Hitchin systems and their quantizations

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1 Lecture 2

1.1 Principal G-bundles on $\mathbb{P}^1$

Last time we proved Grothendieck's theorem: vector bundles of rank n on $\mathbb{P}^1$ are uniquely of the form $\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$ where $m_1 \leq \cdots \leq m_n$ are integers. We will begin today by explaining how this generalizes to an arbitrary connected reductive group. This will require a reformulation of this result.

Recall that rank-n vector bundles are the same as GL_n-bundles, and GL_n is the group of invertible $n \times n$ matrices, so it contains a maximal torus T consisting of diagonal matrices with non-zero entries on the diagonal. Over an algebraically closed field, $T = (\mathbb{G}_m)^n$. Grothendieck's theorem says that every GL_n-bundle on $\mathbb{P}^1$ is associated to a T-bundle, meaning that the structure group of the bundle reduces to the torus. The same GL_n-bundle can come from many different T-bundles, because the data of a T-bundle is sensitive to the ordering of the integers $m_1, \ldots, m_n$. Precisely, if E_1, E_2 are T-bundles on $\mathbb{P}^1$, then

$$E_1 \times_T \mathrm{GL}_n \cong E_2 \times_T \mathrm{GL}_n$$

if and only if there exists a permutation $w \in S_n$ such that $E_1 \cong w(E_2)$. Here $E_i \times_T \mathrm{GL}_n$ is the GL_n-bundle associated to the T-bundle E_i.

If G is a connected reductive group over an algebraically closed field, $T \subset G$ is a maximal torus, and $N(T) \subset G$ is the normalizer of T, let $W := N(T)/T$ be the *Weyl group*.

Theorem. *Any G-bundle on $\mathbb{P}^1$ is associated to a T-bundle E, and given two T-bundles E_1 and E_2,*

$$E_1 \times_T G \cong E_2 \times_T G$$

if and only if $E_1 = w(E_2)$ for some element $w \in W$.

Proof idea. Reduce to the case of vector bundles by considering representations of G. □

Let T be a torus. How do we classify T-bundles on $\mathbb{P}^1$? We know $T \cong (\mathbb{G}_m)^n$, so non-canonically a T-bundle is just an n-tuple of integers. Canonically, let $X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ be the *cocharacter lattice*, and recall that a T-bundle on $\mathbb{P}^1$ is defined by a transition map

$$g \colon U_0 \cap U_\infty = \mathbb{G}_m \to T,$$

which must exactly be a cocharacter after rescaling. So

$$\{T\text{-bundles on } \mathbb{P}^1\} \cong X_*(T).$$

The theorem then says that G-bundles on $\mathbb{P}^1$ are classified by $X_*(T)/W$.

Remark. If $T \subset G$ is a maximal torus, then the cocharacter lattice $X_*(T)$ is equivalently the *character* or *weight lattice* $\Lambda^\vee$ of the Langlands dual group $G^\vee$. Thus the theorem says that G-bundles on $\mathbb{P}^1$ are parameterized by

$$\Lambda^\vee/W \cong \Lambda^\vee_+,$$

the set of dominant integral weights for $G^\vee$, which labels its irreducible representations.

1.2 Double quotient realization of $\mathrm{Bun}_G(X)$

Last time, we considered a smooth irreducible projective curve X and a split connected reductive group G over k. To this pair we attached the moduli stack $\mathrm{Bun}_G(X)$ of principal G-bundles on X. In general, $\mathrm{Bun}_G(X)$ is a very complicated object, but most of these complications will not be relevant for us. We defined $\mathrm{Bun}_G(X)$ via its functor of points, but today we would like to describe $\mathrm{Bun}_G(X)$ in a slightly more explicit way.

