Hitchin systems and their quantization

Pavel Etingof and Henry Liu

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Abstract

This is an expanded version of the notes by the second author of the lectures on Hitchin systems and their quantization given by the first author at the Beijing Summer Workshop in Mathematics and Mathematical Physics "Integrable Systems and Algebraic Geometry" (BIMSA-2024).

1 Introduction

In 1987, Nigel Hitchin introduced a finite dimensional complex integrable system attached to a reductive group G and a smooth projective curve X over $\mathbb{C}$, which is now called the *Hitchin integrable system*. This system lives on (a partial compactification of) the cotangent bundle $T^*\mathrm{Bun}_G^\circ(X)$ to the moduli space $\mathrm{Bun}_G^\circ(X)$ of stable principal G-bundles on X, and admits a direct generalization to the ramified case, when X has marked points with various kinds of level structure at them. In this generalized form, Hitchin systems (even restricted to genus 0 and 1) and their degenerations subsume (upon q-deformation) most of the known finite dimensional integrable systems. Also, Beilinson and Drinfeld showed in [BD1] that Hitchin systems admit a natural quantization, which is a crucial ingredient in their proposed *geometric Langlands correspondence*, finally proved this year for all G (in the stronger categorical form) in [GR, GeL]. Finally, quantum Hitchin systems play a central role in the recently introduced *analytic Langlands correspondence* ([Te, Fr1, EFK1, EFK2, EFK3, EFK4]) and the interpretation of the geometric and analytic Langlands correspondence in supersymmetric 4-dimensional quantum gauge theory ([KW, GW]). This puts Hitchin systems in the center of attention in several areas of mathematics and mathematical physics.

The goal of this paper is to give an accessible introduction to Hitchin systems and their quantization. It is an expanded version of the notes by the second author of the lectures given by the first author at the Beijing Summer Workshop in Mathematics and Mathematical Physics "Integrable Systems and Algebraic Geometry" (BIMSA-2024) dedicated to the memory of Igor Krichever, who did foundational work on Hitchin systems and stressed their importance over several decades.

We begin by recalling the notion of a principal bundle in algebraic geometry (Section 2), and then we discuss the moduli stack $\mathrm{Bun}_G(X)$, where X is a smooth irreducible projective curve and G a split reductive group[1] over some field $\mathbf{k}$ (Section 3). Then we proceed to the construction of $\mathrm{Bun}_G(X)$ as a double quotient of the loop group (Section 4). In Section 5, we introduce a smooth subvariety $\mathrm{Bun}_G^\circ(X) \subset \mathrm{Bun}_G(X)$ of stable bundles and consider its cotangent bundle, which is the space of Higgs pairs. In

[1] The basic examples of G are GL_n, SL_n, and PGL_n, and also the classical groups (symplectic and orthogonal). A particularly important case is the multiplicative group GL_1, and split tori, which are products of several copies of GL_1.

Section 6, we introduce the classical Hitchin system and prove Hitchin's theorem on its integrability (the proof of completeness is given only for type A). This leads us to the important notion of a *spectral curve*. In Section 7, we generalize the Hitchin system to the (tamely) ramified case, and show how some known integrable systems, such as the (twisted) Garnier system and elliptic Calogero-Moser system, arise as examples. Finally, in Section 8 we discuss quantization of Hitchin systems, which is based on the representation theory of affine Lie algebras.

For pedagogical purposes the paper contains problems (supplied with solutions) and exercises (without solutions). Solutions of problems are given in Section 9.

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2 Principal G-bundles

2.1 Definition of a principal G-bundle

Let $\mathbf{k}$ be a field. The basic case for us will be $\mathbf{k} = \mathbb{C}$. Let X be an algebraic variety and G be an affine algebraic group, both over $\mathbf{k}$.

Definition 2.1. A *principal G-bundle* on X, also called a *G-torsor* over X, is an algebraic variety P over $\mathbf{k}$ equipped with a regular map $\pi\colon P \to X$ and a right action $a\colon P \times G \to P$ of G preserving π, which (étale) locally on X is isomorphic to the projection $G \times X \to X$ with the G-action by right multiplication on the G-factor.

