FROM ELLIPTIC GENERA TO ELLIPTIC QUANTUM GROUPS

LECTURES BY ANDREI OKOUNKOV
NOTES BY TIANQING ZHU

ABSTRACT.

CONTENTS

1. LECTURE ONE

In this lecture, we will be discussing some Krichever's early work on topology. Our goal is to give an explanation on elliptic genus and the explanation on how it will be going with what we are doing now.

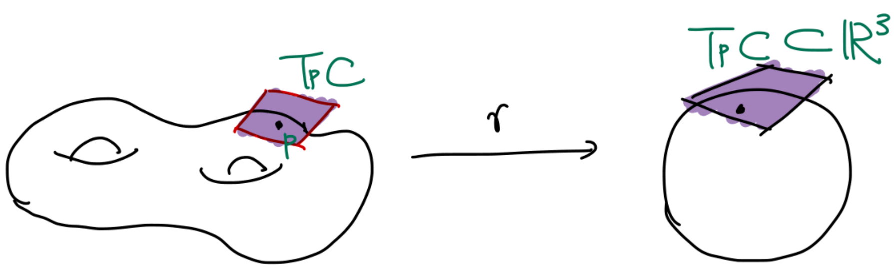
The power of the elliptic functions can be argued geometrically. There is typical argument to show the properties of the elliptic functions in functional analysis and special function theory.

1.1. First glance: Gauss map of oriented surfaces.

At first, our general discussion can get started with the Gauss map, i.e. Let C be an oriented Riemann surface in $\mathbb{R}^3$. Choose a smooth point p in C and its corresponding tangent space T_pC over C. The Gauss map is a map from $r : C \to S^2 = \mathrm{Gr}_+(2, 3, \mathbb{R})$ sending the point $p \in C$ to a flag $\{T_pC \subset \mathbb{R}^3\} \in \mathrm{Gr}_+(2, 3, \mathbb{R})$. We want to compute the degree of the map r

Proposition 1.1. *The degree of the Gauss map r is $1 - g$.*

Here g is the genus of C. The figure of the map is shown below.



The understanding of the map r is to describe the Riemann surfaces C and S^2 via the height function h_C and h_{S^2}. To define the height function h_C for arbitrary oriented Riemann surfaces C, we first embed the oriented Riemann surface $C \hookrightarrow \mathbb{R}^3$ in a vertical way. Then we composite the embedding with the projection $\mathbb{R}^3 \to \mathbb{R}$ to the z-axis, and now we obtain the height function $h_C : C \hookrightarrow \mathbb{R}^3 \hookrightarrow \mathbb{R}$.

The height function h_C is a Morse function, which means that the function h_C has nondegenerate Hessian matrix $\{\partial_i \partial_j h_C|_p\}$ at each point $p \in C$.

The knowledge in Morse theory tells us that the topology of the Riemann surface C can be described by the critical points of the Morse function h_C:

Proposition 1.2. *Given a compact manifold M with a Morse function $f : M \to \mathbb{R}$. We have that:*

$$\chi(M) = \sum_{\lambda} (-1)^{\lambda} C_{\lambda} \tag{1.1}$$

Here C_λ is the number of critical points of the Morse function f with the index λ.

We usually call the right hand side of 1.1 the Morse index. Basic knowledge in Morse theory tells us that the Morse index of h_C is $2-2g$ and for h_{S^2} is 2. In this way we can prove that the degree of the map is given by $1-g$.

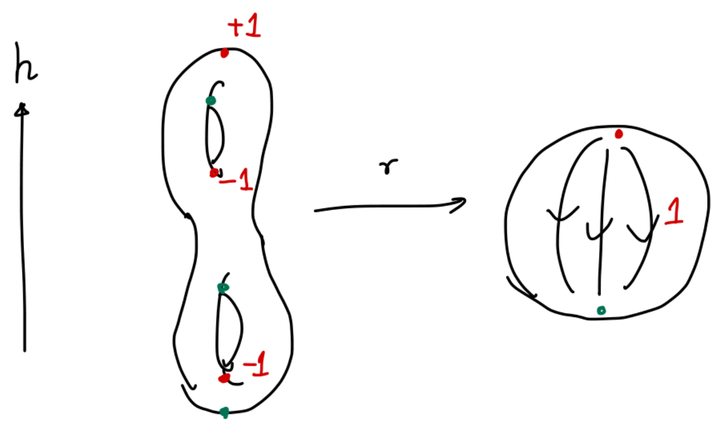


FIGURE 1. Height Morse function for Riemann surfaces C and S^2

The second perspective on the degree of the Gauss map is the following: Choose a vector field $v \in \Gamma(TS^2)$ and consider the pullback r^*v of the vector field to C. Via computation, the index of the vector field v:

$$\text{ind}(v) = \sum_{v(p)=0} \text{ind}_p(v) = (C, C) \tag{1.2}$$

Here (C, C) is the self-intersection number of C inside of TC. Using the Hopf index theorem, we can have that $(C, C) = \chi(C)$.

Now $\text{ind}(r^*v) = \sum_{v(p)=0} \deg(r)\text{ind}_p(v) = \deg(r)\text{ind}(v)$. Since $\chi(C) = \deg(r)\chi(S^2) = 2\deg(r) = 2-2g$, we obtain the degree of r equals $1-g$.

The third way is to consider the graph of the map Γ_r. Choose the north pole p_1 and south pole p_2 of S^2 and choose some flows from p_1 to p_2. In this way the Gauss map r factor through $\wedge^* S^2$ the wedge product of S^2. A minor trouble may be that Γ_r is not smooth. But in some way we can "smoothen" it by the following spirit.

Consider a function $f(x) = x_1^2 + x_2^2 - x_3^2$. Computations and drawings show that $f^{-1}(1)$ is like a tube , $f^{-1}(0)$ is like a cone and $f^{-1}(-1)$ is like the disjoint union of two

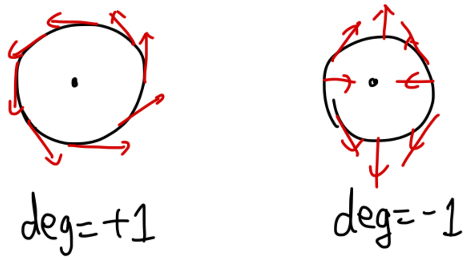


FIGURE 2. Index of the vector field

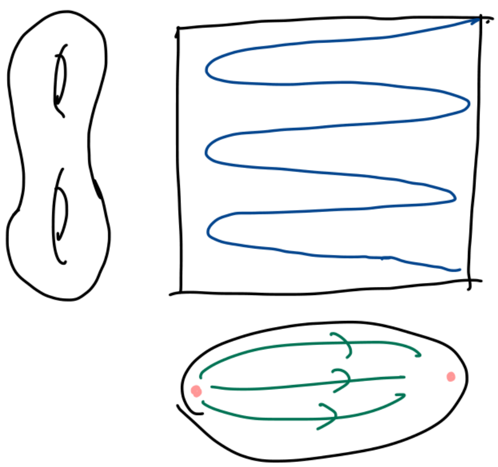


FIGURE 3. The graph of the Gauss map

hemispheres ($\mathbb{C} \sqcup \mathbb{C}$). We can see that it is a cobordism between two smooth manifolds but maybe with some singular spaces in the middle. In this way we can take

$$r(x) = \frac{1}{||x||}(x_1, x_2, -x_3) \tag{1.3}$$

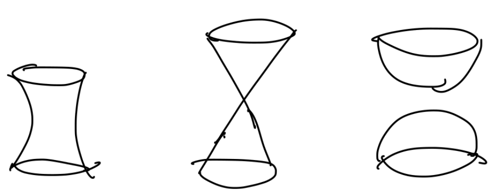


FIGURE 4. The inverse image $f^{-1}(1)$, $f^{-1}(0)$ and $f^{-1}(-1)$

What we can do is that we can perturb the function $f_\epsilon(x) = f(x) - \frac{\epsilon}{||x||}$. In this way we can think about r as a map of $\sqcup_{1-g} S^2$ to S^2.

The fourth way is to consider the index of the operator $\overline{\partial} : C^\infty \to C^\infty \overline{dz}$. Index theorem tells us that:

$$\text{Ind}_C(\overline{\partial}) = (1-g)\text{Ind}_{S^2}(\overline{\partial}) \tag{1.4}$$

It is obvious that $\text{Ker}(\overline{\partial})$ is equal to the space of holomorphic functions, and $\text{Coker}(\overline{\partial})$ is equal to the space of holomorphic differentials.

On the other hand, $\text{ind}(\overline{\partial}) = \chi(\mathcal{O}_C)$, and $\mathcal{O}_C$ fits into the short exact sequence:

$$\mathcal{O}_C \to C^\infty(C) \to C^\infty(C)d\overline{z} \to 0 \tag{1.5}$$

which is the Dolbeault complex.

1.2. **Vector bundles and Grassmannians.** Now we assume X is a compact Hausdorff topological space. A vector bundle over X is a space $p : E \to X$ with a map p such that there is a cover of $X = \cup_i U_i$ such that $p^{-1}(U_i) \cong U_i \times \mathbb{C}^n$ with the transition function $\varphi_{UU'} : U \cap U' \to U \cap U'$ compatible with p.

We can associate $K^0(X)$, i.e. the K-group of the complex vector bundles over X. It is defined as follows:

$$K^0(X) := \frac{\{\text{Set of isomorphism classes of complex vector bundles over } X\}}{\{\text{Short exact sequences}\}} \tag{1.6}$$

The notation of the definition means the following: If we have a short exact sequence of complex vector bundles:

$$0 \to V_1 \to V_2 \to V_3 \to 0 \tag{1.7}$$

In $K^0(X)$ it means that:

$$[V_2] = [V_1] + [V_3] \tag{1.8}$$

This gives $K^0(X)$ an abelian group structure. It is obvious that for the direct sum of complex vector bundles:

$$[V_1 \oplus V_2] = [V_1] + [V_2] \tag{1.9}$$

In the lecture we will always assume that the vector bundle is complex.

The K-group has a ring structure. The tensor product $V_1 \otimes V_2$ of vector bundles makes the set of isomorphism classes of vector bundles into a semiring. Thus we define $K^0(X)$ as the corresponding semiring with the subtraction operation $\ominus$. The symbol $\ominus$ means that:

$$V_1 \ominus V_2 = W_1 \ominus W_2 \text{ is equivalent to } V_1 \oplus W_2 = V_2 \oplus W_1 \tag{1.10}$$

The following proposition implies that every element of the abelian group $K^0(X)$ has an inverse element.

Proposition 1.3. *For any vector bundle E over the space X there is an embedding $V \subset \mathbb{C}^N$ into the trivial bundle $\mathbb{C}^N$ for some large N. Also, for any vector bundle V with N large enough, $V \oplus V^{\perp} = \mathbb{C}^N$.*

In other words, we can assume that for $V_1 \ominus V_2$ with V_2 being trivial. We have that $\ominus V = \ominus \mathbb{C}^N \oplus V^{\perp}$.

We can describe the vector bundle over the space X in a universal way. We introduce the Grassmannian manifold $\mathrm{Gr}(k, n, \mathbb{C})$ to be the topological space consisting of the flags $(\mathbb{C}^k \subset \mathbb{C}^n)$ up to an isomorphism. It is a compact complex manifold of the complex dimension $k(n-k)$.

Now choose $V \subset \mathbb{C}^N$ the vector bundle over X. For each point $p \in X$, we obtian a flag of vector spaces $V_p \subset \mathbb{C}^N_p$. Thus we obtain a continuous map $r : X \to \mathrm{Gr}(\mathrm{rk}(V), N, \mathbb{C})$.

