1. **Grassmannians and Chern classes.**

1.1. Check that the elementary symmetric function $e_k(x_1, x_2, \dots)$ is the unique symmetric function of degree k such that

$$e_k(x_1, \dots, x_k, 0, 0, 0, \dots) = x_1 x_2 \cdots x_k \,.$$

1.2. Let $c(V,t) = \sum_{k \geq 0} t_k c_k(V)$ denote the Chern polynomial of a vector bundle V. Prove that for any subbundle $V' \subset V$ we have

$$c(V,t) = c(V',t)\, c(V/V',t) \,.$$

1.3. Compute the Chern classes of the tangent bundle to $\mathbb{P}^n$. Here and below all projective spaces, Grassmannian, flag varieties etc. denote the corresponding complex manifolds.

1.4. Let $M \subset \mathbb{P}^n$ be a smooth (complex, as always) hypersurface of degree d. Consider the exact sequence of the vector bundles

$$0 \to TM \to T\mathbb{P}^n \Big|_M \to N_{\mathbb{P}^n/M} \to 0$$

on M, where $N_{\mathbb{P}^n/M}$ is the normal bundle. Check that

$$N_{\mathbb{P}^n/M} = \mathscr{O}_{\mathbb{P}^n}(d)|_M$$

and conclude a formula for the Chern classes of M. In particular, when is $c_1(M) = 0$? What is the topological Euler characteristic[1] of M ? What is the genus of the curve M when $n = 2$? Compute $\int_M c_1(M)^{n-1}$.

1.5. Consider the Grassmannian $X = \mathsf{Gr}(k,n)$. Its points correspond to linear subspaces $V \subset \mathbb{C}^n$ of dimension k. Consider the set of pairs

$$\{(V, v), \text{ such that } v \in V\} \subset \mathsf{Gr}(k,n) \times \mathbb{C}^n \,. \qquad (.1)$$

Check that the LHS in (.1) is a vector bundle of rank k over X, called the tautological subbundle of the trivial bundle in the RHS of (.1). By a slight abuse of notation, we denote this tautological bundle by V. Check that

$$TX = V^* \otimes (\mathbb{C}^n/V) \,,$$

where V^* denotes the dual bundle and $\mathbb{C}^n/V$ is the tautological quotient bundle. Express $c_1(TX)$ and $c_2(TX)$ of the tangent bundle TX in terms of the Chern classes of V. If you are familiar with the language of symmetric functions, propose a formula for $c_k(TX)$.

[1] By the Lefschetz hyperplane section theorem, this is equivalent to knowing the dimension of $H^{\text{middle}}(M)$

1.6. Consider the product $X \times X$, where $X = \mathsf{Gr}(k, n)$ as in Problem 1.5. On $X \times X$ we have two tautological bundles V_1 and V_2 pulled back from the two factors. Consider the composed map

$$V_1 \to \mathbb{C}^n \to \mathbb{C}^n / V_2 \,.$$

Show this section of $(\mathbb{C}^n/V_2) \otimes V_1^*$ vanishes precisely over the diagonal in $X \times X$, in other words, the top Chern class of $(\mathbb{C}^n/V_2) \otimes V_1^*$ is the class

$$\Delta \in H^{2 \dim X}(X \times X)$$

of the diagonal. Consider the operation

$$\Phi_\Delta : \gamma \mapsto p_{1,*}(\Delta \cup p_2^*(\gamma))$$

from $H^\bullet(X)$ to $H^\bullet(X)$, where p_1 and p_2 are the projections of $X \times X$ to the respective factors. Show that Φ_Δ is the identity map. Conclude that the cohomology of X is generated by the Chern classes of V.

What is the K-theory analog of these statements ?

1.7. The group $GL(n)$ acts on $X = \mathsf{Gr}(k, n)$ via its defining action on $\mathbb{C}^n$. Describe the orbits of the subgroup $U \subset GL(n)$ formed by lower-triangular matrices with 1s on the diagonal. Show that each orbit contains a unique fixed point for the subgroup

$$\mathsf{A} = \mathrm{diag}(a_1, \dots, a_n)$$

to which all other points are attracted when $a_1/a_2, a_2/a_3, \dots \to \infty$. The orbits are called the Schubert cells and their closures are called Schubert varieties $\mathfrak{S}_\lambda$. They are naturally indexed by partitions λ that fit into $k \times (n-k)$ rectangle. Show they form a basis in integral homology or cohomology of X.

1.8. Show that Schubert cycles $\mathfrak{S}_\lambda^\vee$ for the subgroup U_{opp} of upper-triangular matrices form a basis dual to the basis of Schuber cycles. Translate the equality

$$c_{\mathrm{top}}(V_1^* \otimes (\mathbb{C}^n/V_2)) = \sum_\lambda [\mathfrak{S}_\lambda] \boxtimes [\mathfrak{S}_\lambda^\vee] \in H^\bullet(X \times X)$$

into an identity of symmetric functions.

1.9. Compute the Poincaré polynomial of $\mathsf{Gr}(k, n)$ and compare it with number of points of $\mathsf{Gr}(k, n)$ over a finite field with q elements.

1.10. For $X = \mathbb{P}^{n-1} = \mathsf{Gr}(1, n)$, Schubert classes $\mathfrak{S}_l$, $l = 0, \dots, n-1$, form a chain

$$\mathbb{P}^{n-1} \supset \mathbb{P}^{n-2} \supset \cdots \supset \mathbb{P}^{n-1-l} \supset \dots$$

cut out by the equations $x_1 = \cdots = x_l = 0$. Here x_i are the homogeneous coordinates, or more precisely the components of the natural map

$$(\mathbb{C}^n)^* \otimes \mathscr{O}_X \to \mathscr{O}(1)_X \,.$$

In particular, each individual coordinate x_i is a A-equivariant map

$$\mathscr{O}_X \otimes a_i^{-1} \xrightarrow{x_i} \mathscr{O}(1)_X,$$

and so its zero locus represents the class $\xi + \alpha_i$, where $\xi = c_1(\mathscr{O}(1))$ and $\alpha_i \in H^2_{\mathsf{A}}(\mathrm{pt})$ corresponds to the character a_i. Therefore

$$[\mathfrak{S}_l] = \prod_{i=1}^{l} (\xi + \alpha_i) \in H^\bullet_{\mathsf{A}}(X)\,. \tag{.2}$$

1.11. Verify that the polynomial (.2) is characterized by the following Newton interpolation properties:

- it has degree l in the variables ξ and α_i, corresponding to the fact that $[\mathfrak{S}_l] \in H^{2l}(\mathsf{Gr})$,
- its restriction to A-fixed points not in $\mathfrak{S}_l$ vanishes ,
- its restriction to the A-fixed point in the Schubert cell equals the Euler class of the normal bundle to the Schubert cell.

Generalize this reasoning to compute the classes of the Schubert cells in the $\mathsf{Gr}(k,n)$. Your answer should look like a Schur function in the Chern roots $\xi_1, \ldots, \xi_k$ of the universal bundle, in which the monomials ξ_j^l are replaced by univariate interpolation polynomials of the form (.2). Those unfamiliar with Schur functions will discover them for themselves by solving Problem 1.13

1.12. Generalize the results of Problems 1.10 and 1.11 to equivariant K-theory.

1.13. Let G be the group $GL(n,\mathbb{C})$, $B \subset G$ be the subgroup of the upper-triangular matrices and $\chi : B \to \mathbb{C}^\times$ a character. Consider holomorphic, or meromophic, functions $f(g)$ of $g \in G$ which satisfy

$$f(gb) = f(g)\chi(b)\,, \forall b \in B\,.$$

Interpret them as sections of a holomorphic line bundle $\mathscr{L}_\chi$ on flag manifold

$$\mathrm{Flags}_n = G/B = U(n)/\text{diagonal matrices}\,.$$

Compute the Euler characteristic $\chi(\mathscr{L}_\chi)$ by equivariant localization. Compare your result with the Weyl character formula for G and explain[2] this comparison using the Peter-Weyl decomposition

$$\mathbb{C}[G] = \bigoplus_{\text{irreps } V} V^* \boxtimes V\,, \qquad \text{as } G \times G\text{-modules}\,.$$

[2]For a simple proof of fact that at most one cohomology group of $\mathscr{L}_\chi$ is nonvanishing see [`?Demazure`]

1.14. Let $\mathscr{L}$ be a complex line bundle with a connection ∇ and corresponding curvature $F \in \Omega^2(X, \mathbb{C})$. Show that, nonequivariantly, the form $\frac{i}{2\pi}F$ represents $c_1(\mathscr{L})$. For an equivariant generalization, see Chapter 7 in [?Berline-Getzler-Vergne].

For a rank r vector bundle V, the curvature form is matrix-valued, that is, $F \in \Omega^2(X, \operatorname{End} V)$. Show that

$$\sum_k t^k c_k(V) = \det\left(1 + \frac{it}{2\pi}F\right),$$

nonequivariantly.

2. **Elliptic functions and elliptic curves.**

2.1. Consider holomorphic functions $f(z)$ of $z \in \mathbb{C}^\times$ solving the q-difference equation

$$f(qz) = cz^{-d} f(z)\,, \tag{.3}$$

where q is a fixed complex number such that $|q| < 1$. Compute the dimension of the space of solutions as a function of d (and c, for $d = 0$) in two ways: first by analyzing the Laurent series expansion of f, and then by using the Riemann-Roch formula for the complex elliptic curve $E = \mathbb{C}^\times / q^{\mathbb{Z}}$.

2.2. Consider the function[3]

$$\theta(z) = \prod_{n>0}(1 - q^n z) \prod_{n\geq 0}(1 - q^n z^{-1})$$

and check that is solves the equation

$$\theta(qz) = -q^{-1}z^{-1}\theta(z)\,.$$

Prove that a general solution of (.3) has the form

$$f(z) = \text{const} \prod_{i=1}^{d} \theta(z/w_i)\,, \quad \prod w_i = (-q)^d c\,.$$

Interpret this result as saying that two divisors

$$\sum_{i=1}^{d} w_i, \sum_{i=1}^{d} w_i' \in S^d E$$

[3]In many, many contexts, it is more convenient to use a different normalization of the theta function, namely

$$\vartheta(z) = z^{1/2}\theta(z) = (z^{1/2} - z^{-1/2}) \prod_{n>0}(1 - q^n z)(1 - q^n z^{-1})\,.$$

It has a series expansion in half-integer powers of z, that is, satisfies $\vartheta(e^{2\pi i}z) = -\vartheta(z)$. It is still the unique, up to multiple, section of the line bundle $\mathscr{O}(e)$, where $e = 1 \in E = \mathbb{C}^\times / q^{\mathbb{Z}}$ is the identity, just for a different trivialization of the pullback of this line bundle to $\mathbb{C}^\times$. The extra convenience of using $\vartheta(z)$ is due to its anti-symmetry $\vartheta(z^{-1}) = -\vartheta(z)$.

are linearly equivalent if and only if $\prod w_i = \prod w_i'$ in E, that is, modulo $q^{\mathbb{Z}}$. In other words, the natural map

$$S^d E \to \operatorname{Pic}_d E \cong E$$

is from divisors to line bundles of the same degree is given by the multiplication in the group E. Its fibers are projective spaces for $d > 0$.