In other words, there exists an (étale) open cover $\{U_i\}_{i\in I}$ of X such that for all $i \in I$, there is a G-invariant isomorphism $P|_{U_i} \cong G \times U_i$ under which π goes to the projection $G \times U_i \to U_i$.[2]

The same definition can be made more generally when X (and hence P) is a scheme.

Let $P_x \coloneqq \pi^{-1}(x)$ be the fiber of π at a point $x \in X$. This is a principal homogeneous space for G. Thus a principal G-bundle on X can be thought of as a family of principal homogeneous G-spaces parametrized by X.

[2]An étale chart is more general than a Zariski chart: it is a Zariski open set $\overline{U} \subset X$ along with a finite étale morphism (cover) $U \to \overline{U}$. We will not focus on this too much, because one can work with étale charts in much the same way as with Zariski charts. Namely, by definition, intersections $U_i \cap U_j$ are given by fiber products $U_i \underset{X}{\times} U_j$, and $P|_{U_i} \coloneqq P \underset{X}{\times} U_i$. Naively we can just pretend that the étale charts U_i are ordinary Zariski charts.

2.2 Clutching functions

It follows that a principal G-bundle can be realized concretely by *clutching* (or *transition*) *functions*, namely, regular functions $g_{ij}\colon U_i \cap U_j \to G$ defined when $U_i \cap U_j \neq \emptyset$ such that:

1. $g_{ii} = \mathrm{id}$, $g_{ij} \circ g_{ji} = \mathrm{id}$;
2. the 1-*cocycle condition*: $g_{ij} \circ g_{jk} \circ g_{ki} = \mathrm{id}$ on the triple intersection $U_i \cap U_j \cap U_k$ (when it is non-empty).

This means that the $\{g_{ij}\}$ form a *Čech* 1-*cocycle* on X with coefficients in the sheaf $\mathcal{O}_{X,G}$ of G-valued regular functions on X. The idea is that we glue trivial bundles on each U_i using these clutching functions.[3]

In fact, this data classifies principal G-bundles modulo transformations

$$g_{ij} \mapsto h_i \circ g_{ij} \circ h_j^{-1}$$

for regular functions $h_i\colon U_i \to G$. In other words, cohomologous cocycles define isomorphic bundles, and vice versa. Hence G-bundles on X up to isomorphism are classified by the *first étale cohomology* $H^1_{\text{ét}}(X, \mathcal{O}_{X,G})$.

Note that this cohomology is in general just a set (not a group); it only has a natural group structure if G is abelian (in which case the product is defined by multiplying clutching functions).

2.3 Vector bundles

Definition 2.2. A *vector bundle* of rank n on a scheme X over $\mathbf{k}$ is a scheme P over $\mathbf{k}$ equipped with a regular map $\pi\colon P \to X$ with operations of fiberwise addition $P \underset{X}{\times} P \to P$ and scalar multiplication $\mathbf{k} \times P \to P$ which (étale) locally on X is isomorphic to the projection $\mathbf{k}^n \times X \to X$ with the usual fiberwise addition and scalar multiplication. In this case, for $x \in X$, the n-dimensional vector space $\pi^{-1}(x)$ is called the *fiber* of π at x. A vector bundle of rank 1 is called a *line bundle*.

In other words, a vector bundle on X of rank n is defined by an atlas of étale charts $\{U_i\}$ on X with transition functions $g_{ij}\colon U_i \times U_j \to GL_n$ satisfying the 1-cocycle condition. Thus the notion of a vector bundle of rank n is equivalent to the notion of a principal GL_n-bundle.

It is easy to see that a vector bundle E on X is determined by its *sheaf of sections* $U \mapsto \Gamma(U, E)$ for étale charts U on X, and if U is affine then $\Gamma(U, E)$ is a locally

[3]More precisely, for étale covers, gluing is slightly more complicated than for usual open covers, and one must use a procedure called *faithfully flat descent* ([O], Chapter 4).

free (i.e., projective) $\mathbf{k}[U]$-module[4] of rank r. Moreover, as we will see below, we may consider the usual Zariski charts intead of étale ones. Thus vector bundles of rank n on X are the same as locally free sheaves of the same rank. In particular, if X is affine then a vector bundle on X of rank n is the same thing as a projective $\mathbf{k}[X]$-module of the same rank.