There is a tautological bundle *Tauto* over $\mathrm{Gr}(\mathrm{rk}(V), N, \mathbb{C})$ with the fibre being V over the point $(V \subset \mathbb{C}^N)$. In this way we can see that the vector bundle V can be described as the pullback of the tautological bundle:

$$V = r^* Tauto \to X \tag{1.11}$$

This gives us the following conclusion:

Proposition 1.4. *Given a compact Hausdorff space X, for every complex vector bundle V over X, it is equivalent to give a continuous map $r : X \to Gr(r, N, \mathbb{C})$ from X to the complex Grassmannian manifold $Gr(r, N, \mathbb{C})$ with $r = rk(V)$ and some large enough N. Such that $V = r^* Tauto$.*

We define $\mathrm{Gr}(r, \infty, \mathbb{C}) = \cup_N \mathrm{Gr}(r, N, \mathbb{C})$ the union of the Grassmannian manifold with fixed N. In this case the map r can be written as $r : X \to \mathrm{Gr}(r, \infty, \mathbb{C})$.

In cohomology, the map r induces the map of cohomology group

$$r^* : H^*(\mathrm{Gr}(r, \infty, \mathbb{C})) \to H^*(X) \tag{1.12}$$

The characteristic classes of X is the pullback r^* of the class in $H^*(\mathrm{Gr}(r, \infty, \mathbb{C}))$.

Let us compute $\mathrm{Gr}(r, \infty, \mathbb{C})$. For $r = 1$, $\mathrm{Gr}(1, N)$ is the moduli space of the line $l \subset \mathbb{C}^N$ through the origin point. By definition $\mathrm{Gr}(1, N) \cong \mathbb{P}^{N-1}$ is the complex projective space. The projective space has really nice properties of the cell decomposition. The cell decomposition gives the cohomology:

$$H^*(\mathbb{P}^{N-1}) = \mathbb{Z}[x]/(x^N = 0), \qquad \deg(x) = 2 \tag{1.13}$$

Passing N to the infinity, we have $H^*(\mathrm{Gr}(1, \infty, \mathbb{C})) = \mathbb{Z}[x]$ with each x of the cohomological degree 2. We usually denote $\mathrm{Gr}(1, \infty, \mathbb{C})$ as $\mathbb{P}^{\infty}$.

The meaning of $\mathbb{P}^{\infty}$ is that it is a topological space defined by $\mathbb{C}^{\infty} - \{0\}/GL(1) \cong S^{\infty}/U(1)$. The resulting space is called the classifying space by definition. In general, the Grassmannian has the similar isomorphism

$$\mathrm{Gr}(r, N, \mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N)_{\text{full rank}}/GL_r \tag{1.14}$$

Here $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N)_{\text{full rank}}$ is the space of all the $r \times N$ full-rank matrices. Recall that the given a rank r vector bundle V over X, we can think about it as a principal $U(r)$-bundle P over X. Via the map r, we can realise P as the pullback of $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N)_{\text{full rank}}$. This means that $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N)_{\text{full rank}}$ is the universal bundle $EU(r)$. The Grassmannian $\mathrm{Gr}(r, \infty, \mathbb{C})$ is now the classifying space of $U(r)$:

$$\mathrm{Gr}(r, \infty, \mathbb{C}) \cong BU(r) \tag{1.15}$$

So we have proved the following result:

Proposition 1.5. *The classifying space $BU(r)$ for $U(r)$ is isomorphic to $Gr(r, \infty, \mathbb{C})$.*

Generally for each group H, we have the corresponding classifying space BH. If we have a homomorphism of groups $H_1 \to H_2$, this induces the map of topological spaces $BH_1 \to BH_2$.

We can prove this fact in the case of the classifying space for the unitary groups $U(n)$. So if we have the maximal torus embedding $U(1)^r \to U(r)$ of $U(r)$, we can construct continuous map $BU(1)^r \to BU(r)$ in the language of the Grassmannians.

For the matrices in $r \times N$ of full rank , the matrices such that the $r \times r$ block in $r \times N$ is equivalent to the diagonal matrix forms a subspace of $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N)_{\text{full rank}}$:

$$\mathrm{Hom}(\mathbb{C}, \mathbb{C}^N)^{\oplus r}_{\text{full rank}} \hookrightarrow \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N)_{\text{full rank}} \tag{1.16}$$

Furthermore, the group action of GL_r on $\mathrm{Hom}(\mathbb{C}, \mathbb{C}^N)^{\oplus r}_{\text{full rank}}$ would be reduced to the group action of $GL_1^{\times r}$. This means that we have the following map of topological spaces:

$$\mathrm{Gr}(1, N, \mathbb{C})^{\times r} \to \mathrm{Gr}(r, N, \mathbb{C}) \tag{1.17}$$

which gives the map $BU(1)^r \to BU(r)$ passing N to ∞.

This map induces the map of cohomology $H^*(BU(r)) \to H^*(BU(1))^{\otimes r} \cong \mathbb{Z}[x_1, \cdots, x_r]$ with $\deg(x_i) = 2$. This is an injective map and in fact r^* induces the permutation of the variable $x_1, \cdots, x_r$ This implies that $H^*(BU(r))$ is isomorphic to the space of symmetric polynomials $\mathbb{Z}[x_1, \cdots, x_r]^{S_r}$. Hence we have:

Proposition 1.6. *The cohomology $H^*(BU(r))$ of the classifying space $BU(r)$ is isomorphic to the space of symmetric polynomials $\mathbb{Z}[x_1, \cdots, x_r]^{S_r}$ of r variables.*

The space $\mathbb{Z}[x_1, \cdots, x_r]^{S_r}$ is generated by the elementary symmetric polynomials $\mathbb{Z}[e_1, \cdots, e_r]$. The elementary symmetry polynomials are defined as:

$$e_k(x_1, \cdots, x_r) := \sum_{1 \le i_1 < \cdots < i_k \le r} x_{i_1} \cdots x_{i_k} \tag{1.18}$$

As the element in $H^*(BU(r))$, it is of the cohomological degree $2k$.

Now given a vector bundle V over X with the corresponding map $r: X \to BU(r)$. We define the **k-th Chern class** $c_k(V)$ of the vector bundle V to be:

$$c_k(V) := r^* e_k \in H^{2k}(X) \tag{1.19}$$

1.3. **Geometric interpretation of the Chern classes.** We can give a geometric interpretation of Chern classes. Given σ_l, $1 \leq l \leq n$. The global sections of the rank k vector bundle V over X. We define the degeneration set $D_i(\sigma)$ as:

$$D_i(\sigma) := \{x \in X : \sigma_1(x) \wedge \cdots \wedge \sigma_i(x) = 0\} \subset X \tag{1.20}$$

We now suppose that $\{\sigma_l\}_{1 \leq l \leq k}$ are generic sections of V such that $D_{i+1}(\sigma) - D_i(\sigma)$ is a submanifold of codimension $2(k-i)$. The generic sections give the map

$$i : X \hookrightarrow \mathrm{Gr}(k, n) \tag{1.21}$$

Via sending x to $(\sigma_1(x), \cdots, \sigma_k(x)) \subset V_x$. It is easy to see that $i^*(\text{Tauto})$ is isomorphic to the dual bundle V^*, and it means that:

$$i^*(\text{Tauto}^*) \cong V \tag{1.22}$$

For each $r = 1, \cdots, k$, we define the subspace $V_{n-k+r-1} := \{e_{k-r+1}, \cdots, e_n\} \subset \mathbb{C}^n$. For any $x \in X$, the k-plane $i(x)$ intersects $V_{n-k+r-1}$ of dimension at least r iff the sections $\sigma_1, \cdots, \sigma_{k-r+1}$ are linearly dependent at x. This implies that the image of X in $\mathrm{Gr}(k, n)$ meets the Schubert cycle $\mathfrak{S}_{1,\cdots,1}$ in the degeneracy set D_{k-r+1} of the sections $\sigma_1, \cdots, \sigma_k$.

If we take a cycle $\alpha \in H_{2r}(X)$. Via the computation:

$$\begin{aligned} c_r(V)(\alpha) =& c_r(i^*(\text{Tauto}^*))(\alpha) = c_r(\text{Tauto}^*)(i_*\alpha) \\ =& (i_*\alpha \cdot [\mathfrak{S}_{1,\cdots,1}])^\vee = (\alpha \cdot D_{k-r+1})^\vee \end{aligned} \tag{1.23}$$

In conclusion we have that:

Theorem 1.7. *The Chern class $c_r(V)$ is Poincare dual to $[D_{k-r+1}] \in H_*(X)$*

For example, if we take $c_{top} \in H^{2r}(BU(r))$. This stands for the locus where a section of V^* vanishes. This is equivalent to say that $c_r = e_r = x_1 \cdots x_r$.

1.4. **Analog of Chern classes in K-theory.** In K-theory, we usually take the wedge product of the vector bundle $\wedge^i V \in K^0(X)$ as the analog of $c_i(V) \in H^{2i}(X)$. We can see the similarity of $\wedge^i V$ and $c_i(V)$ from the following perspective.

In cohomology theory, recall that $c_k(V)$ is defined as the pullback of the elementary polynomial $e_k \in H^*(BU(r))$. Each generator $x_i \in H^2(BU(r))$ can be thought of as the first Chern class of some line bundle L_i using the splitting principle:

$$V'' = '' L_1 + \cdots + L_r \tag{1.24}$$

Similarly, in K-theory, if we have the above splitting 1.24, it is easy to check that $\wedge^i V$ corresponds to the i-th elementary polynomial of $L_1, \cdots, L_r$.

In cohomology the integration of the Chern class $\int_X \prod_i c_{k_i}(V)$ is replaced by the Euler characteristic $\chi(\otimes \wedge^{k_i} V)$ in K-theory. They are related by the Grothendieck-Riemann-Roch theorem.

2. Lecture Two

2.1. **Equivariant cohomology theories.** In this lecture we start with the general cohomology theory of some good topological spaces.

Let X be a topological space such that it has finite cells and it is a complex manifold. Some good examples are algebraic varieties, Kahler manifold etc.

Given a continuous map $\pi : X \to Y$, we assume that X is a G-space and Y is an H-space, etc. they admit the group action given by G and H. we say that the map π is **equivariant** with respect to the group action G and H if there is a group homomorphism $\varphi : G \to H$ such that:

$$\varphi(g) f(x) = f(gx), \qquad g \in G, x \in X \tag{2.1}$$

In most of the cases, we prefer both G and H to be a compact Lie group or a reductive complex group. The latter is often used in the case of the algebraic geometry or complex geometry.

Given a pair of CW complexes (X, A) with $A \subset X$ a subcomplex of X. The cohomology of the pair (X, A), which we will denote it by $h^*(X, A)$ for now, is a complex of abelian groups together with a natural transformation $d : h^i(A) \to h^{i+1}(X, A)$ satisfying the following axioms:

- Homotopic maps induce the isomorphism on cohomology.
- The map $\pi : (X, A) \to (Y, B)$ induces the pullback $\pi^* : h^*(Y, B) \to h^*(X, A)$.
- Each pair (X, A) induces a long exact sequence in homology, via the inclusion map $f : A \hookrightarrow X$ and $g : (X, \emptyset) \to (X, A)$:

$$\cdots \longrightarrow h^i(X, A) \xrightarrow{g^*} h^i(X) \xrightarrow{f^*} h^i(A) \xrightarrow{d} h^{i+1}(X, A) \longrightarrow \cdots \tag{2.2}$$

- If X is the union of subcomplexes A and B, the inclusion map $f : (A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism

$$f^* : h^i(X, B) \to h^i(A, A \cap B) \tag{2.3}$$

- If (X, A) is the disjoint union of a set of pairs (X_α, A_α), the inclusion $(X_\alpha, A_\alpha) \hookrightarrow (X, A)$ induce an isomorphism:

$$h^i(X, A) \equiv \prod_\alpha h^i(X_\alpha, A_\alpha) \tag{2.4}$$

One typical example of cohomology theory can be like the singular cohomology, whose component of the complex is given by $\Omega^k = Hom_{\mathbb{Z}}(S_k(X), G)$, here $S_k(X)$ is the chain of complexes over the topological space X with the cochain complex given as:

$$\Omega^0 \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \Omega^2 \xrightarrow{\delta} \cdots \tag{2.5}$$

with Ω^k being the space of $\mathbb{Z}$-module map to G over the k-cell in X. The differential δ is defined as the dual of the cell-differential in the singular homology. Moreover, one can check that $\delta^2 = 0$.