For simplicity, let's assume first that G is semisimple and k is algebraically closed. By Harder's theorem from last lecture, every $E \in \mathrm{Bun}_G(X)$ trivializes once any chosen point is removed from X. So pick a k-point $x \in X$. Cover X by two charts: a disk around x, and $X \setminus x$. In algebraic geometry, we do not have small disks, but we can take a *formal* disk D_x around x instead. To consider bundles using these two charts, there is no 1-cocycle condition, and it suffices to study the transition function on the intersection

$$(X \setminus x) \cap D_x = D_x^\times$$

of the two charts. Here $D_x^\times$ is the *punctured formal disk*. To be precise, let $R \coloneqq \mathcal{O}(X \setminus x)$ be the ring of regular functions on the affine curve $X \setminus x$, and, if t is a formal coordinate at x,

$$\mathcal{O}(D_x) \cong \mathcal{O} \coloneqq k[[t]]$$
$$\mathcal{O}(D_x^\times) \cong K \coloneqq k((t)).$$

Remark. In complex analysis, regular meromorphic functions on the punctured disk are given by convergent Laurent series which are finite in the negative direction. We make these series formal by removing the convergence assumption, and then they make sense over any field.

The inclusion $\mathcal{O} \subset K$ is like a "ring of integers", consisting of elements with valuation ≥ 0. There is also an inclusion

$$R \hookrightarrow K$$

given by taking Laurent series expansion. This description appears to require the choice of the coordinate t, but viewing it geometrically shows that it in fact does not. Bundles E are defined by transition maps $g(t)$ from D_x to $X \setminus x$, or equivalently, elements $g \in G(K)$, up to $g \mapsto h_1 g h_2^{-1}$ where $h_1 \in G(R)$ and $h_2 \in G(\mathcal{O})$. We have thus proved the following proposition.

Proposition. $\mathrm{Bun}_G(X)(k) = G(R)\backslash G(K)/G(\mathcal{O})$.

It is productive to first consider $\mathrm{Gr}_G \coloneqq G(K)/G(\mathcal{O})$, called the *affine Grassmannian*. It is an *ind-variety* — infinite-dimensional, but a nested union of projective varieties of increasing dimension. We see therefore that G-bundles on X correspond to orbits of $G(R)$ on Gr_G.

Remark. The name "affine Grassmannian" is because both the affine Grassmannian and the ordinary Grassmannian are quotients of a Kac–Moody group by a maximal parabolic subgroup.

We can generalize this construction by removing multiple points from X instead of just one. Namely, let $S \subset X$ be a non-empty finite subset, and take the two charts $U_1 \coloneqq X \setminus S$ and $U_2 \coloneqq \bigsqcup_{x\in S} D_x$ where D_x is a formal disk around the point x. Then

$$U_1 \cap U_2 = \bigsqcup_{x\in S} D_x^\times,$$

and therefore, by the same reasoning as above,

$$\mathrm{Bun}_G(X)(k) = G(X \setminus S)\Big\backslash \prod_{s\in S} G(K_x)\Big/ \prod_{x\in S} G(\mathcal{O}_x), \tag{1}$$

where, like before, $\mathcal{O}_x$ and K_x denote the rings of regular functions on D_x and $D_x^\times$ respectively.

Recall that for non-semisimple groups G, e.g. $\mathrm{GL}_1 = \mathbb{G}_m$, there is no finite set S such that all G-bundles are trivialized on $X \setminus S$. So, to generalize the above construction to such groups, we will remove *all* rational points. This sounds like then there will be nothing left, but in fact this is not so; the "Grothendieck generic point" still remains! Indeed, removing a single point in algebraic geometry means to consider rational functions which are allowed to have a pole at that point. So, removing all points means to consider rational functions which are allowed to have poles anywhere, i.e. simply all rational functions. We obtain the presentation

$$\mathrm{Bun}_G(X)(k) = G(k(X))\Big\backslash {\prod_{x\in X}}' G(K_x)\Big/ \prod_{x\in X} G(\mathcal{O}_x). \tag{2}$$

The prime on the product is a technical detail. It denotes the *restricted product* consisting of elements with only finitely many coordinates having poles, i.e. not lying in $G(\mathcal{O}_x)$. The restricted product arises because we are taking a colimit of (1) over *finite* sets S in order to obtain (2).