2.3. Let $f(z)$ be a meromorphic function on E, equivalently a rational function on the algebraic variety E. Show that is has the form

$$f(z) = \text{const} \prod_{i=1}^{d} \frac{\theta(z/a_i)}{\theta(z/b_i)}$$

for some values of a_i and b_j, where $\prod a_i = \prod b_i$ in E, that is, modulo $q^{\mathbb{Z}}$.

2.4. By cutting partitions λ along the diagonal, prove that

$$\sum_{\lambda} q^{\lambda} = \text{coefficient of } z^0 \text{ in } \prod_{n>0} (1 + q^{n-1/2} z)(1 + q^{n-1/2} z^{-1})\,.$$

Deduce that

$$\sum_{n\in\mathbb{Z}} q^{n^2/2} z^n = \prod_{n>0} (1 - q^n)(1 + q^{n-1/2} z)(1 + q^{n-1/2} z^{-1})\,,$$

which is one of the equivalent forms of the Jacobi triple product identity, and of the Macdonald identity for the Lie algebra $\mathfrak{sl}(2)$. Note this means

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{\binom{n+1}{2}} z^n = \theta(z) \prod_{n>0} (1 - q^n)$$

and compare the rate of convergence of two sides.

2.5. Let $\Gamma \subset \mathbb{C}$ be a lattice and $x \in \mathbb{C}$ be a complex number. Form the following product

$$\sigma(x) = x \prod_{0\neq\gamma\in\Gamma} \left(1 - \frac{x}{\gamma}\right) \exp\left(\frac{x}{\gamma} + \frac{x^2}{2\gamma^2}\right)\,,$$

known as the Weierstrass σ-funciton, and show that it represents an odd entire function of x. Consider a vector $\gamma \in \Gamma \setminus 2\Gamma$, which means

$$\sigma(\gamma/2) = -\sigma(\gamma/2) \neq 0\,.$$

For such vector γ, prove that

$$\frac{\sigma(x+\gamma)}{\sigma(x)} = -\exp(\eta_\gamma(x + \gamma/2))$$

for some constant $\eta_\gamma \in \mathbb{C}$. Express the function $\sigma(x)$ in terms of the theta function of the elliptic curve $E = \mathbb{C}/\Gamma$.

2.6. Given a lattice Γ as in Problem 2.5, its holomorphic Eisenstein series are defined by

$$\text{Eisenstein}(\Gamma, n) = \sum_{0 \neq \gamma \in \Gamma} \gamma^{-n},$$

which converges for $n > 2$ and vanishes for n odd. Relate these series to $\ln \sigma(x)$ and express them in terms of the parameter q in the isomorphism $E = \mathbb{C}/\Gamma \cong \mathbb{C}^\times/q^{\mathbb{Z}}$.

2.7. The *theorem of the cube* refers to various generalizations of the following basic statement. Let X and Y be complete algebraic varieties over a field $\Bbbk$, and let Z be an arbitrary variety over $\Bbbk$. Let $x \in X$, $y \in Y$, $z \in Z$ be $\Bbbk$-points. Let $\mathscr{L}$ be a line bundle over $X \times Y \times X$. If $\mathscr{L}$ is trivial when restricted to $\{x\} \times Y \times Z$, $X \times \{y\} \times Z$, and $X \times Y \times \{z\}$, then $\mathscr{L}$ is trivial. Find or read [4] a proof of this or any related statement. In the example $X = Y = E$, check that the corresponding statement for two factors is false.

2.8. Let A be an abelian variety, which for our purposes we will always assume to be of the form $A = E^n$. Let $\operatorname{Pic}(A)$ denote the Picard group of line bundles on A and let $\operatorname{Pic}_0(A)$ be the subgroup of line bundles that are algebraically equivalent to zero [5] Consider the map

$$\phi : \operatorname{Pic}(A) \times A \to \operatorname{Pic}_0(A)$$

that takes

$$(\mathscr{L}, a) \mapsto (\text{translation by } a)^* \mathscr{L} \otimes \mathscr{L}^{-1}.$$

Prove this is a group homomorphism, which is one of the forms of the *theorem of the square*.

2.9. Let $\mathscr{L} \in \operatorname{Pic}(A)$ be homogeneous, that is, $\phi(\mathscr{L}, a) = 0$ for all $a \in A$. Let

$$p_1, p_2, m : A^2 \to A$$

be the two projections and the multiplication map. Prove that

$$p_1^* \mathscr{L} \otimes p_2^* \mathscr{L} = m^* \mathscr{L},$$

and conclude

$$H^\bullet(\mathscr{L}) \otimes H^\bullet(\mathscr{L}) = H^\bullet(\mathscr{L}) \otimes H^\bullet(\mathscr{O}_A).$$

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[4] there are many sources for reading about this result, which goes back to A. Weil, with a classical exposition by Mumford. Among online resources, `https://www.math.ru.nl/personal/bmoonen/BookAV/LineBund.pdf` may be recommended.

[5] Two line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ are algebraically equivalent $\mathscr{L}_1 \sim \mathscr{L}_2$ if there is a line bundle $\widetilde{\mathscr{L}}$ on $A \times B$, where B connected, such that

$$\mathscr{L}|_{A \times \{b_1\}} = \mathscr{L}_1, \quad \mathscr{L}|_{A \times \{b_2\}} = \mathscr{L}_2,$$

for some $b_1, b_2 \in B$. You should check that this is an equivalence relation and

$$\mathscr{L}_1 \sim \mathscr{L}_1', \mathscr{L}_2 \sim \mathscr{L}_2' \quad \Rightarrow \quad \mathscr{L}_1 \otimes \mathscr{L}_2 \sim \mathscr{L}_1' \otimes \mathscr{L}_2'.$$

Since $\dim H^0(\mathscr{O}_A) = 1$, it follows that

$$\dim H^0(\mathscr{L}) \in \{0, 1\}\,.$$

Show that in first case $H^i(\mathscr{L}) = 0$ for all i, while in the second case $\mathscr{L}$ is trivial.

2.10. Show that any $\mathscr{L} \in \mathrm{Pic}_0(A)$ is homogeneous. We will see a converse to this statement below in Problem 2.14

2.11. For an elliptic curve E, check the exact sequence

$$0 \to \mathrm{Pic}_0(E) \to \mathrm{Pic}(E) \xrightarrow{\deg} \mathbb{Z} \to 0$$

and identify $\mathrm{Pic}_1(E)$, and hence $\mathrm{Pic}_0(E)$ with E itself.

Prove that for any B and any line bundle $\widetilde{\mathscr{L}}$ on $E \times B$ whose restrictions to the E-fibers has degree 0 and whose restriction to $0 \times B$ is trivial, there is a map $f : B \to E$ such that

$$\mathscr{L} = (\mathrm{id} \times f)^* \mathscr{P}$$

where $\mathscr{P}$ is the following line bundle on

$$\mathscr{P} = \mathscr{O}(\operatorname{diag} E - E \times \{0\} - \{0\} \times E)$$

on $E \times E$. This realizes E as the *dual* abelian variety $E^\vee = E$, and $\mathscr{P}$ as the Poincaré line bundle on $E \times E^\vee$.

2.12. For an abelian variety of the form $A = E^n$ prove that

$$A^\vee \cong A\,,$$

and construct the Poincare line bundle.

2.13. For elliptic curve E, define the Fourier-Mukai transform

$$\Phi : D^b \operatorname{Coh} E \to D^b \operatorname{Coh} E$$

by the equality

$$\mathscr{F} \mapsto p_{1,*}\left(\mathscr{P} \otimes p_2^*(\mathscr{F})\right)$$

where $p_1, p_2 : E^2 \to E$ are the two projections and the pushforward $p_{1,*}$ is derived. Show that

$$\Phi^2 = (\text{pullback by } a \mapsto -a)\,[-1]\,,$$

where $[-1]$ denotes the cohomological shift of a complex by one step to the right. In particular, Φ is an equivalence. Generalize to an abelian variety of the form $A = E^n$.

2.14. Let $\mathscr{L}$ be a homogeneous line bundle on $A = E^n$ which is not in $\mathrm{Pic}_0(A)$. Using the results of problem 2.9, show that $\Phi(\mathscr{L}) = 0$ and derive a contradiction. This proves that

$$\mathscr{L} \in \mathrm{Pic}_0(A) \quad \Leftrightarrow \quad \phi(\mathscr{L}, —) = 0\,.$$

Prove the exact sequence

$$0 \to \mathrm{Pic}_0(A) \to \mathrm{Pic}(A) \xrightarrow{\phi(\mathscr{L},—)} \mathrm{Hom}_{\text{symmetric}}(A, A^\vee) \to 0\,, \tag{.4}$$

where symmetric means $\phi : A \to A^\vee$ is equal to the pullback map $\phi^\vee : \mathrm{Pic}_0(A^\vee) = A \to \mathrm{Pic}_0(A) = A^\vee$.

2.15. The sequence (.4) is true for all abelian varieties, not just those of the form $A = E^n$, but the proof is more involved. It shows that the map $\phi(\mathscr{L}, —)$ is the correct multivariate generalization of the degree.

What is the degree of the Poincaré bundle on $E \times E$? What is the degree of the line bundle on $A = E^n$ whose section $s(z)$ is given by

$$s(z) = \prod \theta(c_\mu z^\mu)^{m_\mu}$$

where $z^\mu = \prod_{i=1}^n z_i^{\mu_i}$, $c_\mu \in \mathbb{C}^\times$, and $m_\mu \in \mathbb{Z}$. When does such expression give a rational function on A ?