Given a continuous map $f : X \to Y$, we can associate the cone of the map Cone(f) as the quotient of the mapping cyliner $(X \times I) \sqcup_f Y$ with respect to the equivalence relation $\forall x, x' \in X$, $(x, 0) \sim (x', 0)$, $(x, 1) \sim f(x)$.

The good property of the cone of the map is that it gives a cofibration:

$$X \xrightarrow{f} Y \longrightarrow \text{Cone}(f) \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \cdots \tag{2.6}$$

Here $\Sigma X = S^1 \wedge X$ is the suspension of X.

Moreover if X and Y are well-pointed (i.e. the inclusion of the base point is a cofibration). Then 2.6 is co-exact, i.e. we have the following sequence of pointed sets that is exact:

$$\cdots [X, Z]_0 \to [Y, Z]_0 \to [\text{Cone}(f), Z]_0 \to [\Sigma X, Z] \to \cdots \tag{2.7}$$

The continuous map $f : X \to Y$ induces the map of cohomology groups $f^* : H^*(Y) \to H^*(X)$. It is a cochain map, and we define the cone Cone(f^*) of the cochain map f^* as:
(2.8)

$$\text{Cone}(f^*) = \{\ H^{i+1}(Y) \oplus H^i(X) \xrightarrow{d_{C(f)}} H^{i+2}(Y) \oplus H^{i+1}(X)\ \}, \qquad d_{C(f)} = \begin{pmatrix} d_{H*(Y)[1]} & 0 \\ f^*[1] & d_{H^*(X)} \end{pmatrix}$$

We have the property that $h^*(\text{Cone}(f)) = \text{Cone}(f^*)$

For the trivial example $X = pt$, we have an identity map id : $pt \to pt$, and in this case Cone$(id) = pt$. We have $h^*(pt) = \text{Cone}(\ h^*(pt) \xrightarrow{\text{id}} h^*(pt)\ )$.

The cohomology of X can also be described by the suspension of X. Define the cone of X as:

$$CX := (X \times [0, 1])/((x, 0) \sim (x_0, t), t \in (0, 1])) \tag{2.9}$$

Now we embed $X \to CX$ into the cone of X. This will give us a cofibre sequence:

$$X \to CX \to \Sigma X \to \Sigma CX \to \cdots \tag{2.10}$$

Since CX is contractible, this means that $h^*(\Sigma X)$ is the shift cone of $0 \to h^*(X)$, and this means that $h^i(\Sigma X) = h^{i-1}(X)$. The principal implies that, for the induced map $\Sigma^n f : \Sigma^n X \to \Sigma^n Y$.

The suspension has some other nice properties. For example, the mapping space $\mathrm{Map}(\Sigma^n X, Z)$ with $n \geq 1$ from $\Sigma^n X$ to Z has the abelian group structure. ()

2.2. **Equivariant K-theory.** Today our exmaple will be equivariant K-theory. Let X be some compact space. Let V be some vector bundle over X. We will say that V is a **G-equivariant bundle** over X if X is a G-space and the bundle map $\pi : V \to X$ is G-equivariant.

We define the K-group $K^0_G(X)$ of G-equivariant vector bundles over X. We further need that the base point of X should be fixed by the group G. Furthermore:

(2.11)
$$K^0_G(X) := \frac{\{\text{Set of isomorphism classes of } G-\text{equivariant vector bundles over } X\}}{\{\text{Short exact sequences}\}}$$

The quotient means the following: If we have a short exact sequence of G-equivariant vector bundles:

$$0 \to V_1 \to V_2 \to V_3 \to 0 \tag{2.12}$$

we have that:

$$[V_3] = [V_1] + [V_2] \in K^0_G(X) \tag{2.13}$$

This makes $K^0_G(X)$ an abelian group. The tensor product $\otimes$ between G-vector bundles and the rank 1 trivial bundle makes $K^0_G(X)$ a ring.

If G acts on X free, we have $K^0_G(X) = K^0(X/G)$ with X/G the stacky quotient of X by G.

Let us start with the example $X = pt$ and G a compact Lie group. In this case $K^0_G(pt)$ consists of the finite dimensional vector space V who is the G-module. In this case $K^0_G(pt)$ is a representation semiring with operations $(\oplus, \otimes)$. This is in fact the representation ring over G. Let us take $T \subset G$ to be the maximal torus of G with the cocharacter denoted as $z_1, \cdots, z_r$.

Lemma 2.1. *We have the ring isomoprhism $K^0_G(pt) \cong \mathbb{Z}[z_1^{\pm 1}, \cdots, z_r^{\pm 1}]^W$ given by $V \mapsto ch(V)$ the character of V, and here W the Weyl group of G.*

The natural map $X \to pt$ induces the pushforward $K^0_G(pt) \to K^0_G(X)$. This makes $K^0_G(X)$ a $K^0_G(pt)$-module.

Let us define $K^{-1}_G(X) := K^0_G(\Sigma X)$. Here $\Sigma X = S^1 \vee X$ is the suspension of X, as recalled before. The group G will also act on ΣX induced by the action of G on X.

The good property of $K^i_G(X)$ is that it has the Bott periodicity for the equivariant K-theory:

Theorem 2.2. *(Bott periodicity) For the topological space X with a reductive group G action, we have the isomorphism of abelian groups:*

$$K^0_G(\Sigma^2 X) \cong K^0_G(X) \tag{2.14}$$

Given a topological subspace $Y \subset X$, we define $K(X/Y)$ as the abelian groups generated by the complexes of vector bundles on X such that it becomes exact when restricted to Y. This is the relative cohomology in the setting of the K-theory.

2.3. Thom space and Thom isomorphism.

Given a complex vector bundle V over X, we define the Thom space to be the quotient space:

$$\mathrm{Thom}(V) = \mathbb{P}(V \otimes 1)/\mathbb{P}(V) \tag{2.15}$$

By definition, it is the one-point compactification of V. Now the elements of $K_G(\mathrm{Thom}(V))$ are complexes of vector bundles on the total space of V that are exact away from the zero sections X. Via the natural map $\mathrm{Thom}(V) \to X$, we have the morphism $Thom : K^0_G(X) \to K^0_G(Thom(V))$. This makes $K^0_G(Thom(V))$ a $K_G(X)$-module by definition.

We have the analog of the Thom isomorphism in the settings of the equivariant K-theory: The key thing for constructing the Thom isomorphism is that it helps to define the pushforward map in K-theory. Moreover we have the exact sequence:

$$K^0_G(X) \xrightarrow{Thom} K^0_G(Thom(V)) \longrightarrow K^0_G(V) \tag{2.16}$$

The latter map is induced by the inclusion $V \hookrightarrow Thom(V)$. By the natural isomorphism $K^0_G(V) \cong K^0_G(X)$, we have the following statement of the Thom isomorphism:

Theorem 2.3. *$K_G(Thom(V))$ is a free module of rank 1 over $K_G(X)$ generated by Koszul complex of the zero section.*

We are going to give the example of the projective space in the next lecture.

3. Lecture Three

Now given a finite dimensional vector space V, and let us suppose that the corresponding unitary group $U(V)$ acting on the projective space $\mathbb{P}(V)$. So how do we construct the equivariant K-theory $K_{U(V)}(\mathbb{P}(V))$? Now fix $V = \mathbb{C}^n$, we have the diffeomorphism $\mathbb{P}(V) \cong S^{2n-1}/U(1)$, and recall that since $U(1)$ acts on S^{2n-1} freely, we have $K_{U(V)}(S^{2n-1}/U(1)) \cong K_{U(n)\times U(1)}(S^{2n-1})$.

Now suppose that given a Lie group G acting on a homogeneous space $X = G/H$. We have $K_G(G/H) = K_H(pt)$ since given a G-equivariant vector bundle over G/H is the same as considering the action of H acting on the fibre of the vector bundle. Moreover we have $K_H(pt) = \mathrm{Rep}(H) \cong \mathbb{Z}[H^*]$, which is the $\mathbb{Z}$-linear combination of the characters of the irreducible representations of H.

Now back to $K_{U(V)}(\mathbb{P}(V)) \cong K_{U(n)\times U(1)}(S^{2n-1})$, we have a natural embedding $S^{2n-1} \hookrightarrow D^{2n}$. Using the isomorphism $D^{2n} - \{0\} \cong S^{2n-1} \times \mathbb{R}$, this embedding induces the following exact sequence:

(3.1)
$$\cdots \to K^0_{U(n)\times U(1)}(\mathrm{Thom}(0 \hookrightarrow D^{2n})) \to K^0_{U(n)\times U(1)}(D^{2n}) \to K^0_{U(n)\times U(1)}(S^{2n-1}) \to \cdots$$

Note that $K^0_G(D^{2n}) \cong \mathrm{Rep}(G) \cong \mathbb{Z}[G/G] = R$ is just the representation ring of $G = U(n) \times U(1)$, and $K^0_{U(n)\times U(1)}(S^{2n-1}) \cong K^0_{U(n)\times U(1)}(U(n)/U(n-1)) \cong K_{U(n-1)\times U(1)}(pt)$. We can think of $U(n-1) \subset U(n)$ as the stabilizer of the point $v = (1, 0, \cdots, 0)$ in D^{2n} by the $U(n)$-group action. In this case $K^0_G(\mathrm{Thom}(0 \to V))$ consists of the representations of G that are trivial when restricted to $G_v = U(n-1) \times U(1)$.

As the $K^0_{U(n)\times U(1)}(D^{2n})$-module, $K^0_G(\mathrm{Thom}(0 \to V))$ is isomorphic to $K^0_{U(n)\times U(1)}(D^{2n})\det(1 - tg)$ with $t \in U(1), g \in U(n)$, and thus we see that the map $K^0_G(\mathrm{Thom}) \to K^0_G(D^{2n})$ is injective, so it is left exact.

Also since $K^1_{U(n)\times U(1)}(\mathrm{Thom}(0 \hookrightarrow D^{2n})) \cong K^1_{U(n)\times U(1)}(S^{2n}) \cong \tilde{K}^0_{U(n)\times U(1)}(S^{2n+1}) = 0$, the sequence is right exact.

In conclusion, we have that:

$$K_{U(V)}(\mathbb{P}(V)) \cong K_{U(V)}(pt)[t^{\pm 1}]/(\det(1 - tg) = 0) \tag{3.2}$$

Moreover, if we choose the maximal torus $T \subset U(V)$, we have the following result:

Proposition 3.1. *We have the isomorphism:*

$$K_T(\mathbb{P}(V)) \cong \mathbb{Z}[a_1^{\pm 1}, \cdots, a_n^{\pm 1}][t^{\pm 1}]/(\prod_{i=1}^{n}(1 - a_i t) = 0) \tag{3.3}$$

Here we can understand the term $\det(1 - tg)$ in a geometric way. For simplicity we take $\det(1 - t^{-1}g^{-1})$ and $V = \mathbb{C}^2$ for the moment. Consider $\mathcal{O}_0 = \mathbb{C}[x_1, x_2]/(x_1, x_2)$ the structure sheaf of the point $0 \in \mathbb{C}^2$. Recall that as the $\mathbb{C}[x_1, x_2]$-module, it admits the resolution:

$$0 \to e_1e_2R \to e_1R \oplus e_2R \to R \to \mathcal{O}_0 \to 0 \tag{3.4}$$

Here $R = \mathbb{C}[x_1, x_2]$. The second arrow sends fe_1e_2 to $fe_1 - fe_2$. The third arrow sends $fe_1 + ge_2$ to $f + g$. The morphism $R \to \mathcal{O}_0 \cong \mathbb{C}$ sends $f(x_1, x_2)$ to $f(0, 0)$.