Finally, if k is not algebraically closed, we can do the same construction using finite subsets $S \subset X(\bar{k})$ which are Galois-invariant. Here $\bar{k} \supset k$ is the algebraic closure. Let $\Gamma \coloneqq \mathrm{Gal}(\bar{k}/k)$. Then

$$\mathrm{Bun}_G(X)(k) = G(k(X))\Big\backslash {\prod_{x\in X(\bar{k})/\Gamma}}' G(K_x)\Big/ \prod_{x\in X(\bar{k})/\Gamma} G(\mathcal{O}_x)$$

For example, if k is finite, all completions F_v of $F = k(X)$ with respect to valuations v are locally compact topological fields. Such F are called *global fields*. We get

$$\mathrm{Bun}_G(X)(k) = G(F)\Big\backslash G\Big({\prod_{v\in \mathrm{Val}(F)}}' F_v\Big)\Big/ G\Big(\prod_{v\in\mathrm{Val}(F)} \mathcal{O}_v\Big)$$

where $\mathrm{Val}(F)$ is the set of valuations of F. This is called an *arithmetic quotient*. Note that this realization of $\mathrm{Bun}_G(X)(k)$ now holds for any reductive group G, not necessarily semisimple.

For motivation, similar quotients arise in number theory. While global fields of characteristic p have the form $k(X)$, in characteristic 0 they are number fields, i.e. finite extensions of $\mathbb{Q}$.

Definition. If F is a global field, the *ring of adèles* is

$$\mathbb{A} := \mathbb{A}_F := \prod_{v \in \mathrm{Val}(F)}' F_v.$$

Over a number field F, there are two kinds of valuations: Archimedean (embed into $\mathbb{C}$ and take absolute value) and non-Archimedean ones (p-adic valuations). Rings of integers $\mathcal{O}_v \subset F_v$ make sense for non-Archimedean valuations. Let

$$\mathcal{O}_{\mathbb{A}} := \prod_{v \in \mathrm{Val}_{n.a.}(F)} \mathcal{O}_v$$

where $\mathrm{Val}_{n.a.}(F)$ is the set of non-Archimedean valuations. Then we can consider

$$M := G(F)\backslash G(\mathbb{A})/G(\mathcal{O}).$$

This generalizes $\mathrm{Bun}_G(X)$, because if $F = k(X)$, then $M = \mathrm{Bun}_G(X)(k)$.

Example. Let $F = \mathbb{Q}$. Then there are valuations of F with respect to all prime integers, and also the usual ∞ valuation. Then

$$\mathbb{A} = \mathbb{R} \times \prod_{p \text{ prime}}' \mathbb{Q}_p$$
$$\mathcal{O}_{\mathbb{A}} = \prod_{p \text{ primes}} \mathbb{Z}_p,$$

and

$$\mathcal{M} = G(\mathbb{Q})\Big\backslash\Big(G(\mathbb{R}) \times \prod_p{}' G(\mathbb{Q}_p)\Big)\Big/G(\mathbb{Z}_p) = G(\mathbb{Z})\backslash G(\mathbb{R}).$$

For instance, if $G = \mathrm{Sp}_{2n}$, then $\mathcal{M} = \mathrm{Sp}(2n,\mathbb{Z})\backslash \mathrm{Sp}(2n,\mathbb{R})$, and taking a quotient by $U(n)$ on the right gives the moduli space of n-dimensional abelian varieties

$$\mathcal{A}_n = \mathrm{Sp}(2n,\mathbb{Z})\backslash \mathrm{Sp}(2n,\mathbb{R})/U(n).$$

In particular, if $G = \mathrm{SL}_2 = \mathrm{Sp}_2$, then

$$\mathcal{A}_1 = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/U(1)$$

is the moduli space of elliptic curves. Modular forms live on this $\mathrm{SL}_2(\mathbb{R})/U(1)$. For another example, if $G = \mathrm{GL}_1$ and k is any field, then

$$\mathrm{Jac}(X)(k) = k(X)^\times\backslash\mathbb{A}^\times/\mathcal{O}_{\mathbb{A}}^\times.$$

The quotient $\mathbb{A}^\times/\mathcal{O}_{\mathbb{A}}^\times$, modulo $k^\times$, is the group of divisors on X. (Here, $\mathbb{C}^\times$ is the intersection $k(X)^\times \cap \mathcal{O}_{\mathbb{A}}^\times$.)