3. **Krichever's proof of rigidity of the elliptic genus.**

3.1. We begin with a discussion of how to read and interpret localization formulas. Let V be an equivariant vector bundle on X. Define

$$\Lambda^\bullet_t V = \sum_n (-t)^n \Lambda^n V\,, \quad S^\bullet_t = \sum_n t^n S^n V\,,$$

where we interpret the second expression as an element of $K_G(X)[[t]]$. Check that[6]

$$\Lambda^\bullet_t V \otimes S^\bullet_t V = 1\,.$$

3.2. Let $V \in K(X)$ be a vector bundle and assume that $\dim_{\mathbb{Q}} K(X) \otimes \mathbb{Q}$ is a finite-dimensional vector space over $\mathbb{Q}$. Prove that all eigenvalues of the operator of tensor product by V in $K(X) \otimes \mathbb{Q}$ are equal to $\mathrm{rk}\, V$. Moreover, these operators commute for different bundles V_1, V_2. Conclude that all eigenvalues of the operator $\otimes \Lambda^\bullet_t V$ are equal to $(1-t)^{\mathrm{rk}\, V}$, and hence this operator is invertible as a rational function in t with a pole at $t = 1$. As $t \to 0, \infty$, we have

$$(\Lambda^\bullet_t V)^{-1} \sim 1, \quad t \to 0, \qquad (\Lambda^\bullet_t V)^{-1} \sim t^{-\mathrm{rk}\, V} \Lambda^{\mathrm{top}} V^*, \quad t \to \infty\,.$$

[6]you may want to interpret this equality in terms of the Koszul resolution of structure sheaf $\mathscr{O}_0$ of the zero section of V^*

What does this say about series of the form

$$\chi(X, \mathscr{F} \otimes S^\bullet_t V) \in \mathbb{Z}[[t]]$$

where $\mathscr{F} \in K(X)$ is arbitrary ?

3.3. What is the equivariant analog of the results in problem 3.2 ?

3.4. We abbreviate

$$\phi(z) = \prod_{n>0} (1 - q^n z)$$

and define Krichever genus by

$$\mathscr{E}_y(X) = \chi\left(X, \frac{\theta(y \otimes TX)}{\phi(TX)\phi(T^*X)}\right) \quad y \in \mathbb{C}^\times .$$

Here X is compact complex or, more generally, a stably almost complex manifold. Assuming there is S^1-action on X, write make the equivariant localization formula for $\mathscr{E}_y(X)$ explicit. Determine the possible singularities of $\mathscr{E}_y(X)$ as a function on the complexification $\mathbb{C}^\times$ of the group S^1.

3.5. Consider the canonical bundle $\mathscr{K}_X = \Lambda^{\text{top}} T^* X$. Assume that $\mathscr{K}_X$ admits, equivariantly, a root of order N and that $y^N = 1$. (This includes the case when $\mathscr{K}_X$ is trivial and y is arbitrary.) These are the assumptions in the rigidity theorem for $\mathscr{E}_y(X)$. Show that, with these assumptions, $\mathscr{E}_y(X)$ is invariant under $t \mapsto qt$, that is, a function on $E = \mathbb{C}^\times / q^{\mathbb{Z}}$.

3.6. For any $n = 1, 2, \ldots$, let $\mu_n \in S^1$ be the group of elements of order n, and let X_n be the fixed locus of μ_n. Show that it is a smooth and (stably, almost) complex. Denote by N_n the normal bundle of X_n in X. Check that

$$\mathscr{E}_y(X) = \chi\left(X^{(n)}, \frac{\theta(y \otimes TX_n)}{\phi(TX_n)\phi(T^*X_n)} \frac{\theta(y \otimes N_n)}{\theta(N_n)}\right) .$$

Conclude that $\mathscr{E}_y(X)$ is regular at all points of order n in E and, hence, a constant.

. Solutions and hints

1. **Solution to 1.1.** Consider the short exact sequence:

$$0 \to \Omega_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}^n} \to 0 \tag{.1}$$

Take the dual of the short exact sequence and we have:

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(1)^{n+1} \to T\mathbb{P}^n \to 0 \tag{.2}$$

Denote $c_1(\mathscr{O}_X(1)) := H$, via the exact sequence we have that:

$$c_t(T\mathbb{P}^n) = (1+H)^{n+1} \tag{.3}$$

2. **Solution to 1.2.** Since M is the zero locus of a homogeneous polynomial of degree d, i.e. a global section of the line bundle $\mathscr{O}(d)$ over $\mathbb{P}^n$. It defines an ideal sheaf $\mathscr{F}_M$ of $\mathbb{P}^n$, this ideal sheaf is generated by the section $s \in \Gamma(\mathscr{O}_{\mathbb{P}^n}(d))$, thus defines a Cartier divisor $dH \subset \mathbb{P}^n$. In this way we have that $\mathscr{F}_M \cong \mathscr{O}(-dH)$ and $\mathscr{F}_M/\mathscr{F}_M^2 \cong \mathscr{O}(-dH) \otimes \mathscr{O}_M$. Taking the dual and we have that $\mathscr{N}_{\mathbb{P}^n/M} \cong \mathscr{O}_{\mathbb{P}^n}(d)|_M$.

In this way we have that the Chern class of M can be written as:

$$c(M) = \frac{(1+H)^{n+1}}{(1+dH)} \tag{.4}$$

So we have that $c_1(M) = (n+1-d)H$, and when $d = n+1$, $c_1(M) = 0$. The Euler characteristic of M is now:

$$\begin{aligned}\chi(M) = \int_M c_{n-1}(M) = \int_{\mathbb{P}^n} dx \sum_{l=0}^{n-1} \binom{n+1}{l} (-1)^{n+1-l} d^{n+1-l} x^n \\ = (n+1) - \frac{1}{d}[1 + (-1)^n (d-1)^{n+1}]\end{aligned} \tag{.5}$$

In case when M is a projective curve in $\mathbb{P}^2$, $\chi(M) = d(3-d) = 2-2g$, and we have that $g = (d-1)(d-2)/2$.

3. **Solution to 1.3.** Obviously the fibre of the map is isomorphic to $\mathbb{C}^k$. To see the local triviality, given $X \in \mathbb{C}^n$, we can choose an orthogonal projection $p : \mathbb{C}^n \to V$ and we consider the set U of all the k-dimensional spaces V' such that $p(V') \cong V$. Obviously the set U is open in $\mathrm{Gr}(k,n)$. Thus we can construct the map $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ via $(V', v) \mapsto (V', p(v))$, which is clearly an isomorphism.

We use two ways to compute the tangent bundle of $\mathrm{Gr}(k,n)$. Consider the short exact sequence of vector bundles over $\mathrm{Gr}(k,n)$:

$$0 \to V \to \mathbb{C}^n \to \mathbb{C}^n/V \to 0 \tag{.6}$$

Consider the Plucker embedding $\mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}(\bigwedge^k V)$ via $V \mapsto v_1 \wedge \cdots \wedge v_k$. Also note that $\mathscr{O}(-1)$ is the tautological subbundle of $\bigwedge^k \mathbb{C}^n \times \mathbb{P}(\bigwedge^k V)$, in this case we can see that the restriction of $\mathscr{O}(1)$ is isomorphic to $\wedge^k(V^*)$.

Now choose a point $v_1 \wedge \cdots \wedge v_k$ and we consider a map $\phi_c : [-1,1] \to X$ as $\phi_c(t) = (v_1 + t\phi(v_1)) \wedge \cdots \wedge (v_k + t\phi(v_k))$, we have that, with $\phi : V \to \mathbb{C}^n$:

$$\phi_c'(0) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge \phi(v_i) \wedge \cdots \wedge v_k \tag{.7}$$

Thus two curves ϕ_c, ψ_c have the same tangent vector iff their difference lies in the subspace V' spanned by $v_1, \cdots, v_k$, thus it defines a map $V' \to \mathbb{C}^n/V'$, thus isomorphic to $\mathrm{Hom}(V, \mathbb{C}^n/V)$. Thus we have that:

$$TX = \mathrm{Hom}(V, \mathbb{C}^n/V) \cong V^* \otimes \mathbb{C}^n/V \tag{.8}$$

Here we compute the Chern class $c_t(TX)$ of the tangent bundle TX. First we write down:

$$c_t(V^*) = \prod_{i=1}^{k}(1 - tx_i), \qquad c_t(\mathbb{C}^n/V) = \prod_{i=1}^{n-k}(1 + ty_i) = \prod_{i=1}^{k} \frac{1}{(1 + tx_i)} = \sum_{r \geq 0} (-1)^r h_r t^r \tag{.9}$$

In this case we have that:

$$c_t(V^* \otimes \mathbb{C}^n/V) = \prod_{i=1}^{k}\prod_{j=1}^{n-k}(1 - tx_i + ty_j) = \sum_{\lambda,\mu} d_{\lambda\mu} s_\mu(-x) s_{\tilde{\lambda}'}(y) \tag{.10}$$

with $\mu \subset \lambda$. Here $\lambda = (\lambda_1, \cdots, \lambda_k)$ are the partitions such that $\lambda_i \leq n-k$, $\tilde{\lambda}' = (k - \lambda'_{n-k}, \cdots, k - \lambda'_1)$, λ' stands for the transpose of λ. $s_\lambda(y)$ is the Schur polynomial defined as:

$$s_\lambda(y) = \det(e_{\lambda'_i - i + j}(y))_{1 \leq i,j \leq n-k} = \det(h_{\lambda'_i - i + j}(-x))_{1 \leq i,j \leq n-k} = s_{\lambda'}(-x) \tag{.11}$$

The coefficients $d_{\lambda\mu}$ is defined as:

$$d_{\lambda\mu} = \det(\binom{\lambda_i + n - i}{\mu_j + n - j})_{1 \leq i,j \leq n} \tag{.12}$$

Thus for now we have:

$$c_t(V^* \otimes \mathbb{C}^n/V) = \sum_{\lambda,\mu} t^{|\lambda|+|\mu|} d_{\lambda\mu} s_\mu(-x) s_{\tilde{\lambda}}(-x) \tag{.13}$$

Thus we have the formula:

$$c_l(TX) = \sum_{|\lambda|+|\mu|=l, \mu\subset\lambda} d_{\lambda\mu} s_\mu(-x) s_{\tilde{\lambda}}(-x) \tag{.14}$$

4. **Solution to 1.4.** The section of the vector bundle $\mathrm{Hom}(V_1, \mathbb{C}^n/V_2)$ with the restriction to the diagonal is equivalent to $\mathrm{Hom}(V, \mathbb{C}^n/V)$, which is obviously zero. If given the point $(V_1, V_2) \in X \times X$ which is not in the diagonal, it is not zero.

