e.g. if Y is a vector space and the F_i all depend only on the momentum coordinates on $T^*Y = Y \oplus Y^*$.

Recall that $\mathrm{Bun}_G(X) = G(X \setminus x)\backslash G(K)/G(\mathcal{O})$, and denote the pre-image of $\mathrm{Bun}_G^\circ(X)$ by $G^\circ(K) \subset G(K)$. (Recall that $K = \mathbb{C}[D_x^*] \cong \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[D_x] \cong \mathbb{C}[[t]]$.) Then $G(X \setminus x) \times G(\mathcal{O})$ acts on $G^\circ(K)$ with stabilizer $Z(G)$. So we may express

$$T^* \mathrm{Bun}_G^\circ(X) = \begin{array}{c}\text{Hamiltonian reduction of}\\ T^*G^\circ(K) \text{ by } G(X \setminus x) \times G(\mathcal{O})\end{array}.$$

To construct functions on $T^* \mathrm{Bun}_G^\circ(X)$, first trivialize the cotangent bundle of $T^*G^\circ(K)$ by left (or right) multiplication. Then observe that there is a $G(K)$-invariant pairing on the Lie algebra $\mathfrak{g}(K)$, given by

$$\langle a(t), b(t)\rangle_{\mathfrak{g}(K)} \coloneqq \mathrm{Res}_{t=0}\langle a(t), b(t)\rangle_{\mathfrak{g}}\, dt.$$

This allows us to identify $\mathfrak{g}((t))^* \cong \mathfrak{g}((t))\, dt$. Hence

$$T^*G^\circ(K) = G^\circ(K) \times \mathfrak{g}((t))\, dt.$$

Points are pairs $(\tilde{E}, \tilde{\phi})$, and again we can take invariant functions $\{P_i\}_{i=1}^r$ on $\mathfrak{g}$ and apply them to $\tilde{\phi}$ to get functions

$$H_{i,n} \coloneqq \mathrm{Res}(t^n P_i(\tilde{\phi})), \qquad \forall 1 \le i \le r \text{ and } n \in \mathbb{Z}$$

on $T^*G^\circ(K)$. Observe that these $H_{i,n}$ depend only on the momentum coordinates on $T^*G^\circ(K)$, by construction. Note that momenta on T^*G commute like Lie algebra elements, but our elements P_i are invariant, and invariant functions of momenta *do* commute with momenta. Thus

$$\{H_{i,n}, H_{j,m}\} = 0.$$

We have obtained infinitely many functions on $T^*G^\circ(K)$. Of course, when we descend to the finite-dimensional manifold $T^* \mathrm{Bun}_G^\circ(X)$, they will not be functionally independent (in fact they will be linearly dependent), but this is irrelevant for this part of the argument. It is easy to check that the square

$$\begin{tikzcd}
T^*G^\circ(K) \arrow[r, "{(H_1,\ldots,H_r)}"] & \bigoplus_i \mathbb{C}((t))\,(dt)^{d_i} \\
\mu^{-1}(0) \arrow[u, hook] \arrow[d, two heads] & \\
T^* \mathrm{Bun}_G^\circ(X) \arrow[r, "p"] & \bigoplus_i H^0(X, \mathcal{K}_X^{\otimes d_i}) \arrow[uu, hook]
\end{tikzcd}$$

commutes. The vertical map on the right is just (Taylor) series expansion of differentials, and analytic continuation implies the map is injective. So Poisson-commutativity for the H_i downstairs follows from Poisson-commutativity of the $H_{i,n}$ upstairs, which we already know.

1.3 Proof for Hitchin's theorem, step 2, for $G = \mathrm{SL}_n$

Now we prove functional independence, for $G = \mathrm{SL}_n$. This will use an important technique which appears across the field of integrable systems: spectral curves. Take a point

$$b = (b_2, b_3, \dots, b_n) \in \mathcal{B} = \bigoplus_{i=1}^{n-1} H^0(X, \mathcal{K}_X^{\otimes(i+1)}).$$

Recall that the components of the Hitchin map are, by definition, the coefficients of the characteristic polynomial of ϕ. Consider the polynomial

$$\lambda^n + b_2\lambda^{n-2} + \cdots + b_n =: \prod_{i=1}^{n}(\lambda - \lambda_i).$$

Since $\deg b_i = i$, the quantities λ_i are all 1-forms on X. So

$$\{\lambda_1(x), \dots, \lambda_n(x)\} \subset T_x^*X.$$

Varying $x \in X$ produces a subset $C_b \subset T^*X$ associated to b — the graph, or Riemann surface, of the multi-valued function $\lambda(x)$. In fact C_b is an algebraic curve in T^*X, defined by the equation

$$\lambda^n + b_2(x)\lambda^{n-2} + \cdots + b_n(x) = 0$$

where λ is the coordinate along the cotangent fibers. The natural projection $\pi\colon C_b \to X$ has degree n.

Definition. C_b is called the *spectral curve* of b.