1.3 Stable bundles and Higgs fields

Hitchin systems are integrable systems, and integrable systems live on symplectic manifolds. To create a symplectic manifold in our setting, we want to define the cotangent bundle $T^* \operatorname{Bun}_G(X)$. But, as explained, $\operatorname{Bun}_G(X)$ is not a manifold, or even a scheme. It is a stack, however, and the cotangent bundle of a stack is well-defined as another stack. We will avoid this, however, and work only with a particular open set in $T^* \operatorname{Bun}_G(X)$.

Assume $g \geq 2$ and first assume that G is simple and of adjoint type (i.e. the center of G is trivial). A "generic" bundle in $\operatorname{Bun}_G(X)$ has trivial automorphism group, so in the local presentation of $\operatorname{Bun}_G(X)$ as a quotient of an algebraic variety by a group, the group acts freely at such a bundle. The locus of "generic" bundles therefore forms a smooth algebraic variety $\operatorname{Bun}_G^\circ(X)$.

There are many ways to specify what "generic" means. We will use a stability condition. Let $G = \mathrm{GL}_n$. Then G-bundles are rank-n vector bundles. If $E \neq 0$ is a vector bundle on X, there are two integers attached to it: the degree $d(E)$ (given by the first Chern class), and the rank $r(E)$.

Definition. The *slope* of E is
$$\mu(E) \coloneqq d(E)/r(E).$$
We say E is *stable* if for every sub-bundle $0 \neq E' \subsetneq E$,
$$\mu(E') < \mu(E).$$

There is a more technical definition for other reductive groups G, which we will not state.

Exercise. If L is a line bundle, and E is a vector bundle, then E is stable if and only if $E \otimes L$ is stable.

This exercise shows that stability, as we defined it above, is also well-defined for PGL_n-bundles, which are equivalently rank-n vector bundles modulo tensor product with line bundles.

Theorem. *Stable bundles have the trivial group of automorphisms, and form a smooth variety which is an open subset* $\operatorname{Bun}_G^\circ(X) \subset \operatorname{Bun}_G(X)$.

Let $\mathcal{M}_G^\circ(X) \coloneqq T^* \operatorname{Bun}_G^\circ(X)$. The Hitchin system will initially live on $\mathcal{M}_G^\circ(X)$, but actually there is a partial compactification $\mathcal{M}_G(X)$ of $\mathcal{M}_G^\circ(X)$ called the *Hitchin moduli space*, which is still symplectic, to which the Hitchin system naturally extends. This sort of extension to a partial compactification is a natural phenomenon in integrable systems.

For general semisimple G, not necessarily of adjoint type, there is a straightforward extension of this story. A bundle is "generic" if it is *regularly stable*, meaning that it is stable and its group of automorphisms reduces to the center $Z(G)$ (which is the smallest it can be). The resulting $\operatorname{Bun}_G^\circ(X)$ is still a stack with stabilizer $Z(G)$ at every point, but because this stabilizer is the same everywhere, we can rigidify like we did for the Jacobian. In other words, we may ignore the stackiness and just consider the underlying variety.

Before we do anything, we should compute $\dim \operatorname{Bun}_G^\circ(X)$, or, equivalently since it is a smooth variety, the dimension $\dim T_E \operatorname{Bun}_G^\circ(X)$ of the tangent space at a point $E \in \operatorname{Bun}_G^\circ(X)$. This tangent space is just the deformation space of the bundle E.