Note that the resolution terms are free modules over $\mathbb{C}[x_1, x_2]$, i.e. trivial vector bundle over $\mathbb{C}^2$. We can take e_i are coordinates over V, such that $e_1R \oplus e_2R \cong V^* \otimes R$. Taking them as the element in $K_{U(2)\times U(1)}(\mathbb{C}^2)$, we have that:

$$\begin{aligned}[\mathscr{O}_0] =&1 - V^* t^{-1} + \wedge^2 V^* t^{-2} \\ =&1 - \mathrm{Tr}(g^{-1})t^{-1} + \det(g^{-1})t^{-2} = \det(1 - t^{-1}g^{-1})\end{aligned} \tag{3.5}$$

Now let G be a compact Lie group, e.g. $G = U(1)$, and we have that:

$$K_{U(1)}(pt) = \mathrm{Rep}(U(1)) = \mathbb{Z}[t^{\pm 1}] \tag{3.6}$$

The Koszul resolution can be expressed in the category of coherent sheaves over X. In this case the Koszul complex is the resolution of $\mathscr{O}_0$ by free $\mathscr{O}_V$-modules. The resolution can be written as:

$$0 \to \prod_{i=1}^{n} x_i R \to \cdots \to \bigoplus_{i:[k]\to[n]} x_{i_1}\cdots x_{i_k} R \to \cdots \to \bigoplus_{i=1}^{n} x_i R \to R \to \mathscr{O}_0 \to 0 \tag{3.7}$$

In this case we can express $[i_*\mathscr{O}_0] \in K_{U(n)\times U(1)}(V)$ as:

$$[i_*\mathscr{O}_0] = \sum_{i=0}^{n} (-1)^i \wedge^i V^* t^{-i} = \det(1 - t^{-1}g^{-1}) \tag{3.8}$$

3.1. **Equivariant elliptic cohomology.** The definition of the equivariant elliptic cohomology is complicated. Here we give a pedagogical way to say what is the equivariant elliptic cohomology in terms of the geometry.

Recall that $K_{U(1)}(pt) = \mathbb{Z}[t^{\pm 1}]$ can be thought of as the space of holomorphic functions over $\mathbb{C}^*$. This observation tells us that we can think about $K_{U(1)}(pt)$ as the algebra of holomorphic functions over $\mathbb{C}^*$.

For the case of the $U(1)$-equivariant elliptic cohomology, we could have a naive guess:

$$\mathrm{Ell}_{U(1)}(pt) \sim \text{holomorphic functions on } \mathbb{C}^*/q^{\mathbb{Z}} \tag{3.9}$$

But we know that there is no nontrivial holomorphic functions over the elliptic curve. Here we need some modification that we can think about the cohomology theory in this case as the cohomology theory valued in coherent sheaves over some spaces. Under this philosophy, one might take:

$$h_G^*(X) = \text{A quasicoherent sheaf on } \mathrm{Ell}_G(pt) \tag{3.10}$$

and for $G = U(1)$, it is a quasicoherent sheaf on E. Thus $\mathrm{Ell}_{U(1)}(pt)$ is the structure sheaf of the elliptic curve E, or we can just treat it as the elliptic curve E.

For the elliptic cohomology $\mathrm{Ell}_T(\mathbb{P}(V))$, first note that $\mathrm{Ell}_T(pt) = E^n$, i.e. the Cartesian product of n elliptic curves. Then $\mathrm{Ell}_T(\mathbb{P}(V))$ should be:

$$\mathrm{Ell}_T(\mathbb{P}(V)) = \{\text{one of the } a_i \text{ is equal to } t^{-1}\} \subset E^n \times E \tag{3.11}$$

So similarly we have $\mathrm{Ell}_{U(n)}(pt) = \mathrm{Sym}^n E$, and we can think about it as the space of semistable $GL(n)$ degree 0 bundles on E. In this case $\mathrm{Ell}_G(\mathbb{P}(V))$ can be described as:

$$\mathrm{Ell}_G(\mathbb{P}(V)) = \{\text{one of the } a_i \text{ is equal to } t^{-1}\} \subset \mathrm{Sym}^n E \times E \tag{3.12}$$

For the elliptic analog of the Thom isomorphism, recall that in the equivariant K-theory, given V a vector bundle over X, $K_G(\mathrm{Thom}(X \hookrightarrow V))$ is a free rank 1 $K_G(X)$-module generated by complexes of Koszul resolutions of zero sections X in V. i.e. the generators as $\sum_i (-1)^i \wedge^i V^*$. Also note that the vector bundle V over X can be thought as coming from the pullback of the tautological bundle $Taut$ over $BU(n)$ for some map $r : X \to BU(n)$. Inspried by this, we consider the tautological bundle Θ_{univ} over $\mathrm{Ell}_{U(n)}(pt)$ defined via the Chern roots $\prod_i (1 - a_i^{-1})$, i.e. It is defined by the divisor $\{0\} + S^{n-1}E$ in $S^n E = \mathrm{Ell}_{U(n)}(pt)$. Now we take V as the principal $U(n)$-bundle over X. We have the following chain of maps:

$$c : \mathrm{Ell}_G(X) \cong \mathrm{Ell}_{G \times U(n)}(V) \to \mathrm{Ell}_{G \times U(n)}(pt) \to \mathrm{Ell}_{U(n)}(pt) \tag{3.13}$$

The last map uses the fact that given a group homomorphism $G_1 \to G_2$, there is a natural map:

$$\mathrm{Ell}_{G_1}(pt) \to \mathrm{Ell}_{G_2}(pt) \tag{3.14}$$

mapping the G_1-bundles over E to the G_2-bundles over E.

In this way we define the Thom sheaf as:

$$\Theta(V) := c^* \Theta_{univ} \tag{3.15}$$

Thus $\Theta(V)$ is a locally free sheaf of rank 1 over $\mathrm{Ell}_{U(n)}(X)$. We can see that $\Theta(V)$ is no longer a free module.

Generally if we have the closed embedding of complex manifolds $X \hookrightarrow Y$, we have that $i^* i_*$ is equal to the multiplication of the Thom sheaf $\Theta(N_{X/Y})$. Moreover we have the following pushforward map:

$$f_* : \mathrm{Ell}(f)_* \Theta(-N_f) \to \mathscr{O}_{\mathrm{Ell}_T(Y)} \tag{3.16}$$

4. Lecture Four

Recall that we have the torus action $A = (\mathbb{C}^*)^{n+1}$ acting on $\mathbb{P}^n$. We have the equivariant K-theory:

$$K_A(\mathbb{P}^n) = \mathbb{Z}[a_i^{\pm 1}]_{0 \le i \le n}[t^{\pm 1}] / (\prod_{i=0}^{n} (1 - a_i^{-1} t^{-1}) = 0) \tag{4.1}$$

The geometric meaning of the isomorphism can be explained in the following way: For $n = 2$, Write down $\mathbb{P}^2 = \{[x_0 : x_1 : x_2]\}$, and we consider the line $D_i = \{x_i = 0\}$, $i = 0, 1, 2$ in $\mathbb{P}^2$. Each D_i corresponds to a line bundle $\mathscr{O}(D_i)$ on $\mathbb{P}^2$, with the equivariant character given as $1 - a_i^{-1} t^{-1}$ for $i = 1, 2$. If we consider the intersection of D_1 and

D_2, it corresponds to the sheaf $i_*\mathcal{O}_0$, which corresponds to $\prod_{i=1}^{2}(1-a_i^{-1}t^{-1})$, and it just corresponds to the Koszul resolution of $i_*\mathcal{O}_0$.

Now we study Spec $K_A(\mathbb{P}^1)$. By the computation, the scheme corresponds to two hyperplanes $\{t = a_0^{-1}\} \cup \{t = a_1^{-1}\}$ in $(\mathbb{C}^*)^3$. The natural map π : Spec $K_A(\mathbb{P}^1) \to$ Spec $K_A(pt)$ corresponds to the projection map $(a_0, a_1, t) \mapsto (a_0, a_1)$. If (a_0, a_1) is generic, it is easy to see that the fibre should be two points. If $a_0 = a_1$, the fibre corresponds to the double point (a^{-1}, a, a).

What would we learn from these computations? If we look at the fibre of π, we have that the generic fibre over $a \in (\mathbb{C}^*)^3$ is $K((\mathbb{P}^n)^a)$ with $a : \mathbb{C}^* \to (\mathbb{C}^*)^3$ thought of as the character over A. If we choose a such that it is away from the subvariety of the form $a_i = a_j$, we have that $K_A(X)'' = ''K_A(X^A)$.

The interpretation of $K_A(X)'' = ''K_A(X^A)$? is given as follows: Given the inclusion map $i : X^A \hookrightarrow X$, we have the pullback map $i^* : K_A(X) \to K_A(X^A)$ and the pushforward map $i_* : K_A(X^A) \to K_A(X)$ on the equivariant K-theory. In the case of $X = \mathbb{P}^n$, the pullback stands for the following thing: Given $f(a,t) \in K_A(\mathbb{P}^n)$, we have

$$i^*f(t,a) = (f(a_0^{-1}, a), f(a_1^{-1}, a), \cdots, f(a_n^{-1}, a)) \tag{4.2}$$

The pushforward gives as that:

$$i_*(1, 0, \cdots, 0) = \prod_{i>0}(1 - a_i^{-1}t^{-1}) \tag{4.3}$$

Thus by computation we have that:

$$i^*i_*(1, 0, \cdots, 0) = (\prod_{i>0}(1 - a_i^{-1}a_0), 0, \cdots, 0) \tag{4.4}$$

which is the wedge product of the normal bundle $\wedge^* N_{X/X^A}$ that corresponds to the Thom isomorphism. This is the equivariant localisation for $\mathbb{P}^n$.

The equivariant localisation for the equivariant K-theory is the following:

Theorem 4.1. *The support of the cokernel of the map $i_* : K_A(X^A) \to K_A(X)$ lies in the following hyperplane of A^*:*

$$\bigcup_{\mu}\{t^\mu = 1\} \tag{4.5}$$

Moreover it is an isomorphism after localisation on the hyperplane in 4.5.

The proof of the equivariant localisation of K-theory follows the similar philosophy. Take X with the torus action A, and consider the fixed point set X^A of X. This will give us the exact sequence:

$$K_A(\mathrm{Thom}(X^A \hookrightarrow X)) \xrightarrow{i_*} K_A(X) \longrightarrow K_A(X \backslash X^A) \longrightarrow \cdots \tag{4.6}$$

The i_* is injective after localisation since $i^* i_*$ is the multiplication by $\wedge^* N_{X/X^A}$. The K-theory $K_A(X \backslash X^A)$ is built from the torus action from A/A' where $A' \neq A$. We can think of the subtorus $A' \subset A$ as the torus defined by the equation $t^\mu = 1$ for some weight μ. Since $K_A(A/A') = K_{A'}(pt)$, this implies that $K_A(X \backslash X^A)$ is the torsion quotient of $K_A(X)$.

Given a vector bundle $V \in K_A(X)$ and the map $p : K_A(X) \to K_A(pt)$, we can use the equivariant localisation to express $\chi(X, V)$. Note that via $i_* : K_A(X^A) \to K_A(X)$, we can express V via $\frac{i^* V}{\wedge^* N}$, thus we have that:

$$\chi(X, V) = \chi(X^A, \frac{i^* V}{\wedge^* N}) \tag{4.7}$$

4.1. **Krichever genus.** Given that X is a smooth manifold, we assume that there exists $N \geq 0$ such that $TX \oplus \mathbb{R}^N$ has complex structure. Krichever studied the theta class $\Theta(TX)$ corresponding to the tangent bundle TX. Let $z \in U(1)$ acting on the fibre of TX, and we can write down $\Theta(TX \otimes z)$ as the equivariant version of the theta class of TX. In the equivariant K-theory, $\Theta(V) = \sum_l (-1)^l \wedge^l V = \prod_i (1 - x_i)$ corresponds to the alternating sum of the wedge product of V. In the elliptic cohomology theory, $\Theta(V) = \prod_i \theta(x_i)$ corresponds to the product of theta functions of the Chern roots of V.

The Krichever genus of X is defined as:

$$\mathrm{Kr}(X) = (X \to pt)_* \Theta(TX \otimes z) = \chi(X, \Theta(TX \otimes z)) \tag{4.8}$$

The Krichever rigidity says that 4.8 does not depend on the group action $U(1)$ on X.

In K-theory, the theta class is represented as:

$$\Theta(TX \otimes z) = \sum_i (-z^{-1})^i \wedge^i T^* X \tag{4.9}$$

This gives that

$$\chi(\Theta(TX \otimes z)) = \sum_i (-z^{-1})^i \chi(X, \wedge^i T^* X) \tag{4.10}$$

Moreover if X is compact and Kahler, the Hodge theory says that $\sum_i (-z^{-1})^i \chi(X, \wedge^i T^* X) \in H^*(X, \mathbb{C})$.