Let $(E, \phi) \in T^* \mathrm{Bun}_G^\circ(X)$. Applying p produces $p(E, \phi) = (b_2, \dots, b_n)$ where the b_i are just coefficients of the characteristic polynomial of ϕ. In other words,

$$\lambda^n + b_2\lambda^{n-2} + \cdots + b_n = \det(\lambda - \phi),$$

and the $\lambda_i(x)$ are just eigenvalues of $\phi(x)$, for $x \in X$. So $C_{p(E,\phi)}$ is traced out in T^*X by the spectrum of $\phi(x)$ as $x \in X$ varies. It is important to keep in mind that the eigenvalues are 1-forms, so they live in T^*X.

The spectral curve $C(E, \phi) \coloneqq C_{p(E,\phi)}$ is useful because it depends only on $p(E, \phi)$. Can we recover (E, ϕ) from $C(E, \phi)$ and something else?

Theorem (Hitchin)**.** *C is smooth and irreducible for generic $b \in \mathcal{B}$.*

We will not prove this, but it is not a very hard result.

If we have a Higgs field ϕ whose spectral curve is C, then there is an *eigenline bundle* L_ϕ on C. The fiber of L_ϕ at a point $\lambda \in C$ lying over $x \in X$ is the eigenline of $\phi(x)$ with eigenvalue λ, in the generic situation where all eigenvalues are distinct. The vector bundle E is then reconstructed from L_ϕ by

$$E \cong \pi_* L_\phi.$$

Algebraically, modules for $\mathcal{O}_C$ are also modules for $\mathcal{O}_X$ via π. Geometrically, this is because the fiber E_x is a direct sum $\bigoplus_{\lambda \in \pi^{-1}(x)} (L_\phi)_\lambda$, at least when all eigenvalues are distinct. More strongly, we can also recover ϕ from L_ϕ, because the action of ϕ on E is just multiplication by the cotangent coordinate on L_ϕ.

The line bundle L_ϕ on C_b has some degree d which is independent of b since it is a topological invariant. (One can compute d explicitly if desired, but we will not.) Then

$$L_\phi \in \mathrm{Pic}^d(C_b) \cong \mathrm{Jac}(C_b)$$

and so $p^{-1}(b) \subset T^* \mathrm{Bun}_G^\circ(X)$ gets identified with a subset of $\mathrm{Jac}(C_b)$. It remains to show that for generic b,

$$\dim p^{-1}(b) = (n^2-1)(g-1)$$

and no bigger. If this dimension were too big, the dimension of the image of p must be too small and there would be unwanted functional dependence among the functions H_i.

Observe that $p^{-1}(b)$ is actually contained in the kernel of the map $\mathrm{Jac}(C_b) \to \mathrm{Jac}(X)$ given by $L \mapsto \wedge^n \pi_* L$. This is because SL_n-bundles are equivalent to vector bundles of rank n with trivial determinant. The kernel has dimension exactly

$$\dim \mathrm{Jac}(C_b) - \dim \mathrm{Jac}(X) = g(C_b) - g$$

since both $\mathrm{Jac}(C_b)$ and $\mathrm{Jac}(X)$ are group schemes. Here we used that $\dim \mathrm{Jac}(C_b) = g(C_b)$ is the genus of C_b. It remains to show that

$$g(C_b) = (n^2-1)(g-1) + g = n^2(g-1) + 1.$$

Note that while $p^{-1}(b)$ is still an abelian variety, it is not the Jacobian of anything.

Theorem. $g(C_b) = n^2(g-1)+1$.

Proof. Since $g(C_b)$ is deformation-invariant, we can compute it at the point $b_1 = b_2 = \cdots = b_{n-1} = 0$. Then all eigenvalues are distinct except when $b_n(x) = 0$. So $g(C_b)$ may be computed by the Riemann–Hurwitz formula. Since

$$b_n \in H^0(X, \mathcal{K}_X^{\otimes n})$$

is a section of a degree $(2g-2)n$ bundle, generically b_n has $(2g-2)n$ simple zeros. Riemann–Hurwitz says

$$\begin{aligned} \chi(C_b) &= (\chi(X) - (2g-2)n)\, n + (2g-2)n \\ &= (2g-2)(-n^2 - n + n) = -n^2(2g-2). \end{aligned}$$

Since $\chi(X) = 2-2g$, the result is

$$g(C_b) = \frac{2 + n^2(2g-2)}{2} = 1 + n^2(g-1). \qquad \square$$

This concludes the proof of Hitchin's theorem.

Remark. Note that $g(C_b) = \dim \mathrm{Bun}^{\circ}_{\mathrm{GL}_n}(X)$. Without the extra argument above to show that $p^{-1}(0)$ lies in the kernel of $\mathrm{Jac}(C_b) \to \mathrm{Jac}(X)$, the bound given by the theorem is too weak.

One can generalize the Hitchin system to reductive G, allowing $\dim Z(G)$ of the exponents d_i in the Hitchin base to be 1. Since $\dim H^0(X, K_X) = g$, which is one more than $g-1$, the resulting Hitchin base has dimension $(g-1)\dim G + \dim Z(G)$, which is still equal to $\dim \mathrm{Bun}^{\circ}_G(X)$. Then, in the case $G = \mathrm{GL}_n$, the bound given by the theorem is exactly tight enough, and we obtain that $p^{-1}(b)$ is dense in $\mathrm{Jac}(C_b)$.