Finally, recall that on vector bundles over any scheme there are natural operations of taking direct sum, tensor product, and dual bundle $E \mapsto E^\vee$, which are defined by the corresponding operations on the fibers (or clutching functions). For example, $g_{E^\vee,ij} = (g_{E,ij}^{-1})^*$.

2.4 Associated bundles

Let Y be an algebraic variety with a left action of G. Then any principal G-bundle E on X gives rise to the *associated bundle* $E \times_G Y$ on X with fiber Y, with the same clutching functions as E but viewed as acting on Y.

For instance, if $G = \mathrm{GL}_n$ and Y is the vector representation of G, then $E \times_G Y$ is the *vector bundle associated to the principal G-bundle E*. As noted above, this defines a one-to-one correspondence between principal GL_n-bundles and vector bundles of rank n on X.

Similarly, SL_n-bundles bijectively correspond to vector bundles of rank n with trivial determinant, and PGL_n-bundles bijectively correspond to vector bundles of rank n up to tensoring with line bundles. More generally, given a finite-dimensional rational representation

$$\rho\colon G \to \mathrm{GL}(V),$$

every principal G-bundle E on X with clutching functions g_{ij} gives rise to the associated vector bundle $E_\rho = E \times_G V$ of rank n, whose clutching functions are $\rho(g_{ij})$.

2.5 Induced bundles

Let H, G be affine algebraic groups over $\mathbf{k}$, E be a principal H-bundle on X and $\varphi\colon H \to G$ be a homomorphism. Then the associated bundle $F := E \times_H G$ is a principal G-bundle on X called the *induced G-bundle* from E via φ. Reciprocally, if F is a principal G-bundle on X then an *H-structure* on F is an H-bundle E on X with an isomorphism $F \cong E \times_H G$.

Similarly, we can talk about an H-structure on F over a (locally closed) subscheme $X' \subset X$. For example, if $H \subset G$ then an H-structure on F over a point $x \in X$ is just an H-orbit in the fiber F_x. Note that choices of an H-structure on F at X are parametrized by G/H, and that H matters only up to conjugation, because right-multiplication by a group element $g \in G$ transforms an H-orbit into a gHg^{-1}-orbit.

[4] For an affine scheme U over a field $\mathbf{k}$, the ring of regular functions on U will be denoted by $\mathbf{k}[U]$ or $\mathcal{O}(U)$.

2.6 Classification of line bundles

Let $G = \mathrm{GL}_1$ and assume for simplicity that $\mathbf{k}$ is algebraically closed. Then

$$H^1_{\text{ét}}(X, \mathcal{O}_{X,G}) = \mathrm{Pic}(X)$$

(or, more precisely, $\mathrm{Pic}(X)(\mathbf{k})$), the Picard group of X, which classifies line bundles on X. In particular, if X is a smooth irreducible projective curve over $\mathbf{k}$ then there is a (non-canonically split) short exact sequence

$$0 \to \mathrm{Pic}_0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0 \tag{1}$$

in which the projection $\mathrm{Pic}(X) \to \mathbb{Z}$ takes the degree of a line bundle. The kernel $\mathrm{Pic}_0(X)$ of this projection (the group of isomorphism classes of line bundles of degree 0) is exactly the *Jacobian* $\mathrm{Jac}(X)$ of X; for $\mathbf{k} = \mathbb{C}$, as an analytic manifold it is a compact complex torus of dimension g, where g is the genus of X.

For example, let $X = \mathbb{P}^1$. Then $\mathrm{Pic}(X) = \mathbb{Z}$ and all line bundles on X have the form $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$ for $n \in \mathbb{Z}$. To construct $\mathcal{O}(n)$ explicitly, cover $\mathbb{P}^1 = \mathbb{A}^1 \cup \infty$ with two charts, $U_0 := \mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$ and $U_\infty := \mathbb{P}^1 \setminus 0$. Now, every vector bundle on $\mathbb{A}^1$ is trivial; this is because vector bundles on an affine scheme S are just finite projective modules over the ring of regular functions on S, and all finite projective modules over $\mathbf{k}[\mathbb{A}^1] = \mathbf{k}[x]$ are free. So we may use these charts to define vector bundles (in particular, line bundles) on $\mathbb{P}^1$. Their intersection is $U_0 \cap U_\infty = \mathbb{A}^1 \setminus 0 = \mathbb{G}_m$, so we must specify a clutching function