To show that Φ_Δ is an identity map, note that as a class in homology, $p_2^*(\gamma) = \gamma \times [X_1]$ and $p_{1,*}(\Delta \cap p_2^*(\gamma)) = [p_1(\gamma \times \gamma)] = \gamma$

In terms of the cohomology, this means that:

$$\Phi_\Delta(\gamma) = [X_2] \cap (p_2^*(\gamma) \cup c_{top}(TX)) \tag{.15}$$

$c_{top}(TX)$ can be written as $\sum_\lambda s_\lambda(-x)s_{\tilde{\lambda}}(-x)$, which means that γ must be generated by the Chern class $c_i(V_1)$.

The result expression of γ is the linear combination of the Chern class of $c_k(V_1)$, thus we conclude that $H^*(X)$ is generated by $c_k(V)$.

The K-theoretic analog of the statement is that $K(X)$ is generated by $[\bigwedge^* V]$.

5. **Solution to 1.5.** Note that the element in $\mathsf{Gr}(k, n)$ can be represented by a linear map $A : \mathbb{C}^k \to \mathbb{C}^n$ which is injective and of full rank up to the conjugacy of GL_k. In this case the strictly-lower triangular matrices $U \subset GL_n$ acts as the row simplification for the $k \times n$ matrix A. The simplest row form of the matrix A is of the following:

$$\begin{pmatrix} * & * & \cdots & * & * & * & * \\ 0 & 0 & 0 & * & * & \cdots & * \\ 0 & 0 & 0 & 0 & * & * & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & * \end{pmatrix} \tag{.16}$$

i.e. It is determined by the number $(\lambda_1, \cdots, \lambda_k)$ such that $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq n$. Choose a suitable diagonal matrix $U \in GL(k)$, one can further assume that:

$$A = \begin{pmatrix} 0 & 0 & 1 & * & \cdots & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \end{pmatrix} \tag{.17}$$

Thus the U-orbit in $\mathsf{Gr}(k, n)$ is determined by the sequence of numbers $(\lambda_1, \cdots, \lambda_k) \in \mathbb{N}^k$ such that $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq n$. We define the new set of numbers $\lambda_i' = |\lambda_i - i|$, they define a partition λ, and we say thet the matrix A is of the form of the partition λ. The corresponding orbit is denoted by Ω_λ^0, which is an affine space $\mathbb{C}^{k(n-k)-|\lambda|}$. Note that this gives a stratification of $\mathsf{Gr}(k,n)$:

$$\mathsf{Gr}(k,n) = \bigsqcup_{\lambda} \Omega_\lambda^0 \tag{.18}$$

For the fixed point fo the torus action:

$$g = \mathrm{diag}(a_1, \cdots, a_n) \tag{.19}$$

It is obvious that if $A \in \Omega_\lambda^0$, $gA \in \Omega_\lambda^0$. This means that there exists the subtorus $t \in T \subset GL(k)$ such that:

$$At = gA \tag{.20}$$

We can decompose the vector space $\mathbb{C}^k$ into the eigenspace with respect to the T-torus action $\mathbb{C}^k = \mathbb{C}t_1 \oplus \cdots \oplus \mathbb{C}t_k$ and $\mathbb{C}^n$ into the A-torus action eigenspace $\mathbb{C}^n = \mathbb{C}a_1 \oplus \cdots \oplus \mathbb{C}a_n$. By computation it is easy to see that the equation reads:

$$\sum_j t_i A_{ij} w_j = \sum_j A_{ij} a_j w_j \tag{.21}$$

Since A is full rank, the above equation requires that $a_j = t_i$. But since $(a_1, \cdots, a_n)$ is generic, this requires that we can have only at most one a_k such that $a_k = t_i$. Thus we can choose $(a_1, \cdots, a_n)$ such that $a_i = t_i$ for $1 \le i \le k$ and $A(\mathbb{C}t_i) \subset \mathbb{C}a_i$. Since A is full rank, this condition completely determines the matrix A up to $GL(k)$ action. Easy computation can show that each fixed point is in the corresponding Schubert cells Ω_λ.

Now define $\Omega_\lambda := \bigsqcup_{\mu \ge \lambda} \Omega_\lambda^0$. This is the closed subvariety of $\mathsf{Gr}(k,n)$, and the chain of the inclusion $\Omega_{\lambda_1} \subset \Omega_{\lambda_2} \subset \cdots \subset \mathsf{Gr}(k,n)$ gives the CW decomposition for $\mathsf{Gr}(k,n)$, $[\Omega_\lambda]$ is a class in $H_{2k(n-k)-2|\lambda|}(\mathsf{Gr}(k,n))$ and thus generate the integral (co)homology of $\mathsf{Gr}(k,n)$.

6. **Solution to 1.6.** We denote the dual Schubert variety as $[\Omega_\lambda^\vee]$, thus we need to show that:

$$[\mathsf{Gr}(k,n)] \cap [\Omega_\lambda^\vee] \cap [\Omega_\mu] = \delta_{\lambda\mu} \tag{.22}$$

If $\mu \not< \lambda$, we have that $[\Omega_\lambda^\vee] \cap [\Omega_\mu] = 0$. Or otherwise $[\Omega_\lambda^\vee] \cap [\Omega_\mu]$ has dimension $|\lambda| - |\mu|$, which is the class of a Richardson variety. Thus it is left with the case $\mu = \lambda$. In this case $\Omega_\lambda^\vee \cap \Omega_\mu = \delta_{\mu\lambda} T_\lambda$ is equal to the torus fixed point.

By the formula of the Chern class:

$$c_t(V_1^* \otimes (\mathbb{C}^n / V_2)) = \sum_{\lambda,\mu} t^{|\lambda|+|\mu|} d_{\lambda\mu} s_\mu(-x) s_{\tilde{\lambda}}(-x) \tag{.23}$$

The top form in this case is written as:

$$c_{top}(V_1^* \otimes (\mathbb{C}^n/V_2)) = \sum_\lambda s_\lambda(-x)s_{\tilde{\lambda}}(-x) \tag{.24}$$

7. **Solution to 1.7.** The Poincaré polynomial for $\mathsf{Gr}(k,n)$ is computed as:

$$P_t(\mathsf{Gr}(k,n)) = \sum_{l=0}^{k(n-k)} t^{2l}\lambda_{n,k}(l) = \frac{(1-t^{2n})\cdots(1-t^{2n-2k+2})}{(1-t^2)\cdots(1-t^{2k})} \tag{.25}$$

Here $\lambda_{n,k}(l)$ stands for the number of partitions of l into $\leq n-k$ parts, each of size $\leq k$.

For the number of $\mathbb{F}_q$ points on $\mathsf{Gr}(k,n)$. Note that we have an action of $GL_n(\mathbb{F}_q)$ on $\mathsf{Gr}(k,n)(\mathbb{F}_q)$. This is a transitive action, and the stabiliser is given by the matrices $A = (a_{ij}) \in GL_n(\mathbb{F}_q)$ such that $a_{ij} = 0$ for $k+1 \leq i \leq n$ and $1 \leq j \leq k$. Then the number of elements in the stabliser is

$$\#GL_k(\mathbb{F}_q)\cdot \#GL_{n-k}(\mathbb{F}_q)\cdot \#M_{k,n-k}(\mathbb{F}_q) \tag{.26}$$

The number of points in $GL_k(\mathbb{F}_q)$ is given by:

$$\#GL_k(\mathbb{F}_q) = q^{\frac{k(k-1)}{2}}(q^k-1)(q^{k-1}-1)\cdots(q-1) \tag{.27}$$

Thus we conclude that:

$$\begin{aligned}\#\mathsf{Gr}(k,n)(\mathbb{F}_q) =& \frac{\#GL_n(\mathbb{F}_q)}{\#GL_k(\mathbb{F}_q)\cdot \#GL_{n-k}(\mathbb{F}_q)\cdot \#M_{k,n-k}(\mathbb{F}_q)} \\ =& \frac{(q^n-1)\cdots(q^{n-k+1})}{(q^k-1)\cdots(q-1)}\end{aligned} \tag{.28}$$

8. **Solution to 1.8 1.9.** Denote the character of the torus action T on $\mathsf{Gr}(k,n)$ as $\alpha_1,\cdots,\alpha_n$. Denote the Schubert basis of $\mathsf{Gr}(k,n)$ as Ω_λ, and the corresponding T-fixed point as p_λ.

It is known that the Euler class of $T\mathsf{Gr}_{k,n}$ can be written as:

$$e(T_{p_\lambda}\mathsf{Gr}(k,n)) = \sum_\lambda s_\lambda(-x)s_{\tilde{\lambda}}(x) = \prod_{i\in\lambda, j\notin\lambda}(\alpha_i-\alpha_j) \tag{.29}$$

We have that:

$$[\Omega_\lambda]|_{p_\lambda} = e(T_{p_\lambda}\Omega_\lambda) = \prod_{i\in\lambda, j\notin\lambda, i>j}(\alpha_j-\alpha_i) \tag{.30}$$

Here we use $\lambda = (i_1 < \cdots < i_d) \subset \{1,\cdots,n\}$. Sincw we know that $\Omega_\mu \subset \Omega_\lambda$ iff $\mu \supset \lambda$, we can conlcude that:

$$[\Omega_\lambda]|_{p_\mu} = 0 \qquad \text{unless } \mu \geq \lambda \tag{.31}$$

This gives the Newton interpolation solution for the Schubert class $[\Omega_\lambda]$

We claim that this two condition gives a unique solution. Now suppose there are two classes α, α' satisfying the above condition. Let $\beta = \alpha - \alpha'$. Since we have known that $\beta|_{p_\mu} = 0$ unless $\mu \geq \lambda$. Let $\eta \geq \lambda$ be a minimal element such that $\beta|_{p_\eta} \neq 0$. Obviously by the construction of β, $\eta > \lambda$. Now choose a T-equivariant curve C connecting p_η and other fixed points p_β such that $\beta \not< \eta$, in this we can see that $\beta|_{p_\eta}$ must be divisible by

$$\prod_{i\in\eta, j\neq\eta, i>j} (\alpha_i - \alpha_j) \tag{.32}$$

which is equal to $[\Omega_\eta]|_{p_\eta}$. So $\beta|_{p_\eta}$ has degree at least $|\eta|$. Since $\eta > \lambda$, we have $|\eta| > |\lambda|$, which is a contradiction.

The corresponding fact is that the equivariant Schubert class $[\Omega_\lambda]$ corrresponds to the double Schur function $s_\lambda(x|\alpha)$ defined as:

$$s_\lambda(x|y) = \frac{\det[(x_i|y)^{\lambda_j+d-j}]_{1\leq i,j\leq d}}{\det[(x_i|y)^{d-j}]_{1\leq i,j\leq d}}, \qquad (x_i|y)^p = (x_i - y_1)\cdots(x_i - y_p) \tag{.33}$$

The double Schur function satsify the condition .30 .31 above, thus it is the unique solution.