Exercise. Deformations of the bundle E are classified by $H^1(X, \operatorname{ad} E)$, where $\operatorname{ad} E$ is the *adjoint bundle* of E. This is the vector bundle associated to E given by the adjoint representation.

Example. Let $G = \mathrm{GL}_n$. Deformations of a vector bundle E are classified by $\operatorname{Ext}^1(E, E)$; this is unsurprising because, affine-locally, we are just deforming modules. Since

$$\operatorname{Ext}^1(E, E) = \operatorname{Ext}^1(\mathcal{O}, E^* \otimes E) = H^1(X, \operatorname{ad} E),$$

this agrees with our claim that deformations are classified by $H^1(X, \operatorname{ad} E)$.

Let $\mathfrak{g}$ be the Lie algebra of G. The invariant pairing on $\mathfrak{g}$ may be used to identify $(\operatorname{ad} E)^* \cong \operatorname{ad} E$. Using this and Serre duality, it follows that

$$\begin{aligned} T_E^* \operatorname{Bun}_G^\circ(X) = H^1(X, \operatorname{ad} E)^* &\cong H^0(X, \mathcal{K}_X \otimes (\operatorname{ad} E)^*) \\ &= H^0(X, \mathcal{K}_X \otimes \operatorname{ad} E) \end{aligned}$$

Elements of $H^0(X, \mathcal{K}_X \otimes \operatorname{ad} E)$ have a very vivid geometric interpretation: they are 1-forms on X with coefficients in $\operatorname{ad} E$. From physics, they have the name *Higgs fields* on E.

It remains to compute $\dim H^0(X, \mathcal{K}_X \otimes \operatorname{ad} E)$. Let's assume for simplicity that G is simply-connected, in which case $\operatorname{Bun}_G(X)$ is irreducible. (The result extends to the non-simply-connected case as well.) The Euler characteristic

$$\chi(X, \mathcal{K}_X \otimes \operatorname{ad} E) \coloneqq \dim H^0(X, \mathcal{K}_X \otimes \operatorname{ad} E) - \dim H^1(X, \mathcal{K}_X \otimes \operatorname{ad} E)$$

is deformation-invariant, and $\dim H^1(X, \mathcal{K}_X \otimes \operatorname{ad} E) = \dim H^0(X, \operatorname{ad} E) = 0$, where the first equality is Serre duality and the second equality is because stable bundles have no (infinitesimal) automorphisms.[1] In particular, we may compute the Euler characteristic for the trivial bundle $E = \mathcal{O}_X$:

$$\begin{aligned} \chi(X, \mathcal{K}_X \otimes \mathfrak{g}) &= \Big(\dim H^0(X, \mathcal{K}_X) - \dim H^1(X, \mathcal{K}_X)\Big) \cdot \dim G \\ &= (g-1) \cdot \dim G. \end{aligned}$$

Since $\operatorname{Bun}_G(X)$ is connected, any bundle can be connected to the trivial bundle by a path. For generic bundles E, we have therefore computed that

$$\dim H^0(X, \mathcal{K}_X \otimes \operatorname{ad} E) = (g-1) \cdot \dim G.$$

For $G = \mathbb{G}_m$, we know $\operatorname{Bun}_G(X) = \operatorname{Jac}(X)$ has dimension g. So for general reductive G,

$$\dim \operatorname{Bun}_G^\circ(X) = (g-1) \dim \mathfrak{g} + \dim Z(\mathfrak{g}).$$

Example. For $G = \mathrm{GL}_n$, the dimension is $(g-1)n^2 + 1$.

[1] This is where we use that G is semisimple, because otherwise generic G-bundles still have automorphisms, and so $\dim H^1$ will not vanish in the Euler characteristic.