$$g \colon \mathbb{G}_m \to \mathrm{GL}_1 = \mathbb{G}_m$$

from U_∞ to U_0, i.e., a non-vanishing regular function on $\mathbb{G}_m$. Such non-vanishing functions are all of the form $g(z) = cz^n$ for some non-zero constant c and integer $n \in \mathbb{Z}$. The constant c may be absorbed by rescaling, for example on U_0, so we may take $g(z) = z^n$. The clutching function $g(z) = z^n$ defines the line bundle $\mathcal{O}(n)$. This confirms that the line bundles on $\mathbb{P}^1$ are $\mathcal{O}(n)$, $n \in \mathbb{Z}$.

By definition, sections of $\mathcal{O}(n)$ are pairs of polynomials $x_0(z)$ and $x_\infty(z^{-1})$ such that $x_0(z) = z^n x_\infty(z^{-1})$. Hence $\mathcal{O}(n)$ has an $(n+1)$-dimensional space of sections if $n \geq 0$, and no non-zero sections if $n < 0$. Also $\mathcal{O}(n)^\vee \cong \mathcal{O}(-n)$.

2.7 Smooth and analytic G-bundles, Serre's GAGA theorem

Definition 2.1 is completely parallel to the one used in differential geometry and topology, in which case G is a real Lie group and we consider C^∞ functions instead of regular functions. The same is true for complex analytic geometry, where G is a complex Lie group and we use holomorphic functions. (Of course, in these cases, one may consider usual open sets instead of étale ones.)

One of the most important basic theorems in geometry of G-bundles is *Serre's GAGA theorem*, named after his paper "Géométrie Algébrique et Géométrie Analytique" ([S1]) where it was proved.

Theorem 2.3 (Serre's GAGA theorem)**.** *If X is a complex projective variety, then there is an equivalence of categories*

$$\Big(\text{algebraic } G\text{-bundles on } X\Big) \xrightarrow{\sim} \Big(\text{analytic } G\text{-bundles on } X\Big)$$

obtained by analytification: we view the algebraic variety X as an analytic variety X^{an}, and the same for G-bundles on X.

Remark 2.4. The equivalence of Theorem 2.3 extends to coherent sheaves and therefore preserves cohomology of such sheaves.

Problem 1. Let L be a non-trivial line bundle of degree 0 on an elliptic curve X over $\mathbb{C}$. Show that

(i) $L|_{X\setminus 0}$ is non-trivial as an algebraic bundle, but

(ii) $L|_{X\setminus 0}$ is trivial as an analytic bundle.

Conclude that the GAGA theorem fails for the (non-projective) curve $X \setminus 0$.

2.8 Étale vs Zariski charts

Finally, let us explain why we need to consider étale charts instead of Zariski charts. The reason is that in usual topology, points have neighborhoods which are contractible and therefore have trivial cohomology, while in the Zariski topology, such neighborhoods do not exist in general (e.g., think about Zariski open sets on the complex line, which are just complements of finite sets). For this reason, principal G-bundles on algebraic varieties need not be Zariski locally trivial.

Example 2.5. Let $X := \mathbb{C}^\times$ and $P := \mathbb{C}^\times$ with map $\pi\colon P \to X$ given by $z \mapsto z^2$. This is a principal μ_2-bundle, where $\mu_2 = \{1, -1\}$. But it does not trivialize on any non-empty Zariski open set in $\mathbb{C}^\times$. Indeed, the monodromy around $0 \in \mathbb{C}$ is multiplication by -1, and removing a finite number of points will not change the existence of this monodromy.

Even if G is connected reductive, this may still happen, say, when X is a surface. This is a much more subtle phenomenon which we will not talk about. But for curves, which is the case we are interested in, the situation is simpler, due to the following (non-trivial) theorem.

Theorem 2.6 (Borel–Springer [BS], Steinberg [St])**.** *If X is a smooth curve and G is a connected reductive group, then any principal G-bundle E on X is Zariski-locally trivial. In other words, for any $x \in X$, E trivializes after removing a finite subset not containing x from X (possibly depending on E).*

Remark 2.7. Note that for some connected reductive groups, like $G = \mathrm{GL}_n$ or SL_n, this is actually true for any X (not just smooth curves) by the non-abelian version of *Hilbert Theorem 90* ([S2]).