9. **Solution to 1.10.** Under the setting of the equivariant K-theory, denota a_i the character of the torus T. So we have that:

$$[\Omega_\lambda]|_{p_\mu} = 0 \qquad \text{unless } \mu \geq \lambda \tag{.34}$$

$$[\Omega_\lambda]|_{p_\lambda} = \prod_{i\in\lambda, j\notin\lambda, i>j} (1 - \frac{a_i}{a_j}) \tag{.35}$$

The uniqueness comes from the similar reason, but we replace the degree condition by the Newton polytope condition. The corresponding symmetric function is still the double Schur polynomial, which is almost all the same instead:

$$(x_i|y)^p = (1 - y_1 x_i^{-1})\cdots(1 - y_p x_i^{-1}) \tag{.36}$$

10. **Solution to 1.11.** Using the Huzbrecht-Riemann-Roch theorem:

$$\begin{aligned}
\chi(\mathscr{L}_\lambda) &= \int_{G/B} \mathrm{ch}(\mathscr{L}_\chi)\mathrm{Td}(G/B) = \int_{G/B} e^\lambda \prod_{\alpha\in\Phi_+} \frac{\alpha}{1 - e^{-\alpha}} \\
&= \sum_{w\in W} \frac{1}{e(T_w(G/B))} e^\lambda \prod_{\alpha\in\Phi_+} \frac{\alpha}{1 - e^{-\alpha}} \\
&= \sum_{w\in W} \frac{e^{w\lambda}}{\prod_{\alpha\in\Phi_+}(1 - e^{-w(\alpha)})} = \sum_{w\in W} \frac{(-1)^{l(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi_+}(e^{\alpha/2} - e^{-\alpha/2})}
\end{aligned} \tag{.37}$$

The Weyl character formula states that for the irreducible representation L_λ of the semisimple Lie algebra, we have that:

$$ch_{L_\lambda}(a) = \mathrm{tr}_{\mathrm{End}(L_\lambda)}(e^a) = \sum_{w\in W} \frac{(-1)^{l(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi_+}(e^{\alpha/2} - e^{-\alpha/2})} \tag{.38}$$

The interpretation of the coincidence of the formula can be seen as follows. The global section of $\mathscr{L}_\lambda$ can be interpreted as the holomorphic function $f(g)$ over G such that $f(gb) = f(g)\lambda(b)$, thus it is a subspace of $\mathbb{C}[G]$. While the Peter-Weyl isomorphism:

$$\mathbb{C}[G] = \bigoplus_\lambda L_\lambda^* \boxtimes L_\lambda \tag{.39}$$

implies that the algebraic holomorphic function over G is generated by $\langle v_1, gv_2\rangle$ with $v_1 \in L_\lambda^*$, $v_2 \in L_\lambda$. If we fix λ and the vacuum vector $v_1 \in L_\lambda^*$, it is easy to see that $\langle v_1, gbv_2\rangle = \lambda(b)\langle v_1, gv_2\rangle$. Thus the function satisfying the above equivariant property should lie in L_λ. Thus we have that $\Gamma(\mathscr{L}_\lambda) = L_\lambda$.

11. **Solution to 1.12.** Given the vector bundle $p : V \to X$, via the splitting principle, there exists a manifold $\pi : F(V) \to X$ such that $F(V)$ is a fibration over X with the fibre over $x \in X$ isomorphic to the full flag manifold of $p^{-1}(x)$, and the pullback of V over $F(V)$ is that

$$\pi^*(V) = L_1 \oplus \cdots \oplus L_r \tag{.40}$$

Now using the identity $F_{\pi^*\nabla} = \pi^* F_\nabla$, we have the decomposition $\pi^* F = F_1 \oplus \cdots \oplus F_r$

$$\pi^*\det(1 + \frac{it}{2\pi}F) = \prod_{i=1}^r (\det(1 + \frac{it}{2\pi}F_i)) = \prod_{i=1}^r (1 + \frac{it}{2\pi}F_i) = \prod_{i=1}^r (1 + tc_1(L_i)) \tag{.41}$$

This means that $\pi^*(\sum_k t^k c_k(V) - [\det(1 + \frac{it}{2\pi}F)] = 0 \in H^*(F(V), \mathbb{C})$. Now since the the fibre of the map $F(V) \to X$ is flag manifold associated to the fibre of V over X. While the flag manifold does not contain the odd cohomology classes, this means that the map $\pi^* : H^*(X, \mathbb{C}) \to H^*(F(V), \mathbb{C})$ is injective on even degrees. Thus the proof is finished.

12. **Solution to 2.1.** First we analyze the Laurent series of $f(z)$.

Write down the Laurent expansion of $f(z)$ as:

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \tag{.42}$$

Taking the formula into the q-difference equation and we obtain that:

$$\sum_{n\in\mathbb{Z}} a_n q^n z^n = \sum_{n\in\mathbb{Z}} c a_n z^{n-d} \tag{.43}$$

This requires that $d \in \mathbb{Z}$ and it gives us the recursion relations:

$$a_{n+d} = c^{-1}q^n a_n, \qquad a_{n+kd} = c^{-k}q^{kn+\frac{k(k-1)d}{2}}a_n \tag{.44}$$

This means that the space of solutions has the dimension d as long as $c \neq 0$. The solution can be written as:

$$\sum_{k\in\mathbb{Z}} c^{-k}q^{\frac{k(k-1)d}{2}+kn}z^{dk} = \sum_{k\in\mathbb{Z}}(q^d)^{\frac{k^2}{2}}(\frac{z^d q^{n-d/2}}{c})^k \tag{.45}$$

It has the zeroes over $-z^d c^{-1} \sim q^{\mathbb{Z}}$.

For $d < 0$, one can check that the convergence radius is zero. So we always assume $d > 0$.

The second method is that one could construct a line bundle over $E = \mathbb{C}^\times/q^{\mathbb{Z}}$:

$$\mathscr{L}_{c,d} := \mathbb{C}^\times \times \mathbb{C}/\sim, \qquad (z, f(z)) \sim (qz, cz^{-d}f(z)) \tag{.46}$$

It can be seen that the line bundle has the corresponding divisor over the points $[p_i]$ such that $\mathscr{L} = \mathscr{O}([p_1] + \cdots + [p_d])$. Using the Riemann Roch:

$$\dim(\Gamma(\mathscr{L})) = \deg(D) = d \tag{.47}$$

For $d = 0$, it is easy to see that the solution is 0 if $c \neq 1$. If $c = 1$, $f(z)$ is a constant. Thus the dimension of the solution space is 0 if $c \neq 1$ and 1 if $c = 1$. The corresponding line bundle is the structure sheaf over the elliptic curve for $c = 1$. For $c \neq 1$, there exists the meromorphic solutions, which means that the corresponding line bundle is not ample, and thus has zero global sections.

13. **Solution to 2.2.**

$$\theta(qz) = \prod_{n>0}(1-q^{n+1}z)\prod_{n\geq 0}(1-q^{n-1}z^{-1}) = \frac{1-q^{-1}z^{-1}}{1-qz}\theta(z) = -q^{-1}z^{-1}\theta(z) \tag{.48}$$

Consider another solution $g(z)$ such that $g(qz) = cz^{-d}g(z)$. Since we have known that the solution space has the elements containing only the simple zeros. We can assume that $g(z)$ has the same zeros as $f(z) = \prod_i \theta(z/w_i)$. In this case we have that $g(z)/f(z)$ is a holomorphic function such that:

$$\frac{f(qz)}{g(qz)} = \text{Const}\cdot\frac{f(z)}{g(z)} \tag{.49}$$

Thus we have that $f(z)/g(z) = \text{Const}\cdot z^n$ for some n.

Geometrically speaking, consider the natural map $\mathrm{Sym}^d E \to E$, $(w_1, \cdots, w_d) \mapsto \prod_{i=1}^d w_i$. The equation $\prod w_i = (-q)^d c$ defines a divisor in $\mathrm{Sym}^d E$. They correspond to the same line bundle on E iff $\prod w_i = \prod w_i'$.

14. **Solution to 2.3.** Note that given an ample line bundle $\mathscr{L}$ corresponding to the q-difference equation $P(qz) = cz^{-d}P(z)$ has the solution of the form $\prod_i \theta(z/a_i)$ with $\prod_i a_i = (-q)^d c$. This gives a projective embedding $E \hookrightarrow \mathbb{P}^{d-1}$ via $z \mapsto (P_1(z), \cdots, P_d(z))$ with $P_i(z)$ the function of the form $\prod_i \theta(z/a_i)$. Thus the rational function over E can be written as:

$$\frac{\prod_i \theta(z/a_i)}{\prod_i \theta(z/b_i)}, \qquad \prod_i a_i = \prod_i b_i \tag{.50}$$

15. **Solution to 2.4.** Do the Laurent expansion of $\prod_{n>0}(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1})$:

(.51)
$$\prod_{n>0}(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1}) = \Big(\sum_{k_1<\cdots<k_d, d\geq 0} z^d q^{k_1+\cdots+k_d-d/2}\Big)\Big(\sum_{k_1<\cdots<k_d, d\geq 0} z^{-d} q^{k_1+\cdots+k_d-d/2}\Big)$$

we have that the coefficient of z^0 is given as:

$$\sum_{1\leq k_1<\cdots<k_d, d\geq 0} q^{(k_1+\cdots+k_d)-d} = \sum_{0\leq k_1<\cdots<k_d, d\geq 0} q^{k_1+\cdots+k_d} = \sum_\lambda q^{|\lambda|} \tag{.52}$$

Define the function:

$$f(z) = \prod_{n>0}(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1}) \tag{.53}$$

we have that:

(.54)
$$f(qz) = \prod_{n>0}(1+q^{n+1/2}z)\prod_{n>0}(1+q^{n-3/2}z^{-1}) = \frac{1+q^{-1/2}z^{-1}}{1+q^{1/2}z} f(z) = q^{-1/2}z^{-1}f(z)$$

It is easy to see that the dimension of the solution space is 1, and the solution can be written as const $\cdot \sum_{k\in\mathbb{Z}} q^{k^2/2}z^k$, thus we have that:

$$\sum_{k\in\mathbb{Z}} q^{k^2/2}z^k = F(q)\prod_{n>0}(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1}) \tag{.55}$$