For example, if $X = \mathbb{P}^1$, $H^*(\mathbb{P}^1, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}[2]$. The $H^0 \cong H^0(\mathcal{O}_{\mathbb{P}^1})$, and $H^2 \cong H^1(T^*_{\mathbb{P}^1})$.

Now let S^1 acts on X, and let a as the coordinate of $\mathbb{C}^*$. Using the localisation formula:

$$\chi(X, \Theta(TX \otimes z)) = \chi(X^A, \frac{i^* \Theta(TX \otimes z)}{\Theta(N_{X/X^A})}) = \chi(X^A, \Theta(TX^A \otimes z) \frac{\Theta(N_{X/X^A} \otimes z)}{\Theta(N_{X/X^A})}) \tag{4.11}$$

Note that S^1 does not act on $\Theta(TX^A \otimes z)$, and it only acts on $\Theta(N \otimes z)/\Theta(N)$, and we can write down $\Theta(N \otimes z)/\Theta(N)$ as:

$$\prod_i \frac{1 - w_i^{-1} a^{n_i} z}{1 - w_i^{-1} a^{n_i}} \tag{4.12}$$

Given the S^1 action on TX, this action induces the action on the cohomology group $H^i(X, TX)$. It makes $H^i(X, TX)$ the finite-dimensional representation of S^1. In this case, note that $\chi(X, \Theta(TX \otimes z))$ represents the character of a finite dimensional representation of S^1. This means that the character $\chi(X, \Theta(TX \otimes z))$ is the Laurent polynomial of a. If we take the limit $a \to 0, \infty$, for each term in 4.12:

$$\lim_{a \to 0} \frac{1 - w^{-1} a^n z}{1 - w^{-1} a^n} = \begin{cases} 1 & \text{If } n > 0 \\ \frac{1 - w^{-1} z}{1 - w^{-1}} & \text{If } n = 0 \\ z & \text{If } n < 0 \end{cases} \tag{4.13}$$

Thus it is a constant over the variable a.

Theorem 4.2. *(Krichever,1990) Suppose that $c_1(TX) = 0$ in $H^2(X, \mathbb{Z})$ or $N | c_1(TX)$ and $z^N = 1$, and X compact stably almost complex. Then $Kr(X)$ is a function of z only.*

Here we list the proof when $c_1(TX) = 0$, for the proof of the latter case, this will be an exercise.

Consider the following diagram:

$$\begin{array}{ccc} TX & & \text{Taut} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\gamma} & BU(n) \end{array} \tag{4.14}$$

Passing to the equivariant elliptic cohomology, we have the theta class $\Theta(\text{Taut})$ over $\text{Ell}_{U(n)}(pt) \cong S^n E$, and we have $\Theta(TX) = \gamma^* \Theta(\text{Taut})$. Moreover if we consider that the map $X \to BU(n)$ factors through $X \to BSU(n)$, note that we have $\text{Ell}_{SU(n)}(pt) = \{\text{n-tuples that sum to } 0\} \subset \text{Sym}^n E$. This is isomorphic to the projective space $\mathbb{P}^{n-1}$. We have the map:

$$\mathbb{P}^{n-1} \hookrightarrow \text{Sym}^n E \to E \to 0 \tag{4.15}$$

If we identify $\text{Sym}^n E$ as the space of divisors of degree n in E, this means that $\Theta(TX \otimes z)$ is a line bundle over $\mathbb{P}^{n-1}$. Since the line bundle $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(n)$, which is discrete, this implies that $\Theta(TX \otimes z)$ is a section of the same bundle as $\Theta(TX)$. This implies that $\chi(X, \Theta(TX \otimes z))$ is a section of trivial bundle, and hence constant.

4.2. **Krichever's genus for the instanton moduli space.** One example that is often used is the Krichever's genus for the instanton moduli space:

(4.16)
$$M(r,n) := \{\text{Torsion-free coherent sheaves on } \mathbb{P}^2 \text{ trivialised over the line } H \cong \mathbb{P}^1 \subset \mathbb{P}^2 \text{ of rank } r \text{ and second Chern class } = n\}$$

It has the natural $\mathbb{C}^*_{t_1} \times \mathbb{C}^*_{t_2}$ action induced by the action on $\mathbb{P}^2$, and the natural GL_r-action induced by the GL_r-action on the framing of the rank r torsion-free coherents sheaves.

The fixed points of the torus action are in one-to-one correspondence with the r-tuples of partitions:

$$M(r,n)^{A\times\mathbb{C}^*_{t_1}\times\mathbb{C}^*_{t_2}} = \{\text{r-tuples of partitions } \boldsymbol{\lambda} \text{ such that } |\boldsymbol{\lambda}| = n\} \tag{4.17}$$

We consider the following generating function:

$$\Psi(z) = \sum_{n\geq 0} z^n \mathrm{Kr}(M(n,r)) = \sum_{n\geq 0} z^n \chi(M(r,n), \Theta(TM(n,r)\otimes t_3)) \tag{4.18}$$

We consider the theta class in K-theory, using the localisation theory, we have that:

$$\begin{aligned}\Psi(z) &= \sum_{n\geq 0} z^n \chi(M(r,n)^{A\times\mathbb{C}^*_{t_1}\times\mathbb{C}^*_{t_2}}, \frac{\wedge^*(TM(n,r)\otimes t_3)}{\wedge^*(TM(n,r))}) \\ &= \sum_{n\geq 0} z^n \sum_{\boldsymbol{\lambda}\in r\text{-tuples of partitions}, |\boldsymbol{\lambda}|=n} \frac{\wedge^*(T_{\boldsymbol{\lambda}}M(n,r)\otimes t_3)}{\wedge^*(T_{\boldsymbol{\lambda}}M(n,r))}\end{aligned} \tag{4.19}$$

In the case of the equivariant elliptic cohomology, we can use the definition in the exercise 3.4. In this case, we can check that the generating function can be written as

$$\sum_n z^n \sum_{\boldsymbol{\lambda}\in r\text{-tuples of partitions}} \frac{\Theta(T_{\boldsymbol{\lambda}}M(n,r)\otimes t_3)}{\Theta(T_{\boldsymbol{\lambda}}M(n,r))} \tag{4.20}$$

This coincides with the Nikita's result on the instanton partition function.

5. Lecture Five

5.1. **Box counting and boundary conditions.** The geometry we are going to consider today is mainly on the Hilbert scheme of points over $\mathbb{C}^2$:

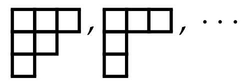
$$\mathrm{Hilb}_n(\mathbb{C}^2) := \{I \text{ the ideal of } \mathbb{C}[x,y] \text{ such that } \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n\} \tag{5.1}$$

The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ has a natural torus action $\mathbb{C}^*_{t_1}\times\mathbb{C}^*_{t_2}$ induced by the torus action on $\mathbb{C}^2$. The classical results show that:

$$\mathrm{Hilb}_n(\mathbb{C}^2)^{\mathbb{C}^*_{t_1}\times\mathbb{C}^*_{t_2}} = \{\lambda \text{ the partition of } n\} \tag{5.2}$$

The partition can be depicted by the Young diagram:

(5.3)



Another Hilbert scheme that we consider is the Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^3, \text{curves})$ of curves in $\mathbb{C}^3$:

(5.4)

$$\mathrm{Hilb}(\mathbb{C}^3, \text{curves}) : \{I \text{ the ideal of } \mathbb{C}[x, y, z] \text{ such that } \mathrm{Spec}(\mathbb{C}[x, y, z]/I) \text{ is an algebraic curve}\}$$

The components $\mathrm{Hilb}(\mathbb{C}^3, [C], \chi)$ of $\mathrm{Hilb}(\mathbb{C}^3, \text{curves})$ is determined by the degree and the arithmetic genus of C:

$$([C], \chi(\mathscr{O}_C)) \in H_2(\mathbb{C}^3, \mathbb{Z}) \oplus H_0(\mathbb{C}^3, \mathbb{Z}) \cong \mathbb{Z} \tag{5.5}$$

Thus its components can be written as $\mathrm{Hilb}_n(\mathbb{C}^3) := \mathrm{Hilb}(\mathbb{C}^3, 0, n)$.

The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^3)$ has a natural $\mathbb{C}^*_{t_1} \times \mathbb{C}^*_{t_2} \times \mathbb{C}^*_{t_3}$ action induced by the torus action on $\mathbb{C}^3$. The torus fixed points are in one-one correspondence with the plane partitions.

We can proceed to the plane partitions, it means that we line all the partitions into a three-dimensional space. The plane partition would be like a sequence of partitions $(\lambda_1, \lambda_2, \cdots)$. These plane partitions are in one-one correspondence with the ideal in $\mathbb{C}[x_1, x_2, x_3]$ with some specific length conditions. This means that each plane partition is a fixed point in the Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^3, \text{curves})$ under the torus action $\mathbb{C}^*_{\epsilon_1} \times \mathbb{C}^*_{\epsilon_2} \times \mathbb{C}^*_{\epsilon_3}$.

Now we fix a divisor $D \subset \mathbb{C}^3$ determined by the equation f in $\mathbb{C}^3$ and the curve class C, there is an open locus

$$\mathrm{Hilb}_n(\mathbb{C}^3)^o \subset \mathrm{Hilb}_n(\mathbb{C}^3) \tag{5.6}$$

consisting of the ideal I such that the multiplication map $f\cdot : \mathscr{O}_C \to \mathscr{O}_C$ is injective. This means that the intersection $C \cap D$ is transverse, i.e. $\dim_{\mathbb{C}}(C \cap D) = 0$. In this way we have a natural map given by

$$\cap D : \mathrm{Hilb}_n(\mathbb{C}^3)^o \to \mathrm{Hilb}_{\#C\cap D}(D) \qquad I \mapsto I \cap (f) \tag{5.7}$$

The map 5.7 admits a resolution by the following diagram:

$$\begin{array}{ccccc} & & \mathrm{Hilb}_n(\mathbb{C}^3/D) & & \\ & \nearrow & & \searrow^{\cap_{rel} D} & \\ \mathrm{Hilb}_n(\mathbb{C}^3)^o & & \xrightarrow{\quad \cap D \quad} & & \mathrm{Hilb}_{\#C\cap D}(D) \end{array} \tag{5.8}$$

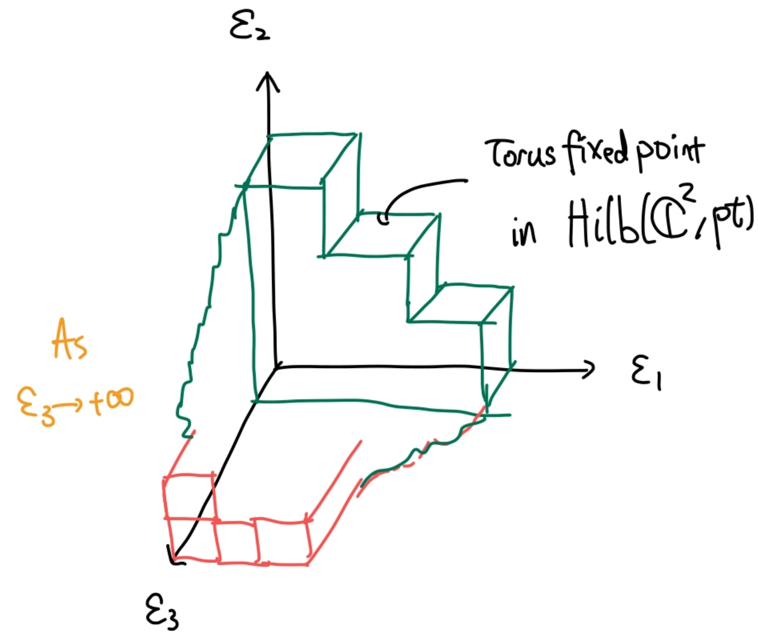


FIGURE 5. Plane partitions, i.e. torus fixed points of $\mathrm{Hilb}(\mathbb{C}^3, \text{curves})$

The divisor $D \subset \mathbb{C}^3$ is called the boundary condition. The map $\cap D$ in 5.7 is called the nonsingular boundary condition. The map $\cap_{rel} D$ in 5.8 is called the relative boundary condition.