Problem 2 ([Pr])**.** Let $A \colon \mathbb{C}^\times \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ be a meromorphic function such that $\det A \not\equiv 0$. Let $q \in \mathbb{C}^\times$ with $|q| < 1$ and consider the q-difference equation

$$f(qz) = A(z)f(z) \tag{2}$$

for $f \colon \mathbb{C}^\times \to \mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that (2) has a meromorphic solution $f(z)$ such that $\det f \not\equiv 0$. (Hint: use the GAGA theorem and Hilbert theorem 90.)

For semisimple groups, one can make an even stronger statement:

Theorem 2.8 (Harder, [H])**.** *If G is semisimple, then a G-bundle on any smooth affine curve X is trivial, i.e. on a general irreducible smooth curve it is trivialized if you remove any one point.*[5]

Note that Theorem 2.8 is false for non-semisimple reductive groups! For instance, (1) implies that on $X \setminus x$, we have an identification

$$\mathrm{Pic}(X \setminus x) = \mathrm{Pic}(X)/\langle \mathcal{O}(x)\rangle \cong \mathrm{Jac}(X).$$

So if $g > 0$, line bundles on X are generally not trivialized by removing a point, and one must remove several points. Furthermore, without knowing the line bundle, we don't know which points to remove so that the bundle becomes trivial. Said differently, given any finite set of points on X, there exists a line bundle on X which is not trivialized after removing those points.

Problem 3. Let $\mathcal{L}$ be a line bundle of non-zero degree on a complex elliptic curve X. Show that there exists a point $x \in X$ such that $\mathcal{L}|_{X\setminus x}$ is trivial. Is the same true on a genus 2 curve?

3 Moduli of G-bundles on smooth projective curves

3.1 The stack of G-bundles

For any X and G, let

$$\mathrm{Bun}_G(X)(\mathbf{k}) := \left\{ \begin{array}{c} \text{isomorphism classes of principal } G\text{-bundles} \\ \text{on } X \text{ defined over } \mathbf{k} \end{array} \right\}.$$

[5] Later this theorem was generalized to families by Drinfeld and Simpson, [DS].

More generally, we can take an extension of scalars: if A is a commutative $\mathbf{k}$-algebra, then there is similarly the set $\mathrm{Bun}_G(X)(A)$ of isomorphism classes of G-bundles on X defined over A, with clutching maps now being regular functions $U_i \cap U_j \to G$ defined over A. One can also say that

$$\mathrm{Bun}_G(X)(A) \coloneqq \mathrm{Map}(\mathrm{Spec}\, A, \mathrm{Bun}_G(X)),$$

which is, by definition, the set of isomorphism classes of principal G-bundles on the product $\mathrm{Spec}\, A \times X$. In other words, there is a *functor*

$$\mathrm{Bun}_G(X)\colon A \mapsto \mathrm{Bun}_G(X)(A)$$

from the category of commutative $\mathbf{k}$-algebras to the category of sets. This is the kind of functor we consider in algebraic geometry when we define schemes — Grothendieck's *functor of points*. However, our functor $\mathrm{Bun}_G(X)$ is not representable by a scheme; rather, it is only representable by a more complicated object called an *algebraic stack*. The reason is that bundles, unlike points of a scheme, can have nontrivial automorphisms.

To keep this text short, we will not systematically discuss the notion of a stack, and will keep the exposition somewhat informal (a detailed discussion can be found, for example, in [O]). As we've just mentioned, the main difference between schemes and stacks is that in schemes, points are just points, but in stacks, every point comes with a group of automorphisms.

Example 3.1. Let G be a finite group and let $*$ denote a point. Then there is a stack $[*/G]$, also sometimes denoted BG, called the *classifying stack for* G. Its functor of points is given by

$$\mathrm{Map}(S, [*/G]) = \{\text{principal } G\text{-bundles on } S\}.$$

The right hand side is a category: it has one object, but this object has many automorphisms. So, geometrically, $[*/G]$ has one point, but this point has automorphisms.

This suggests that we should think of $\mathrm{Bun}_G(X)(A)$ as a category (more specifically, a groupoid, i.e., a category where all morphisms are isomorphisms) rather than just a set: its objects are G-bundles on X defined over A and morphisms are isomorphisms of such bundles.