To determine $F(q)$, it remains to compare the coefficients on both sides. Note that for the coefficient of z^0:

$$F(q)\sum_\lambda q^{|\lambda|} = 1 \tag{.56}$$

While we have that $\sum_\lambda q^{|\lambda|} = \prod_{n>0} \frac{1}{1-q^n}$, in this way we have that:

$$F(q) = \prod_{n>0}(1-q^n) \tag{.57}$$

16. **Solution to 2.5.** Oddness is trivial. For the entireness, note that:

$$\begin{aligned} |\sigma(x)| =& |\exp(\log(x) + \sum_{\gamma\in\Gamma^*} \log(1-\frac{x}{\gamma}) + \sum_{\gamma\in\Gamma^*} \frac{x^2}{2\gamma^2})| \\ \leq& \exp(|\log(x)| + \sum_{\gamma\in\Gamma^*} |\log(1-\frac{x}{\gamma})| + |\sum_{\gamma\in\Gamma^*} \frac{x^2}{2\gamma^2}|) \end{aligned} \tag{.58}$$

Since $\sum_{\gamma\in\Gamma^*} 1/\gamma^2$ is convergent, it remains to analyze $\sum_{\gamma\in\Gamma^*} |\log(1-\frac{x}{\gamma})|$. Note that:

$$\sum_{\gamma\in\Gamma^*} |\log(1-\frac{x}{\gamma})| = \sum_{(m,n)\neq(0,0)} |\log(1-\frac{x}{m\omega_1+n\omega_2})| \leq \sum_{k\geq 1}\frac{x^{2k}}{2k} \sum_{(m,n)\neq(0,0)} \frac{1}{|m\omega_1+n\omega_2|^{2k}} \tag{.59}$$

While $\sum_{(m,n)\neq(0,0)} \frac{1}{|m\omega_1+n\omega_2|^{2k}} \leq \frac{C}{k^2}$, thus:

$$\sum_{\gamma\in\Gamma^*} |\log(1-\frac{x}{\gamma})| \leq C\sum_{k\geq 1}\frac{x^{2k}}{k^3} \tag{.60}$$

which is convergent for finite x, and thus $\sigma(x)$ is an entire function over $\mathbb{C}$.

Via computation:

$$\begin{aligned} \sigma(x+\gamma) =& (x+\gamma)\prod_{\gamma'\in\Gamma^*}(1-\frac{x+\gamma}{\gamma'})\exp(\frac{x+\gamma}{\gamma'}+\frac{(x+\gamma)^2}{2\gamma'^2}) \\ =& \gamma(1+\frac{x}{\gamma})\prod_{\gamma'\in\Gamma^*}\frac{\gamma'-\gamma}{\gamma'}(1-\frac{x}{\gamma'-\gamma})\exp(\frac{x}{\gamma'}+\frac{x^2}{2\gamma'^2})\exp(\frac{\gamma}{\gamma'}+\frac{\gamma^2}{2\gamma'^2}+\frac{x\gamma}{\gamma'^2}) \\ =& -x\prod_{\gamma'\in\Gamma^*}(1-\frac{x}{\gamma'})\exp(\frac{x}{\gamma'}+\frac{x^2}{2\gamma'^2})\prod_{\gamma'\in\Gamma^*}\exp(\frac{\gamma}{\gamma'}+\frac{\gamma^2}{2\gamma'^2}+\frac{x\gamma}{\gamma'^2}) \\ =& -\prod_{\gamma'\in\Gamma^*}\frac{\gamma'-\gamma}{\gamma'}\exp(\frac{\gamma}{\gamma'}+\frac{\gamma^2}{2\gamma'^2}+\frac{x\gamma}{\gamma'^2})\sigma(x) \\ =& -\exp(\sum_{\gamma'\in\Gamma^*}\log(\gamma'-\gamma)-\log(\gamma')+\frac{\gamma}{\gamma'}+\frac{\gamma^2}{2\gamma'^2}+\frac{x\gamma}{\gamma'^2})\sigma(x) \\ =& -\exp(\eta(\gamma)(x+\frac{\gamma}{2}))\sigma(x), \qquad \eta(\gamma) := \gamma\sum_{\gamma'\in\Gamma^*}\frac{1}{\gamma'^2} \end{aligned} \tag{.61}$$

17. **Solution to 2.6.** For simplicity we write $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$, in this case we write down the Eisenstein series as:

$$\mathrm{Eis}(\Gamma, k) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}} = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\tau} \tag{.62}$$

First we give the following formula without proof:

$$\pi\cot(\pi\tau) = \frac{1}{\tau} + \sum_{m\in\mathbb{Z}, m\neq 0} \left(\frac{1}{\tau+m} - \frac{1}{m}\right) \tag{.63}$$

We denote $q = e^{2\pi i\tau}$. The left hand side of the formula can be written as:

$$\pi\cot(\pi\tau) = -\pi i\left(1 + 2\sum_{r=1}^{\infty} q^r\right) \tag{.64}$$

Thus we have that:

$$-\pi i\left(1 + 2\sum_{r=1}^{\infty} q^r\right) = \frac{1}{\tau} + \sum_{m\in\mathbb{Z}, m\neq 0} \left(\frac{1}{\tau+m} - \frac{1}{m}\right) \tag{.65}$$

Differentiating with respect to τ we have that:

$$-\sum_{m\in\mathbb{Z}} \frac{1}{(m+\tau)^2} = -(2\pi i)^2 \sum_{r=1}^{\infty} r q^r \tag{.66}$$

One can compute that:

$$\sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau)^{2k}} = \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{r=1}^{\infty} r^{2k-1} q^{nr} \tag{.67}$$

For now we have:

$$\begin{aligned}
\sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}} &= \sum_{m\in\mathbb{Z}, m\neq 0} \frac{1}{m^{2k}} + \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \left(\frac{1}{(m+n\tau)^{2k}} + \frac{1}{(m-n\tau)^{2k}}\right) \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau)^{2k}} \\
&= 2\zeta(2k) + 2\cdot\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} r^{2k-1} q^{nr} \\
&= 2\zeta(2k) + (-1)^k 2\cdot\frac{(2\pi)^{2k}}{(2k-1)!}\sum_{r=1}^{\infty} \frac{r^{2k-1} q^r}{1-q^r}
\end{aligned} \tag{.68}$$

18. **Solution to 2.7.** Consider the line bundle over $E \times E$ with the global section written as:

$$F(z, w) = \frac{\theta(zw)}{\theta(zw_0)\theta(z_0 w)} \tag{.69}$$

It satisfies the q-difference equations:

$$F(qz, w) = w^{-1} w_0 F(z, w), \qquad F(z, qw) = z^{-1} z_0 F(z, w) \tag{.70}$$

with z_0, w_0 some points in E. It is easy to see that the restriction to $\{z_0\} \times E$ gives the function $F(z_0, w)$, i.e. $F(z_0, qw) = F(z_0, w)$ and $F(qz, w_0) = F(z, w_0)$ when restricted to $E \times \{w_0\}$. It is obvious that the restricted ones are trivial, while the original one is not trivial.

19. **Solution to 2.8.** Note that

$$\begin{aligned} \phi((\mathscr{L}_1, a_1)(\mathscr{L}_2, a_2)) &= \tau(a_1 a_2)^*(\mathscr{L}_1 \otimes \mathscr{L}_2) \otimes (\mathscr{L}_1 \otimes \mathscr{L}_2)^{-1} \\ \phi(\mathscr{L}_1, a_1)\phi(\mathscr{L}_2, a_2) &= \tau(a_1)^*\mathscr{L}_1 \otimes \mathscr{L}_1^{-1} \otimes \tau(a_2)^*\mathscr{L}_2 \otimes \mathscr{L}_2^{-1} \end{aligned} \tag{.71}$$

So we need to prove that $\tau(a_1 a_2)^*(\mathscr{L}_1 \otimes \mathscr{L}_2)$ is algebraic equivalent to $\tau(a_1)^*\mathscr{L}_1 \otimes \tau(a_2)^*\mathscr{L}_2$, and this can be done by chosing a line bundle $\mathscr{L}$ over $A \times A^2$ such that $\mathscr{L}|_{A \times (z_1 \times z_2)} \cong \tau(z_1)^*\mathscr{L}_1 \otimes \tau(z_2)^*\mathscr{L}_2$. Thus $\tau(a_1 a_2)^*(\mathscr{L}_1 \otimes \mathscr{L}_2)$ is the fibre over $A \times (a_1, a_2)$ and $\tau(a_1)^*\mathscr{L}_1 \otimes \tau(a_2)^*\mathscr{L}_2$ is the fibre over $A \times (a_1 a_2, a_1 a_2)$.

20. **Solution to 2.9.** We have that $m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1}|_{A \times \{a\}} \cong p_2^*\mathscr{L}|_{A \times \{a\}} \cong \mathscr{O}$ and $m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1}|_{\{0\} \times A} \cong p_2^*\mathscr{L}_{\{0\} \times A} \cong \mathscr{L}$. Then using the Seesaw principle, we have the isomorphism.

For the cohomology, note that:

$$H^k(A^2, m^*\mathscr{L}) = H^k(A^2, p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}) \cong \bigoplus_{i+j=k} H^i(A, \mathscr{L}) \otimes H^j(A, \mathscr{L}) \tag{.72}$$

Now we consider the map $s : A \to A^2$, $x \mapsto (x, 0)$, and we can see that $m \circ s = \mathrm{id}$. Also $(m \circ s)^* : H^k(A, \mathscr{L}) \to H^k(A, \mathscr{L})$ factors through $H^k(A^2, m^*\mathscr{L})$.

Now we set $H^0(\mathscr{L}) = 0$, and in the above settings, we suppose that $i + j = k \geq 1$. We further suppose that k is the smallest number such that $H^k \neq 0$. Thus we can see that it is a contradiction, which gives $H^k = 0$.

21. **Solution to 2.10.** Now given $\mathscr{L} \in \mathrm{Pic}_0(A)$, by definition there is a line bundle $\widetilde{\mathscr{L}}$ over $A \times B$ such that $\widetilde{\mathscr{L}}|_{A\times b_1} = \mathscr{L}, \widetilde{\mathscr{L}}|_{A\times b_2} = \mathscr{O}$.

Consider the line bundle $\mathscr{L}'$ on $A \times A \times B$

$$\mathscr{L}' = (m \otimes \mathrm{id})^* \widetilde{\mathscr{L}} \otimes p_{13}^* \widetilde{\mathscr{L}}^{-1} \otimes p_{23}^* \tilde{\mathscr{L}}^{-1} \tag{.73}$$

It is easy to see that $\mathscr{L}'$ is trivial on $A \times 0 \times B$ and $0 \times A \times B$ and $A \times A \times b_2$. Then using the theorem of the cube, we can see that $\mathscr{L}'$ is trivial. Its restriction to $A \times A \times b_1$ is $m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$. Now since $m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1}|_{A\times a} \cong \tau(a)^*\mathscr{L} \otimes \mathscr{L}^{-1} \cong p_2^*\mathscr{L}^{-1}|_{A\times a} \cong \mathscr{O}$, this implies that $\tau(a)^*\mathscr{L} \cong \mathscr{L}$.