Now we consider the open locus:

$$\mathrm{Hilb}_{n,D_1}(\mathbb{C}^3/D_2) := \mathrm{Hilb}_n(\mathbb{C}^3)^{o,D_1} \cap \mathrm{Hilb}_n(\mathbb{C}^3/D_2) \tag{5.9}$$

This Hilbert scheme can be interpreted as the moduli space of curves in $\mathbb{C}^3$ with nonsingular boundary condition over D_1 and relative boundary condition over D_2. It admits the following map:

$$(- \cap D_1) \times (- \cap_{rel} D_2) : \mathrm{Hilb}_{n,D_1}(\mathbb{C}^3/D_2) \to \mathrm{Hilb}_{\#C\cap D_1}(D_1) \times \mathrm{Hilb}_{\#C\cap D_2}(D_2) \tag{5.10}$$

We can use this map to define the generating function of the K-theoretic DT counting with the boundary condition D_1, D_2:

(5.11)

$$\Psi(z, D_1, D_2) = ((- \cap D_1) \times (- \cap_{rel} D_2))_* \sum_{n \geq 0} z^n \hat{\mathcal{O}}_{\mathrm{Hilb}_n(\mathbb{C}^3),\mathrm{vir}} \in K_T(\mathrm{Hilb}(D_1) \times \mathrm{Hilb}(D_2))_{loc}[[z]]$$

This pushforward map is well-defiend after doing the equivariant localisation. For simplicity we consider the divisor D_1 $x = 0$ in $\mathbb{C}^3$. This corresponds to the yz-plane $\mathbb{C}^2$ in $\mathbb{C}^3$. In this case $\Psi(z, D_1, D_2)$ is an element in $K_T(\mathrm{Hilb}_n(\mathbb{C}^2))^{\otimes 2}_{loc}[[z]]$. The boundary condition D_1 looks like we have a plane partition in three-dimensional space such that its boundary at $x = 0$ looks like a partition. The boundary condition D_2 looks like the plane partition has the asymptotic boundary as a partition when x goes to infinity. Thus it is a matrix with each component denoted by two partitions.

One of the generalisation of the Hilbert scheme is the instanton moduli space. Precisely speaking:

(5.12)
$$\begin{aligned} M(r,n) :=&\{\text{Torsion-free coherent sheaves on } \mathbb{P}^2 \text{ trivialised over the line } H \cong \mathbb{P}^1 \subset \mathbb{P}^2 \\ &\text{of rank } r \text{ and second Chern class} = n\} \end{aligned}$$

It has the natural $\mathbb{C}^*_{t_1} \times \mathbb{C}^*_{t_2}$ action induced by the action on $\mathbb{P}^2$, and the natural GL_r-action induced by the GL_r-action on the framing of the rank r torsion-free coherents sheaves.

Take $A \subset GL_r$ as the maximal torus of GL_r, if we choose the cocharacter $\sigma : \mathbb{C}^* \to A$ such that it gives the action on the framing as $a_1 r_1 + \cdots + a_k r_k$, the fixed point of the action is given by:

$$M(r,n)^\sigma = \bigsqcup_{n_1+\cdots+n_k=n} M(r_1,n_1) \times \cdots \times M(r_k,n_k) \tag{5.13}$$

When $r = 1$, $M(1,n)$ is isomorphic to the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{C}^2)$.

Furthermore, take the generic cocharacter $\sigma : \mathbb{C}^* \to A$ with the torus action from $\mathbb{C}^*_{t_1} \times \mathbb{C}^*_{t_2}$, we have the fixed point as:

$$M(r,n)^{A\times\mathbb{C}^*_{t_1}\times\mathbb{C}^*_{t_2}} = \{\text{r-tuples of partitions } \boldsymbol{\lambda} \text{ such that } |\boldsymbol{\lambda}| = n\} \tag{5.14}$$

Denote $M(r) = \sqcup_n M(n,r)$. We have $K_T(M(r))_{loc} \cong K_T(M(1))^{\otimes r}_{loc}$ such that the basis of $K_T(M(r))_{loc}$ can be denoted by the fixed point basis $|\boldsymbol{\lambda}\rangle$.

Similarly, one can also define the generating function $\Psi(z, D_1, D_2)$ as in 5.11 for the localised equivariant K-theory of the instanton moduli space. In this case $\Psi(z, D_1, D_2) = \Psi(z, t_1, t_2, t_3, a_1, \cdots, a_r)$ is a function with the equivariant variables $a_1, \cdots, a_r$.

One of the key fact is that the generating function is a solution of the qKZ equation for the quantum toroidal algebra $U_{t_1,t_2}(\hat{\hat{\mathfrak{gl}}}_1)$ and the corresponding dynamical equation in the variable z:

$$\Psi(qa_1, a_2) = (z^{deg} \otimes 1)R(\frac{a_1}{a_2})\Psi(a_1,a_2) \in \mathrm{End}(V_1(a_1) \otimes V_2(a_2)) \tag{5.15}$$

Here $V_1(a_1), V_2(a_2)$ are both $K_T(M(1))_{loc}$, and the quantum toroidal algebra $U_{t_1,t_2}(\hat{\hat{\mathfrak{gl}}}_1)$ acts on $K_T(M(1))_{loc}$ as the Fock space module.

The computation and construction of Ψ always require the rigidity of the characteristic class, and this rigidity is there in K-theory. Recall that Krichever rigidity requires $c_1(TX) = 0$ or $N | c_1(TX)$ or $(\kappa)^N = 1$.

The tangent bundle of the instanton moduli space can be thought as the following:

$$T\text{Moduli} = \text{Deformation} - \kappa(\text{Deformation})^* \tag{5.16}$$

Here κ is some equivariant parametre $t_1 t_2 t_3$, which is the canonical class of $\mathbb{C}^3$. In other words, this means that:

$$c_1(T\text{Moduli}) = 2c_1(\text{Deformations}) \neq 0 \tag{5.17}$$

Thus it is not zero but divisible by two. In this way we have that $\kappa^2 = 1$, thus this satisfies the rigidity condition.

For the case of $\text{Hilb}_n(\mathbb{C}^2, pt)$, this is a symplectic variety and thus $c_1(TX) = 0$.

5.2. **Construction of R-matrices.** The construction of the qKZ equation needs the R-matrix of the modules for $U_{t_1,t_2}(\hat{\hat{\mathfrak{gl}}}_1)$. The R-matrix is an automorphism map:

$$R(\frac{a_1}{a_2}) : V_1(a_1) \otimes V_2(a_2) \to V_1(a_1) \otimes V_2(a_2) \tag{5.18}$$

satisfying the quantum Yang-Baxter equation with spectral parametres:

$$R_{13}(a_1/a_3)R_{23}(a_2/a_3)R_{12}(a_1/a_2) = R_{12}(a_1/a_2)R_{23}(a_2/a_3)R_{13}(a_1/a_3) \tag{5.19}$$

In the category of representations for the quantum groups, the R-matrix represents the braiding between two different modules. It can be seen in the following diagram:

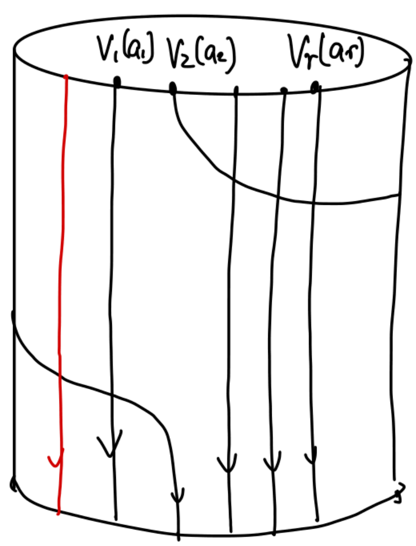


FIGURE 6. Braiding of modules

The quantum Yang-Baxter equation can also be interpreted as the following diagram:

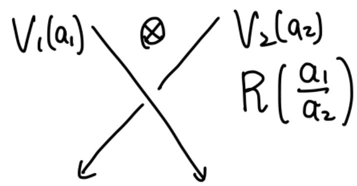


FIGURE 7. R-matrix as braidings of two modules

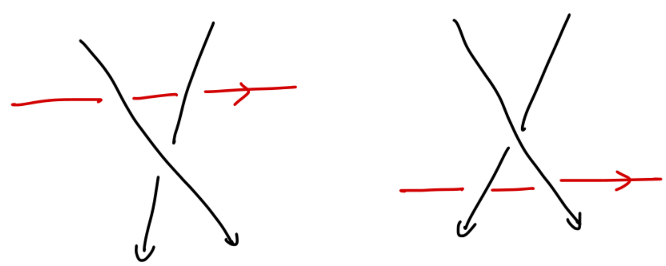


FIGURE 8. YBE as three-particle scattering

It also should commute with the element Z acting on as $z^{\deg}$ in the Cartan subalgebra of $U_{t_1,t_2}(\hat{\hat{\mathfrak{gl}}}_1)$ such that:

$$[Z \otimes Z, R] = 0 \tag{5.20}$$

In the two-dimensional quantum integrable field theory, R-matrix stands for the scattering of two particles collide together at some time from the past to the future. Graphically:

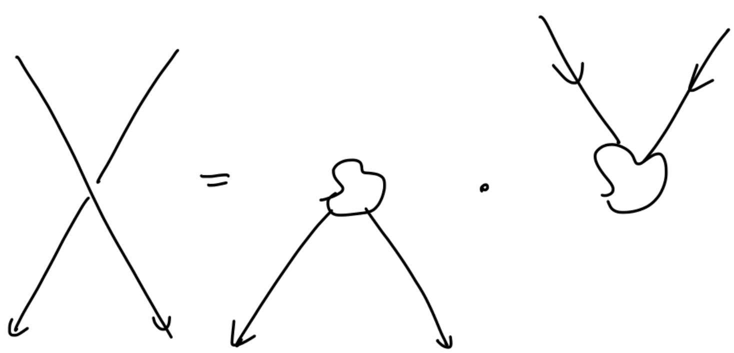


FIGURE 9. Factorisation of Scattering: Two particles $\to$ Collision $\to$ Two particles

The Fock space $\text{Fock}(a)$ is the localised equivariant K-theory of the Hilbert scheme of points $K_T(\text{Hilb}(\mathbb{C}^2, pts))$. The tensor product of the Fock space $\text{Fock}(a_1) \otimes \cdots \otimes \text{Fock}(a_r)$

is isomorphic to the localised equivariant K-theory of instanton moduli space $K_T(M(r))$. Physically speaking, we can think about $K_T(M(r))$ as the state space of the particle.

So in the scattering process, we take the state space of "Two particles" as $K_T(M(r_1)) \otimes K_T(M(r_2))$. Now what is the "Collison"?

Now we consider the embedding of instanton moduli space:

$$M(r_1) \times M(r_2) \hookrightarrow M(r_1 + r_2) \tag{5.21}$$

Here $M(r_1) \times M(r_2)$ is the fixed points of $M(r_1 + r_2)$ under the torus action $\sigma : \mathbb{C}^* \to A$ given by $a_1 r_1 + a_2 r_2$. The torus action gives the torus moment map $\mu : M(r_1 + r_2) \to \mathbb{C}$, and the fixed point of the torus action can be thought of as the critical points of the moment map for the torus action. Usually we denote $M(r_1 + r_2)^\sigma = M(r_1) \times M(r_2)$.

The attracting subspace Attr_σ is defined as:

(5.22)
$$\mathrm{Attr}_\sigma := \{(x, y) \in M(r_1 + r_2)^\sigma \times M(r_1 + r_2) | \lim_{t \to 0} \sigma(t) \cdot y = x\} \subset M(r_1 + r_2)^\sigma \times M(r_1 + r_2)$$

It is an affine bundle over the fixed point set $M(r_1 + r_2)^\sigma$. If we denote $M(r_1 + r_2)^\sigma = \bigsqcup_\alpha F_\alpha$, we use the notation $\mathrm{Attr}_\sigma(F_\alpha)$ as the restriction of Attr_σ to F_α.