1.4 The Hitchin integrable system

We are ready to define the Hitchin integrable system. Let $G = \mathrm{SL}_n$. In this case, $\dim \mathrm{Bun}_G^\circ(X) = (n^2-1)(g-1)$, and elements in $\mathrm{Bun}_G^\circ(X)$ are pairs (E, ϕ) where E is a stable bundle and $\phi \in \Omega^1(X, \operatorname{End} E)$ (with trace zero) is a Higgs field.

Definition. The *Hitchin map* is

$$\begin{aligned} p\colon T^* \mathrm{Bun}_G^\circ(X) &\to \bigoplus_{i=1}^{n-1} H^0(X, K_X^{\otimes(i+1)}) =: \mathcal{B} \\ (E, \phi) &\mapsto (\operatorname{tr} \wedge^2 \phi, \operatorname{tr} \wedge^3 \phi, \dots, \operatorname{tr} \wedge^n \phi). \end{aligned}$$

The target is called the *Hitchin base*. By Riemann–Roch, $\dim H^0(X, K_X^{\otimes(i+1)}) = (2i+1)(g-1)$, and so the dimension of the Hitchin base is

$$\sum_{i=1}^{n-1} (2i+1)(g-1) = (n^2-1)(g-1) = \dim \mathrm{Bun}_G^\circ(X).$$

Theorem (Hitchin)**.** *p is an integrable system, i.e. coordinates (given by choosing a basis of the Hitchin base) are Poisson-commuting and functionally independent.*

2 Problem session 2

Problem 1. Let L be a non-trivial line bundle of degree 0 on an elliptic curve E over $\mathbb{C}$. Show that

1. $L|_{E \setminus 0}$ is non-trivial as an algebraic bundle, but
2. $L|_{E \setminus 0}$ is trivial as an analytic bundle.

So GAGA fails for the (non-projective) curve $E \setminus 0$.

Solution. 1. If $L|_{E\setminus 0}$ were the trivial line bundle, then it has a nowhere-vanishing global section s. Since transition maps for L are meromorphic, s extends to a meromorphic section of L with poles allowed only at $0 \in E$. This is impossible since L is a non-trivial line bundle of degree 0, and in particular has no global sections.

2. Let $\theta(z)$ be the theta function of E, and $\zeta(z) := \theta'(z)/\theta(z)$. These functions are periodic with period 1 and

$$\theta(z + \tau) = Ce^{2\pi i z}\theta(z)$$

for some constant C. Thus $\zeta(z+\tau) = \zeta(z) - 2\pi i$. Hence the *Lame–Hermite function*

$$H(z) := e^{a\zeta(z)} \frac{\theta(z-a)}{\theta(z)}$$

is doubly-periodic, i.e. is a holomorphic function on $E \setminus 0$. It has a simple zero at a and no other zeros and poles, but it has an essential singularity at 0. Thus H may be viewed as a non-vanishing holomorphic section of the analytic line bundle $\mathcal{O}(a)^\vee$ over $E \setminus 0$. Hence $\mathcal{O}(a)$ is trivial on $E \setminus 0$. But the given line bundle L is of the form $L = \mathcal{O}(a) \otimes \mathcal{O}(0)^\vee$ for some $0 \neq a \in E$, so we are done.

Problem 2. Show that any GL_n-bundle E on a smooth projective curve X admits a B-structure, i.e. is associated to a (non-unique) B-bundle. Here $B \subset \mathrm{GL}_n$ is the subgroup of upper-triangular matrices.

Solution. Recall that GL_n-bundles E are equivalently vector bundles. A B-structure on the vector bundle E is equivalently a filtration by sub-bundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that E_{i+1}/E_i are line bundles. We construct such a filtration by induction on the rank n. Let L be an ample line bundle on X. Then

$$H^0(X, E \otimes L^{\otimes N}) \neq 0, \qquad \forall N \gg 0.$$

Let s be a non-zero section, equivalently a non-zero morphism $L^{\otimes -N} \to E$. If s vanishes at points $p_1, \ldots, p_s$ to orders $r_1, \ldots, r_s$, then we get a non-vanishing map $L' \hookrightarrow E$ where