22. **Solution to 2.11.** Since each line bundle corresponds to the divisor $D = \sum_i n_i p_i$, the degree map gives $\deg(D) = \sum_i n_i$. For $\mathscr{L} \in \mathrm{Pic}_0(E)$, it has $\tau(a)^*\mathscr{L} \cong \mathscr{L}$, which means that $\sum_i n_i(p_i + a) = \sum_i n_i p_i$, and it is equivalent to $\sum_i n_i = 0$, i.e. it is the kernel of deg.

Note that the property of the line bundle $\widetilde{\mathscr{L}}$ gives the natural map $f : B \to E$ by the following. Since $\widetilde{\mathscr{L}}|_{E\times b} = \mathscr{L}_b \in \mathrm{Pic}_0(E)$, using the isomorphism $\mathrm{Pic}_0(E) \cong E$, we construct a map $f : B \to E$ via $b \mapsto \mathscr{L}_b$.

On the other hand, for the line bundle $\mathscr{P}$ over $E \times E$, it is easy to check that $\mathscr{P}|_{E\times e} = \mathscr{O}(e - o)$, and here $e = \sum_i n_i p_i$ and $o = \sum_i n_i \infty$. In this way we can see that:

$$\mathscr{P}|_{E\times f(b)} \cong \widetilde{\mathscr{L}}|_{E\times b} \tag{.74}$$

23. **Solution to 2.12.** For the case when $A = E^n$, a quick way to prove that $A^\vee \cong A$ is to note that for the corresponding lattice Λ of E^n, which is $(\mathbb{Z} \oplus \tau\mathbb{Z})^{\oplus n}$, we have the group isomorphism $\mathrm{Hom}(\Lambda, U(1)) \to A^\vee$ via $\chi \mapsto \mathscr{L}(\chi)$ such that $\mathscr{L}(\chi)$ corresponds to the q-difference equations:

$$F(z + u) = \chi(u)F(z), \qquad z \in \mathbb{C}^{2n}, u \in \Lambda \tag{.75}$$

To endow with the complex structure, note that there is a natural injecive map $\mathrm{Pic}(E) \times \cdots \times \mathrm{Pic}(E) \hookrightarrow \mathrm{Pic}(E^n)$, this induce the injective map $\mathrm{Pic}_0(E) \times \cdots \times \mathrm{Pic}_0(E) \hookrightarrow \mathrm{Pic}_0(E^n)$. Note that since $\mathrm{Pic}_0(E) \cong E$ as algebraic varieties, this is an injective map of algberaic varieties $E^n \hookrightarrow \mathrm{Pic}_0(E^n)$, but we know that $\mathrm{Pic}_0(E^n) \cong E^n$ as groups, thus this injective map is also an isomorphism.

To construct te Poincare bundle, note that the isomorphism implies that the degree 0 line bundle $\mathscr{L}$ over E^n is determined by $\mathrm{pr}_1^*\mathscr{L}_1 \otimes \cdots \otimes \mathrm{pr}_n^*\mathscr{L}_n$ for $\mathscr{L}_i \in E^\vee$, this means that for the Poincare bundle $\mathscr{P}$ such that:

$$\mathscr{P}|_{E^n\times(\mathscr{L}_1,\cdots,\mathscr{L}_n)} \cong \mathrm{pr}_1^*\mathscr{L}_1 \otimes \cdots \otimes \mathrm{pr}_n^*\mathscr{L}_n \tag{.76}$$

We can define $\mathscr{P} := \mathrm{pr}_1^*\mathscr{P}_1 \otimes \cdots \otimes \mathrm{pr}_n^*\mathscr{P}_n$, and one can check that it satisfies the required conditions.

24. **Solution to 2.13.** For the Poincare bundle over $E \times E^{\vee}$, its corresponding Hermitian form can be written as:

$$H = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \\ 1 & i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix} \tag{.77}$$

this matrix has 1 positive eigenvalues and 1 negative values. By the general result in the study of cohomology of line bundles of complex tori. We have that $H^0(\mathscr{P}) = H^2(\mathscr{P}) = 0$. Thus it remains to compute $H^1(\mathscr{P})$. Then by the analytic Riemann-Roch, we have $\chi(\mathscr{P}) = -\mathrm{Pf}(\mathrm{Im}(H)) = -1$, which implies that $H^1 = \mathbb{C}$.

Now back to the computation of Φ^2, by computation we can see that:

$$\Phi^2 = \mathbf{R}p_{12,*}(p_{13}^*\mathscr{P} \otimes p_{23}^*\mathscr{P}) \cong \mathbf{R}p_{12,*}(m \otimes 1)^*\mathscr{P} \cong m^*\mathbf{R}p_{1,*}\mathscr{P} \tag{.78}$$

Now using the Leray spectral sequence, it is known that $H^i(\mathscr{P}) = (R^i p_{1,*}\mathscr{P})_0$ and the coherent sheaf $R^i p_{1,*}\mathscr{P}$ vanishes over $E - \{0\}$. Now we have $R^i p_{1,*}\mathscr{P} \cong \delta_{i1}\mathbb{C}[-1]$, which is supported on $\{0\}$. Hence we have that Φ^2 is isomorphic to $\mathscr{O}_E[-1]$, with E being the off-diagonal subvariety $(-x, x) \in E \times E$. Thus $\Phi^2 = (-1_E)^*[-1]$.

For the general case E^n, note that since $\mathscr{P} = \mathrm{pr}_1^*\mathscr{P}_E \otimes \cdots \otimes \mathrm{pr}_n^*\mathscr{P}_E$, we have:

$$H^k(E^n \times E^n, \mathscr{P}) \cong \bigoplus_{i_1 + \cdots + i_n = k} H^{i_1}(\mathscr{P}_E) \otimes \cdots \otimes H^{i_n}(\mathscr{P}_E) = \begin{cases} \mathbb{C} & k = n \\ 0 & \text{otherwise} \end{cases} \tag{.79}$$

Thus similar calculation shows that:

$$\Phi^2 = (-1_{E^n})^*[-n] \tag{.80}$$

25. **Solution to 2.14.** By computation:

$$\begin{aligned} \Phi(\mathscr{L}) =& Rp_{1,*}(\mathscr{P} \otimes p_2^*(\mathscr{L})) \\ =& Rp_{1,*}(\mathscr{P} \otimes p_1^*(\mathscr{L}^{-1}) \otimes m^*(\mathscr{L}^{-1})) \\ =& Rp_{1,*}(\mathscr{P} \otimes m^*(\mathscr{L}^{-1})) \otimes \mathscr{L}^{-1} \sim R\Gamma(\mathscr{L}^{-1}) = 0 \end{aligned} \tag{.81}$$

It remains to show that $\mathscr{L} \mapsto \phi(\mathscr{L}, -)$ gives a surjective map from $\mathrm{Pic}(A)$ to $\mathrm{Hom}_{sym}(A, A^{\vee})$. Since we only deal with the case $A = E^n$, it is equivalent to show that every group scheme map $\phi : E^n \to E^n$ such that the induced map $\phi^* : E^n \to E^n$ is the same as ϕ.

For the group scheme map $\phi : E^n \to E^n$, it is easy to see that it is equivalent to a linear isomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ which preserves the lattice $\phi(\Lambda) \subset \Lambda$, and in our case it is equivalent to $\phi \in GL_n(\mathbb{Z})$. Thus for now the induced map $\phi^* : \mathrm{Pic}_0(E^n) \to \mathrm{Pic}_0(E^n)$, the induced line bundle $\phi^*\mathscr{L}$ has the automorphy form as:

$$\phi^*\chi = \phi(\chi) \tag{.82}$$

This can be seen in the following way: Given the line bundle $\mathscr{L} \in \mathrm{Pic}_0(E^n)$ such that the section $F(z)$ satisfies the difference equation:

$$F(q^{e_i}z) = \chi_i(q)F(z) \tag{.83}$$

While the section of $\phi^*\mathscr{L}$ can be written as:

$$F(\phi(z)) = F(z_1^{a_{11}} \cdots z_n^{a_{1n}}, \cdots, z_1^{a_{n1}} \cdots z_n^{a_{nn}}) \tag{.84}$$

which means that:

$$F(\phi(q^{e_i}z)) = F(q^{a_{1i}}z_1^{a_{11}} \cdots z_n^{a_{1n}}, \cdots, q^{a_{ni}}z_1^{a_{n1}} \cdots z_n^{a_{nn}}) = \chi_1^{a_{1i}} \cdots \chi_n^{a_{ni}} F(\phi(z)) \tag{.85}$$

Thus as long as $\phi : E^n \to E^n$ is a group scheme map, the induced map ϕ^* is the same as ϕ.

Now given arbitrary $\mathscr{L} \in \mathrm{Pic}(E^n)$, its section satisfies the difference equations:

$$F(q^{e_i}z) = \chi_i(z)F(z) \tag{.86}$$

with $\chi_i(z)$ the monomials of z. Thus the sections of $\tau(a)^*\mathscr{L}$ is given as $F(a_1z_1, \cdots, a_nz_n)$, with the automorphy given as $\chi_i(az)$. Thus the automorphy of $\tau(a)^*\mathscr{L} \otimes \mathscr{L}^{-1}$ is given as:

$$\chi_i(az)/\chi_i(z) = \xi_i(a) \tag{.87}$$

which is a monomial of a with coefficients being 1. The power of a lies in $\mathbb{Z}^n$, which means that it also gives arbitrary matrix in $GL_n(\mathbb{Z})$. Thus it is obviously surjective.