For the fixed point components one can define the partial order on the fixed point components $\{F_\alpha\}$:

$$F_\beta \geq F_\alpha \Longleftrightarrow F_\alpha \subset \overline{\mathrm{Attr}_\sigma(F_\beta)} \tag{5.23}$$

The K-theoretic stable envelope Stab_σ is a K-theory class

$$\mathrm{Stab}_\sigma \in K_T(X \times X^\sigma) \tag{5.24}$$

such that

- $\mathrm{supp}\, \mathrm{Stab}_\sigma \subset \mathrm{Attr}_\sigma^f$
- $\mathrm{Stab}|_{F_\alpha \times F_\alpha} = (-1)^{\mathrm{rk} N_{>0}^{1/2}} (\frac{\det N_{<0}}{\det N^{1/2}})^{1/2} \mathscr{O}_{\mathrm{Attr}}$

The σ-degree of Stab_σ is determined by the fractional line bundle $\mathscr{L} \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ such that:

$$\deg_\sigma \mathrm{Stab}_{\sigma,\mathscr{L}}|_{F_\beta \times F_\alpha} \otimes \mathscr{L}|_{F_\alpha} \subset \deg_\sigma \mathrm{Stab}_{\sigma,\mathscr{L}}|_{F_\beta \times F_\beta} \otimes \mathscr{L}|_{F_\beta} \tag{5.25}$$

Dear diagonal, $\mathrm{Stab}_{\pm\sigma}$ looks like $i_* p^*$:

$$\begin{array}{ccccc} & & \mathrm{Attr}_\sigma & & \\ & \overset{i}{\nearrow} & & \overset{p}{\searrow} & \\ M(r_1 + r_2) & & & & M(r_1) \times M(r_2) \end{array} \tag{5.26}$$

The stable envelope satisfies the triangle lemma. Namely, consider a subtorus $A' \subset A$ such that $X^{A'} \neq X^A$. A chamber $\mathscr{C}_A \subset \mathrm{Lie}(A)$ determines a pair of chambers

$$\begin{aligned} \mathscr{C}_{A'} &= \mathscr{C}_A \cap \mathrm{Lie}(A') \subset \mathrm{Lie}(A') \\ \mathscr{C}_{A/A'} &= \mathrm{Image}(\mathscr{C}_A) \subset \mathrm{Lie}(A/A') \end{aligned} \tag{5.27}$$

The triangle lemma is the following diagram of maps

$$\begin{array}{ccc} K_T(X^A) & \xrightarrow{\mathrm{Stab}_{\mathscr{C}_A}} & K_T(X) \\ & {\scriptstyle \mathrm{Stab}_{\mathscr{C}_{A/A'}}} \searrow \quad \nearrow {\scriptstyle \mathrm{Stab}_{\mathscr{C}_{A'}}} & \\ & K_T(X^{A'}) & \end{array} \tag{5.28}$$

The R-matrix can be defined as the following:

$$R_\sigma : K_T(M(r_1) \times M(r_2)) \xrightarrow{\mathrm{Stab}_\sigma} K_T(M(r_1 + r_2)) \xrightarrow{\mathrm{Stab}^{-1}_{-\sigma}} K_T(M(r_1) \times M(r_2)) \tag{5.29}$$

The R-matrix R_σ can be checked that it is a rational function over the variable a_1/a_2. Using the triangle lemma we can check that it satisfies the Yang-Baxter equation. Using the FRT formalism, we can construct the quantum affine algebra $U_\hbar(\hat{\mathfrak{g}})$ via the R-matrix.

5.3. **Elliptic Stable Envelopes.** In the elliptic cohomology, the stable envelope map looks like Ell(Fixed locus) $\to$ Ell(Ambient space), but we know that elliptic cohomology is not a ring, instaed it is a scheme. Thus the corresponding map in the elliptic cohomology is the section $\mathrm{Stab}^{Ell}_\sigma$ of some line bundle on Ell(Stable manifold), that is, some Θ-bundle, such that the section is written in terms of the theta functions. These sections can let us know how to extend the fixed locus near the neighborhood of one component of the fixed locus.

To get the feeling about the subtlety, we can first consider $\mathrm{Ell}_A(\mathbb{P}^n)$. By the knowledge in the previous lectures, it is a subscheme of $\mathrm{Ell}_{A\times U(1)}(pt) \cong E^n \times E$ defined by the equation $\prod_i (1 - a_i t) = 0$, and it has a natural projection to $\mathrm{Ell}_A(pt) \cong E^n$. For each line $ta_0 = 1$ comes from $i_* p^*$ as above.

We can see that $\mathrm{Ell}_A(\mathbb{P}^n)$ is a projective variety, and this means that the holomorphic global section of its structure sheaf is trivial. Also the vector bundle over the scheme is not trivial, and in this case it is much more natural for us to consider the meromorphic sections of some sheaf over the projective variety.

Now let Y be an algebraic variety, and $\mathscr{L}$ a line bundle, D is a divisor. If we look at the sections $H^0(Y, \mathscr{L})$ of the line bundle. We can restrict the section to the divisor $H^0(Y, \mathscr{L}) \to H^0(D, \mathscr{L})$, and this will give a exact sequence:

$$H^0(Y, \mathscr{L}(-D)) \to H^0(Y, \mathscr{L}) \to H^0(D, \mathscr{L}) \to H^1(Y, \mathscr{L}(-D)) \tag{5.30}$$

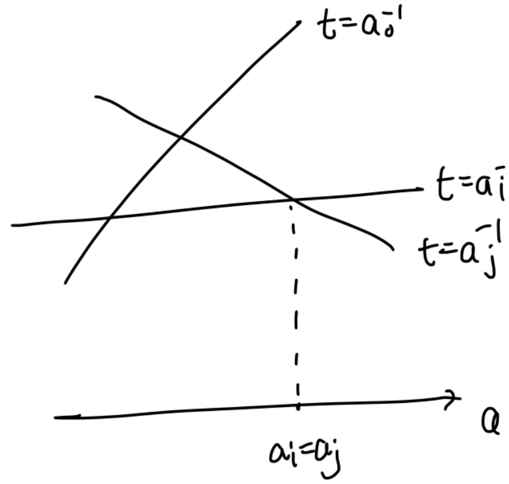


FIGURE 10. Fibration of $\mathrm{Ell}_A(\mathbb{P}^n) \to \mathrm{Ell}_A(pt) \cong E^n$

So if we know a section of $\mathscr{L}$ on the divisor, the splitting of the restriction $H^0(Y, \mathscr{L}) \to H^0(D, \mathscr{L})$ can be given by the interpolation $H^0(D, \mathscr{L}) \to H^0(Y, \mathscr{L})$. Here $H^1(Y, \mathscr{L}(-D))$ will be the obstruction to interpolation. $H^0(Y, \mathscr{L}(-D))$ stands for the ambiguity in the interpolation. So if $H^0(Y, \mathscr{L}(-D)) = H^1(Y, \mathscr{L}(-D)) = 0$, this means that the sections are totally determined by the interpolation in the divisors D.

For us in the lecture, Y will be the abelian variety, $\mathscr{L}$ and D will be some theta bundle Θ. If $\deg(\mathscr{L}(-D)) = 0$, either $\mathscr{L}(-D)$ is the trivial bundle or $H^i(\mathscr{L}(-D)) = 0$. The condition that $\mathscr{L}(-D)$ can be thought of as some resonance condition.

Back to our case, since the map Ell(fixed locus) $\to$ Ell(ambient space) is determined by sections of some line bundle, we further twist the line bundle by some degree zero line bundle in $\mathrm{Pic}_0(\mathrm{Ell}(X)) \cong \mathrm{Pic}(X) \otimes E$. We denote coordinate in $\mathrm{Pic}_0(\mathrm{Ell}(X))$ as z, and this will give the dynamical variable in the elliptic quantum groups. This z is the same as $z^{\deg}$ in the partition function before, and this is usually called the Kahler variables.

Given the symplectic variety X with a Hamiltonian torus T-action. We first fix the polarisation of its tangent bundle:

$$TX = T^{1/2}X + \hbar^{-1}(T^{1/2}X)^\vee, \qquad \hbar \in \mathrm{Char}(T) \tag{5.31}$$

We also define the index bundle by

$$\mathrm{ind} = T^{1/2}(X) - (T^{1/2}X)^\vee \tag{5.32}$$

We define the Kahler torus by:

$$\mathscr{E}_{\mathrm{Pic}(X)} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} E \tag{5.33}$$

The coordinates on this abelian variety is usually called the Kahler variables.

We define the extended equivariant elliptic cohomology by:

$$E_T(X) := \mathrm{Ell}_T(X) \times \mathscr{E}_{\mathrm{Pic}(X)} \tag{5.34}$$

It is a scheme over $\mathscr{B}_{T,X} := \mathrm{Ell}_T(pt) \times \mathscr{E}_{\mathrm{Pic}(X)}$.

Now pick up a line bundle $\mathscr{L} \in \mathrm{Pic}(X)$. By the map c defined in 3.13, it can induce the following map of schemes:

$$\tilde{c} : \mathrm{Ell}_T(X) \to \mathscr{E}^{\vee}_{\mathrm{Pic}(X)}, \qquad \mathscr{E}^{\vee}_{\mathrm{Pic}(X)} := \mathrm{Hom}(\mathrm{Pic}(X), E) \tag{5.35}$$

Here $\mathscr{E}^{\vee}_{\mathrm{Pic}(X)}$ is the dual abelian variety of $\mathscr{E}_{\mathrm{Pic}(X)}$

We have the map $(\tilde{c} \times 1) : \mathrm{Ell}_T(X) \times \mathscr{E}_{\mathrm{Pic}(X)} \to \mathscr{E}^{\vee}_{\mathrm{Pic}(X)} \times \mathscr{E}_{\mathrm{Pic}(X)}$.

On the product of two dual abelian varieties there is a universal line bundle $\mathscr{U}_{\mathrm{Poincaré}'}$. The sections of $\mathscr{U}_{\mathrm{Poincaré}}$ on $E^{\vee} \times E$ is given by

$$\frac{\theta(sz)}{\theta(z)\theta(s)} \tag{5.36}$$

We define

$$\mathscr{U} := (\tilde{c} \times 1)^* \mathscr{U}_{\mathrm{Poincaré}} \tag{5.37}$$

Fix a homomorphism between abelian varieties $g : \mathrm{Ell}_T(pt) \to \mathscr{E}_{\mathrm{Pic}(X)}$, we can associate au automorphism $\tau(g) : \mathscr{B}_{T,X} \to \mathscr{B}_{T,X}$ by

$$(\tau, z) \mapsto (t, z + g(t)), \qquad t \in \mathrm{Ell}_T(pt) \tag{5.38}$$

The following isomorphism can be checked in the exercise:

$$\frac{\tau(g_1)^*\mathscr{U}}{\mathscr{U}} \otimes \frac{\tau(g_2)^*\mathscr{U}}{\mathscr{U}} \cong \frac{\tau(g_1 g_2)^*\mathscr{U}}{\mathscr{U}} \tag{5.39}$$

Moreover, given an line bundle $\mu \in \mathrm{Pic}(X)$, it can be thought of as a map $E \to \mathscr{E}_{\mathrm{Pic}(X)}$. The T-character $\lambda \in \mathrm{Char}(T)$ can also be thought of as a map $\mathrm{Ell}_T(pt) \to E$. In this case we can think of $\mu\lambda$ as a map $\mu\lambda : \mathscr{B}_{T,X} \to \mathscr{B}_{T,X}$.