$$L' \coloneqq L^{\otimes -N} \otimes \mathcal{O}(\sum_{i=1}^{s} r_i p_i).$$

(Alternatively, assuming $n > 1 = \dim X$, a sufficiently generic section s is already nowhere vanishing.) Complete this inclusion into a short exact sequence

$$0 \to L' \to E \to E' \to 0.$$

Since $\operatorname{rank} E' < \operatorname{rank} E$, by the induction hypothesis, E' has a filtration by sub-bundles

$$0 = E'_0 \subset E'_1 \subset \cdots E'_{n-1} = E'.$$

Adding L' to this filtration, we obtain the desired filtration of E.

Problem 3. Suppose G is connected and reductive. Show, using Čech 1-cocycles, that the tangent space to $\mathrm{Bun}_G^\circ(X)$ at E is $H^1(X, \operatorname{ad} E)$.

Solution. Pick a finite cover $X = \bigcup_{i \in I} U_i$ such that the principal G-bundle E is trivialized on each U_i. (We may take a Zariski cover by the hypothesis on G.) Recall that E is determined by transition functions, which are regular functions

$$g_{ij} \colon U_i \cap U_j \to G,$$

satisfying $g_{ij} \circ g_{ji} = \mathrm{id}$ and the 1-cocycle condition $g_{ij} \circ g_{jk} \circ g_{ki} = \mathrm{id}$. Therefore, an infinitesimal deformation of E is given by the modification

$$g_{ij} \mapsto \tilde{g}_{ij} \coloneqq g_{ij} \cdot (1 + \exp(\epsilon \xi_{ij}))$$

where $\epsilon^2 = 0$, for a choice of regular function $\xi_{ij} \colon U_i \cap U_j \to \mathfrak{g}$ for each $i, j \in I$. Here $\mathfrak{g}$ denotes the adjoint representation of G. The conditions that $\{\tilde{g}_{ij}\}_{i,j \in I}$ is still a set of valid transition functions, namely that $\tilde{g}_{ij} \circ \tilde{g}_{ji} = \mathrm{id}$ and $\tilde{g}_{ij} \circ \tilde{g}_{jk} \circ \tilde{g}_{ki} = \mathrm{id}$, hold if and only if

$$\begin{aligned} g_{ij} \xi_{ij} g_{ij}^{-1} &= -\xi_{ji}, \\ g_{ij} \xi_{ij} g_{ij}^{-1} + g_{ik} \xi_{jk} g_{ik}^{-1} + \xi_{ki} &= 0. \end{aligned}$$

These equations exactly express that the element

$$(\xi_{ij})_{i,j\in I} \in \bigoplus_{i,j\in I} H^0(U_i \cap U_j, \operatorname{ad} E)$$

lies in the kernel of the Čech differential. Similarly, recall that two sets $\{g_{ij}\}_{i,j\in I}$ and $\{g'_{ij}\}_{i,j\in I}$ of transition functions are equivalent if and only if there exist regular functions $\{h_i \colon U_i \to G\}_{i\in I}$ such that $g'_{ij} = h_i g_{ij} h_j^{-1}$. A similar reasoning shows that the deformed transition functions defined by two different $(\xi_{ij})_{i,j\in I}$ and $(\xi'_{ij})_{i,j\in I}$ are equivalent if and only if they differ by the image, under the Čech differential, of an element

$$(\xi_i)_{i\in I} \in \bigoplus_{i\in I} H^0(U_i, \operatorname{ad} E).$$

Putting it all together, $T_E \operatorname{Bun}_G^\circ(X)$ is the cohomology at the middle term of the Čech complex

$$\bigoplus_{i\in I} H^0(U_i, \operatorname{ad} E) \to \bigoplus_{i,j\in I} H^0(U_i \cap U_j, \operatorname{ad} E) \to \bigoplus_{i,j,k\in I} H^0(U_i \cap U_j \cap U_k, \operatorname{ad} E),$$

which is $H^1(X, \operatorname{ad} E)$ by definition.