26. **Solution to 2.15.** The global section of the Poincare bundle $\mathscr{P}$ on $E \times E$ has the automorphy as:

$$F(qz, w) = w^{-1}F(z, w), \qquad F(z, qw) = z^{-1}F(z, w) \tag{.88}$$

Thus for the corresponding map $\phi(\mathscr{P}, a) = \tau(a)^*\mathscr{P} \otimes \mathscr{P}^{-1}$, the resulting bundle $\tau(a)^*\mathscr{P} \otimes \mathscr{P}^{-1}$ has the global section with the automorphy:

$$F(qz, w) = a_2^{-1}F(z, w), \qquad F(z, qw) = a_1^{-1}F(z, w) \tag{.89}$$

Thus we have $\phi(\mathscr{P}, a_1, a_2) = (a_2^{-1}, a_1^{-1})$, and the corresponding matrix is given by:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{.90}$$

For the line bundle on E^n with the section of the form:

$$s(z) = \prod \theta(c_\mu z^\mu)^{m_\mu}, \qquad z^\mu = \prod_{i=1}^{n} z_i^{\mu_i}, c_\mu \in \mathbb{C}^\times, m_\mu \in \mathbb{Z} \tag{.91}$$

This gives a linear map:

$$\phi(a_i) = \prod_\mu (a_1^{\mu_1} \cdots a_n^{\mu_n})^{-m_\mu \mu_i} \tag{.92}$$

whose corresponding lattice map is given by:

$$\phi(e_i) = -\sum_\mu (m_\mu \mu_1 \mu_i, \cdots, m_\mu \mu_n \mu_i) \tag{.93}$$

27. **Solution to 3.1.** A possibly simple proof is that for the vector bundle V over X, we construct the flag fibration $\pi : F(V) \to X$ over X such that $\pi^* V = L_1 \oplus \cdots \oplus L_n$, so we have that:

$$\pi^* \bigwedge_t^* V = \prod_{i=1}^{n} (1 - tL_i), \qquad \pi^* S_t^* V = \sum_{i=1}^{n} (1 - tL_i)^{-1} \tag{.94}$$

While since π^* is injective on $K_0(X) \to K_0(F(V))$, the problem is solved.

28. **Solution to 3.2.** We first enlarge the vector space $K(X) \otimes \mathbb{Q}$ to $K(X) \otimes \mathbb{C}$. Now with the tensor product, $K(X) \otimes \mathbb{C}$ itself becomes a commutative finite-dimensional algebra over $\mathbb{C}$. Moreover, $K(X) \otimes \mathbb{C}$ is also a $K(X) \otimes \mathbb{C}$-module. Since $K(X) \otimes \mathbb{C}$ is commutative, all of its irreducible representation is 1-dimensional, thus we can decompose the representation $\rho = \rho_1 \oplus \cdots \oplus \rho_n$. This means that $V \otimes V_i = \lambda_i V_i$, and now comparing the rank $\mathrm{rk}(V \otimes V_i) = \mathrm{rk}(V)\mathrm{rk}(V_i)$, this implies that $\lambda_i = \mathrm{rk}(V)$.

Easy computation shows that $\wedge_t^* V$ has eigenvalues as $(1 - t)^{\mathrm{rk}(V)}$.

Given a proper map $p : X \to pt$, using the Grothendieck-Riemann-Roch we have that:

$$\begin{aligned} \chi(X, \mathscr{F} \otimes S_t^* V) =& \mathrm{ch}(p_!(\mathscr{F} \otimes S_t^* V)) = p_*(\mathrm{ch}(\mathscr{F} \otimes S_t^* V)\mathrm{Td}(X)) \\ =& p_*(\mathrm{ch}(\mathscr{F})\mathrm{ch}(S_t^* V)\mathrm{Td}(X)) \\ =& p_*(\mathrm{ch}(\mathscr{F})\mathrm{ch}(\bigwedge_t^* V)^{-1}\mathrm{Td}(X)) \\ =& p_*(\frac{\mathrm{ch}(\mathscr{F})}{\prod_{i=1}^{t}(1 - tL_i)}\mathrm{Td}(X)) = \int_X \frac{\mathrm{ch}(\mathscr{F})}{\prod_{i=1}^{t}(1 - tL_i)}\mathrm{Td}(X) \end{aligned} \tag{.95}$$

29. **Solution to 3.3.** We assume that X has a Lie group G action such that $X^G = \bigsqcup_\alpha F_\alpha$. By the equivariant localization we have that:

(.96)

$$\int_X \frac{\mathrm{ch}(\mathscr{F})}{\mathrm{ch}(\wedge_t^* V)}\mathrm{Td}(X) = \sum_\alpha \int_{F_\alpha} \frac{\mathrm{ch}(\mathscr{F})}{\mathrm{ch}(\wedge_t^* V)e(N_{F_\alpha})}\mathrm{Td}(X) = \sum_\alpha \int_{F_\alpha} \frac{\mathrm{ch}(\mathscr{F})\mathrm{Td}(N_{F_\alpha})}{\mathrm{ch}(\wedge_t^* V)e(N_{F_\alpha})}\mathrm{Td}(F_\alpha) = \sum_\alpha \chi(F_\alpha, \mathscr{F} \otimes S_t^* V$$

30. **Solution to 3.4.** If we write $TX = L_1 + \cdots + L_n$, i.e. $T^*X = L_1^{-1} + \cdots + L_n^{-1}$, we can express Krichever genus as:

$$\begin{aligned} & p_*(\mathrm{Td}(X)\mathrm{ch}(\frac{\theta(y \otimes TX)}{\phi(TX)\phi(T^*X)})) = \prod_{i=1}^n p_*(\frac{x_i}{1-e^{-x_i}}\frac{\theta(yx_i)}{\phi(x_i)\phi(x_i^{-1})}) \\ & = \prod_{i=1}^n p_*(x_i \prod_{k=1}^\infty \frac{(1-q^n ye^{yx_i})(1-q^n y^{-1}e^{-x_i})}{(1-q^n e^{x_i})(1-q^n e^{-x_i})}) \end{aligned} \tag{.97}$$

Denote $X^{S^1} = \bigsqcup_\alpha F_\alpha$, using the equivariant localization we have that:
(.98)

$$\begin{aligned} & \int_X \prod_{i=1}^n (x_i \prod_{k=1}^\infty \frac{(1-q^n ye^{yx_i})(1-q^n y^{-1}e^{-x_i})}{(1-q^n e^{x_i})(1-q^n e^{-x_i})}) = \sum_\alpha \int_{F_\alpha} e_{S^1}(TF_\alpha) \prod_{i=1}^n (\prod_{k=1}^\infty \frac{(1-q^n ye^{yx_i})(1-q^n y^{-1}e^{-x_i})}{(1-q^n e^{x_i})(1-q^n e^{-x_i})}) \\ & = \sum_\alpha \int_{F_\alpha} e_{S^1}(TF_\alpha) \prod_{i=1}^n (\prod_{k=1}^\infty \frac{(1-q^n yz^{n_i})(1-q^n y^{-1}z^{-n_i})}{(1-q^n z^{n_i})(1-q^n z^{-n_i})}) \\ & = \sum_\alpha \int_{F_\alpha} \mathrm{Td}(F_\alpha)\mathrm{ch}(\frac{\theta(y\otimes N_{F_\alpha})\theta(y \otimes TF_\alpha)}{\theta(N_{F_\alpha})\theta(TF_\alpha)\theta(T^*F_\alpha)}) = \sum_\alpha \chi(F_\alpha, \frac{\theta(y\otimes N_{F_\alpha})\theta(y \otimes TF_\alpha)}{\theta(N_{F_\alpha})\theta(TF_\alpha)\theta(T^*F_\alpha)}) \end{aligned}$$

Thus the equivairant parametre z only occurs in the term $\theta(y \otimes N_{F_\alpha})/\theta(N_{F_\alpha})$. It has the possible singularities over $z^{k_i} = q^n$ with $n \in \mathbb{Z}$.

31. **Solution to 3.5.**

$$\begin{aligned} \mathscr{E}(qz) & = \sum_\alpha \chi(F_\alpha, \frac{\theta(yqN_{F_\alpha})}{\theta(qN_{F_\alpha})}\frac{\theta(yTF_\alpha)}{\theta(TF_\alpha)\theta(T^*F_\alpha)}) \\ & = \sum_\alpha y^{-S^1-\text{weight of } N_{F_\alpha}} \chi(F_\alpha, \frac{\theta(yN_{F_\alpha})}{\theta(N_{F_\alpha})}\frac{\theta(yTF_\alpha)}{\theta(TF_\alpha)\theta(T^*F_\alpha)}) \end{aligned} \tag{.99}$$

So if $\mathscr{K}_X = \mathscr{L}^{\otimes N}$ for some line bundle $\mathscr{L}$, this means that:

$$S^1\text{-weight of } N_{F_\alpha} = S^1\text{-weight of } TX|_{F_\alpha} = N(S^1\text{-weight of } \mathscr{L}) \tag{.100}$$

with the fact that $y^N = 1$, this gives that $\mathscr{E}(qz) = \mathscr{E}(z)$.

32. **Solution to 3.6.** Note that $\mu_n \subset S^1$ is a normal subgroup of S^1, this means that $S^1/\mu_n \cong S^1$, which is still a Lie group.

We first denote $X^{\mu_n} = \sqcup_I X_I$ as the fixed point component of X. For each X_I there is a Lie group S^1/μ_n action. In this case we have:

$$\chi(X_I, \frac{\theta(y \otimes TX_I)}{\phi(TX_I)\phi(T^*X_I)}) = \sum_{\alpha \in I} \chi(F_\alpha, \frac{\theta(y \otimes N_{F_\alpha/F_I})\theta(y \otimes TF_\alpha)}{\theta(N_{F_\alpha/F_I})\theta(TF_\alpha)\theta(T^*F_\alpha)}) \tag{.101}$$

Now go back to the Krichever genus of X:

$$\begin{aligned}\chi(X, \frac{\theta(y \otimes TX)}{\phi(TX)\phi(T^*X)}) &= \sum_I \sum_{\alpha \in I} \chi(F_\alpha, \frac{\theta(y \otimes N_{F_\alpha/X})\theta(y \otimes TF_\alpha)}{\theta(N_{F_\alpha/X})\theta(TF_\alpha)\theta(T^*F_\alpha)}) \\ &= \sum_I \sum_{\alpha \in I} \chi(F_\alpha, \frac{\theta(y \otimes N_{F_\alpha/F_I})\theta(y \otimes N_{F_I/X})\theta(y \otimes TF_\alpha)}{\theta(N_{F_\alpha/F_I})\theta(N_{F_I/X})\theta(TF_\alpha)\theta(T^*F_\alpha)}) \\ &= \sum_I \chi(X_I, \frac{\theta(y \otimes TX_I)\theta(y \otimes N_{X_I})}{\phi(TX_I)\phi(T^*X_I)\theta(N_{X_I})})\end{aligned} \tag{.102}$$

If we assume the result, we can see that the formula can be changed via the variable change $z \mapsto z^n$. This means that $\mathscr{E}_y(X)$ does not have poles of order n. Thus we can see that $\mathscr{E}_y(X)$ is holomorphic on z.