Now fix a cocharacter $\sigma : \mathbb{C}^* \subset T$ of T and consider the inclusion of the fixed locus

$$i : X^A \to X, \qquad A \subset T \tag{5.40}$$

The elliptic stable envelope is a meromorphic section of a line bundle over $E_T(X \times X^A)$:

(5.41)

$$\mathrm{Stab}_\sigma \in \Gamma(E_T(X \times X^A) \backslash \Delta, (\Theta(T^{1/2}X) \otimes \mathscr{U}) \boxtimes (\Theta(T^{1/2}X^A) \otimes (1 \times i^*)^* \tau(-\hbar \det \mathrm{ind})^* \mathscr{U} \otimes \Theta(\hbar)^{-\mathrm{rk}\,\mathrm{ind}}))$$

It gives a map:

$$\mathrm{Stab}_\sigma : \Theta(T^{1/2}X^A) \otimes (1 \times i^*)^* \tau(-\hbar \det \mathrm{ind})^* \mathscr{U} \otimes \Theta(\hbar)^{-\mathrm{rk}\,\mathrm{ind}} \to \Theta(T^{1/2}X) \otimes \mathscr{U} \tag{5.42}$$

which is supported over Attr^f_σ such that the diagonal looks like:

$$\mathrm{Stab}_{\mathscr{C}}(F_\alpha)|_{F_\alpha} = (-1)^{\mathrm{rk}\,\mathrm{ind}}\theta(N^{<0}_{F_\alpha}) \tag{5.43}$$

The symbol Δ stands for the resonant locus in $\mathscr{B}_{T/A,X} = \mathrm{Ell}_{T/A}(pt) \times \mathscr{E}_{\mathrm{Pic}(X)}$. It contains the information of the poles of the elliptic stable envelopes. It is a closed subspace in $E_T(X \times X^A)$.

5.4. **Integral representation of the generating function.** Back to $\Psi(a_1, \cdots, a_r, t_1, t_2, t_3, z) \in K_T(\mathrm{Hilb}_n(\mathbb{C}^2))^{\otimes 2}[[z]]$. If we treat it as the morphism from $K_T(\mathrm{Hilb}_n(\mathbb{C}^2))$ to $K_T(\mathrm{Hilb}_n(\mathbb{C}^2))$, it turns out that it can be expressed as:

(5.44)
$$\langle\alpha|\Psi(a_1, \cdots, a_r, t_1, t_2, t_3, z)|\beta\rangle = \int_{C\subset\text{Maximal torus}} \prod_i \frac{dx_i}{2\pi i x_i} f_\alpha(x) \cdot g_\beta(x) \cdot \prod \frac{\Gamma_q(\cdots)}{\Gamma_q(\cdots)}$$

Here $\alpha, \beta \in K_T(M(r_1)) \otimes K_T(M(r_2))$. The function $f_\alpha(x) = \mathrm{Stab}_\sigma(\alpha)$ is the stable envelope for the class α, it is a rational function coming from the quantum affine algebra $U_\hbar(\hat{\mathfrak{g}})$ constructed via the R-matrix defined in 5.29. This can be thought of as the off-shell Bethe vectors.

The function $g_\beta(x) = \mathrm{Stab}^{Ell}_\sigma(\beta)$ is the elliptic stable envelope for the class β, it is the elliptic functions come from the sections of some line bundles of the elliptic cohomology. The product of the q-Gamma function $\frac{\Gamma_q(\cdots)}{\Gamma_q(\cdots)}$ is from the Bethe equations.

6. Appendix: Quantum groups

In this appendix we give some basic introduction to the quantum groups.

6.1. **Definition of Hopf algebra.** The Hopf algebra $(A, m, \Delta, \eta, \epsilon)$ is a bialgebra over field k together with a k-linear antipode map $S : A \to A$ such that it satisfies the following properties:

(1) Multiplication m and comultiplication Δ:

$$\begin{array}{ccccc} A\otimes A & \xrightarrow{m} & A & \xrightarrow{\Delta} & A\otimes A \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \uparrow{\scriptstyle m\otimes m} \\ A\otimes A\otimes A\otimes A & & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & & A\otimes A\otimes A\otimes A \end{array} \tag{6.1}$$

(2) Multiplication m and counit ϵ:

$$\begin{tikzcd} A\otimes A \arrow[rr, "m"] \arrow[dr, "\epsilon\otimes\epsilon"'] & & A \arrow[dl, "\epsilon"] \\ & k\otimes k\cong k & \end{tikzcd} \tag{6.2}$$

(3) Comultiplication Δ and unit η:

$$\begin{tikzcd} & k\otimes k\cong k \arrow[dl, "\eta\otimes\eta"'] \arrow[dr, "\eta"] & \\ A\otimes A & & A \arrow[ll, "\Delta"] \end{tikzcd} \tag{6.3}$$

(4) Unit η and counit ϵ:

$$\begin{tikzcd} k \arrow[dd, "\mathrm{id}"] \arrow[dr, "\eta"] & \\ & A \arrow[dl, "\epsilon"] \\ k & \end{tikzcd} \tag{6.4}$$

(5) Coassociativity:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A\otimes A \arrow[d, "\mathrm{id}\otimes\Delta"] \\ A\otimes A \arrow[r, "\Delta\otimes\mathrm{id}"] & A\otimes A\otimes A \end{tikzcd}, \quad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & A\otimes A \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ A\otimes A \arrow[r, "\epsilon\otimes\mathrm{id}"] & k\otimes B\cong B\cong B\otimes k \end{tikzcd} \tag{6.5}$$

(6) Compatibility with the antipode map

$$\begin{tikzcd} & A\otimes A \arrow[r, "S\otimes\mathrm{id}"] & A\otimes A \arrow[dr, "m"] & \\ A \arrow[ur, "\Delta"] \arrow[r, "\epsilon"] \arrow[dr, "\Delta"] & k \arrow[rr, "\eta"] & & A \\ & A\otimes A \arrow[r, "\mathrm{id}\otimes S"] & A\otimes A \arrow[ur, "m"] & \end{tikzcd} \tag{6.6}$$

One basic example is the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. The coproduct is given by $\Delta(x) = x\otimes 1 + 1\otimes x$ for every $x\in\mathfrak{g}$. The counit map is given by $\epsilon(x) = 0$ for all $x\in\mathfrak{g}$, and the antipode map is given by $S(x) = -x$. Note that the coproduct here is cocommutative.

6.2. **Quantum groups and Hopf algebra deformation.** Given a Kac-Moody Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, a loop Lie algebra is the vector space of holomorphic maps $\mathrm{Map}(C \to \mathfrak{g})$ from C is a curve like $\mathbb{C}, \mathbb{C}^*$ and E, to the Lie algebra $\mathfrak{g}$. A quantum loop group is, roughly speaking, a Hopf algebra deformation of the universal enveloping algebra $U(\mathrm{Map}(C \to \mathfrak{g}))$.

The interpretation of the universal enveloping algebra $U(\mathfrak{g})$ can be seen as the following: Consider the Lie group G corresponding to the Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ can be thought of as the left-invariant vector field over the function on G. So it is a linear map $\mathrm{Fun}(G) \to \mathrm{Fun}(G)$. There is a natural pairing $U(\mathfrak{g}) \times \mathrm{Fun}(G) \to \mathbb{C}$ given by $(X, f) \mapsto (X \cdot f)(e)$, and it induces the map $U(\mathfrak{g}) \to \mathrm{Fun}(G)^*$.

Then the algebra of functions $\mathrm{Fun}(G)$ on G is Hopf dual to the universal enveloping algebra $U(\mathfrak{g})$, and in this case $U(\mathfrak{g})$ can be thought of as the space of distributions suppoerted at $1 \in G$.

The algebra $\mathrm{Fun}(G)$ has a natural coproduct structure given by $[\Delta(f)](g_1, g_2) = f(g_1 g_2)$, and the antipode map $(Sf)(g) = f(g^{-1})$. Note that the coproduct is not cocommutative.

Now we focus on the case when $C = \mathbb{C}$, and the corresponding universal enveloping algebra is $U(\mathfrak{g}[t])$. In this case the quantum loop algebra is the Yangian algebra $Y_\hbar(\mathfrak{g})$. The algebra $U(\mathfrak{g}[t])$ admits the natural evaluation map $\mathrm{ev}_a : U(\mathfrak{g}[t]) \to U(\mathfrak{g})$ via $t \mapsto a$. If V is an $U(\mathfrak{g})$-module, it has a natural $U(\mathfrak{g}[t])$-module structure via the evaluation map, and we denote the corresponding module as $V(a)$. The Hopf deformation $Y_\hbar(\mathfrak{g})$ gives an R-matrix as the braiding operation

$$R(a_1 - a_2) : V_1(a_1) \otimes V_2(a_2) \to V_1(a_1) \otimes V_2(a_2) \tag{6.7}$$

such that $R(a_1 - a_2)$ satisfies the Yang-Baxter equation with the spectral parametre.

Now suppose that we have matrices $R_{V_i, V_j}(a)$ solving the Yang-Baxter equation for some collections of modules $V_1, V_2, \cdots$. The R-matrix can be extended to:

$$R_{\otimes_i V_{k_i}, \otimes_j V_{m_j}} : \bigotimes_i V_{k_i} \otimes \bigotimes_j V_{m_j} \to \bigotimes_i V_{k_i} \otimes \bigotimes_j V_{m_j} \tag{6.8}$$

We consider a category in which objects are $\bigoplus \bigotimes V_{k_i}$ with objects being the quotients, and morphisms are all the linear maps which commutes with the R-matrix R. In this way we can define the quantum group to be the linear operators on $\bigoplus \bigotimes V_{k_i}$ that commute with matrix elements of R.

To explain the philosophy. Let us work on the example $\mathfrak{g} = \mathfrak{gl}(n)$. The Lie algebra $\mathfrak{gl}(n)[t]$ is spanned by $E_{ij} t^k$. For the Hopf deformation $Y_\hbar(\mathfrak{gl}(n))$, we take $V_1 = V_2 = \mathbb{C}^n$, and in this case we have

$$R = 1 - \frac{\hbar}{a_1 - a_2}(12) \tag{6.9}$$

We take the basis $e_i, e_j \in \mathbb{C}^n$, it is easy to see that the coefficients of e_j in $R(e_i \otimes v)$ with $v \in V_2$ is equal to an operator $T_{ij}(a_1)$ acting on $V_2(a_2)$. We can do the expansion of $T_{ij}(a)$ over $a = \infty$:

$$T_{ij}(a) = \tilde{E}_{ij} + \frac{\tilde{E}_{ij}^{(1)}}{a} + \frac{\tilde{E}_{ij}^{(2)}}{a^2} + \cdots \tag{6.10}$$

The operator $T_{ij}(a)$ satisfies the RTT equation:

$$R(a_1 - a_2)(T(a_1) \otimes T(a_2)) = (T(a_1) \otimes T(a_2))R(a_1 - a_2) \in Y_\hbar(\mathfrak{gl}_n) \otimes \mathrm{End}(V_1(a_1) \otimes V_2(a_2)) \tag{6.11}$$

In other words, the Yangian $Y_\hbar(\mathfrak{gl}_n)$ is generated by the matrix coefficients of $T(a)$. It has the coproduct and counit given by:

$$\Delta(T(a)) = T(a) \otimes T(a), \qquad \epsilon(T(a)) = 1 \tag{6.12}$$

The RTT equation implies that the generators $T_{ij}(a)$ can be realised by the representations of the Yangian.

We consider the following representations of the Yangian:

$$\pi \otimes \cdots \otimes \pi : Y_\hbar(\mathfrak{gl}_n) \to \mathrm{End}(V(a_1)) \otimes \cdots \otimes \mathrm{End}(V(a_N)), \qquad V \cong \mathbb{C}^n \tag{6.13}$$

such that each V is the fundamental representation. The representation is given by:

$$T(a) \mapsto R_{01}(a - a_1) \cdots R_{0N}(a - a_N) \in \mathrm{End}(V(a)) \otimes \mathrm{End}(V(a_1)) \otimes \cdots \otimes \mathrm{End}(V(a_N)) \tag{6.14}$$

Denote $V_N = V(a_1) \otimes \cdots \otimes V(a_N)$, the above map gives us an algebra morphism:

$$Y_\hbar(\mathfrak{gl}_n) \to \prod_N \mathrm{End}(V_N) \tag{6.15}$$

One can check that this map is an injective map. This explains the philosophy that the quantum group to be the linear operators on $\bigoplus \bigotimes V_{k_i}$ that commute with matrix elements of R