Algebraic stacks are a sort of globalization of the notion of the quotient of a variety (or, more generally, a scheme) by a group action. In particular, there is the important special case of a *quotient stack*. Namely, if Y is a variety and H is an affine algebraic group acting on Y, then there is the stack $[Y/H]$ with functor of points given by

$$\mathrm{Map}(S, [Y/H]) = \left\{ \begin{array}{c} \text{principal } H\text{-bundles } P \to S \\ \text{with an } H\text{-equivariant map } P \to Y \end{array} \right\}.$$

This data can be written as the Cartesian square

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ S & \longrightarrow & [Y/H]. \end{array}$$

If the H-action on Y is sufficiently nice (e.g., free), then Y/H is a nice topological space, but in general it is not (it has poor separation properties), and the idea of stacks is to never take the quotient and to work equivariantly instead.

Of course, a representation of a stack as $[Y/H]$, if exists, is not unique, as $[Y/H] = [Y \times_H H'/H']$ whenever $H \subset H'$. In particular, given an inclusion $H \hookrightarrow \mathrm{GL}_n$, we can write $[Y/H]$ as $[\widetilde{Y}/\mathrm{GL}_n]$, where $\widetilde{Y} := Y \times_H \mathrm{GL}_n$. Thus we may always assume without loss of generality that $H = \mathrm{GL}_n$.

It turns out that $\mathrm{Bun}_G(X)$ is not a quotient stack of an algebraic variety by an algebraic group; for example $\mathrm{Bun}_{\mathrm{GL}_1}(\mathbb{P}^1) = \mathbb{Z}$, so it has infinitely many components. Yet this stack can be written as a nested union of open substacks $\mathrm{Bun}_G(X)_n$, $n \geq 0$, which are of the form $[Y_n/H_n]$ where Y_n is a smooth variety and H_n and affine algebraic group (and we may arrange that $H_n = \mathrm{GL}_{m_n}$).

Thus $\mathrm{Bun}_G(X)(\mathbf{k})$ has a natural topology (taking quotient topology on $\mathrm{Bun}_G(X)_n$ and then the inductive limit in n), but this topology is very non-separated; in particular, $\mathbf{k}$-points do not have to be closed. This is in contrast to schemes, where $\mathbf{k}$-points are closed by definition.

Example 3.2. We see that if $G = \mathrm{GL}_1$ then $\mathrm{Bun}_G(X) = \mathrm{Pic}(X)$ is a stack which is not a scheme, because $\mathrm{Aut}(\mathcal{L}) = \mathbb{C}^\times$ for any line bundle $\mathcal{L}$; it is usually called the *Picard groupoid* of X (this is the category whose objects are line bundles on X and morphisms are isomorphisms of line bundles). But this stack is not so different from a scheme: every point has the *same* automorphism group, and we can *rigidify* to get a scheme. To "rigidify" here means to consider bundles along with the extra data of an element in the fiber over some fixed point $x_0 \in X$. This extra data kills the automorphism group.

In contrast, if $G = \mathrm{PGL}_2$, and $X = \mathbb{P}^1$, the simplest possible curve, then we can have automorphism groups of arbitrarily large dimension. The more degenerate the bundle becomes, the bigger the automorphism group will be. So in this case $\mathrm{Bun}_G(X)$ is already a "true" stack.

Indeed, consider the PGL_2-bundle $E_n := \mathcal{O}(n) \oplus \mathcal{O}(0)$ for $n > 0$, whose clutching function is the diagonal matrix $g(z) = \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix}$. Automorphisms of E_n have the form $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where $a_{11}, a_{22} \in \mathbb{C}$, $a_{12} \in \mathrm{Hom}(\mathcal{O}(0), \mathcal{O}(n)) = H^0(\mathcal{O}(n))$ lives in an $(n+1)$-dimensional space, and $a_{21} \in \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(0)) = H^0(\mathcal{O}(-n)) = 0$. The constraint that the automorphism is an element in PGL_2 means that we can set $a_{22} = 1$. Putting it all together, we get that $\dim \mathrm{Aut}(E_n) = (n+1) + 1 = n + 2$.

Remark 3.3. If the genus of X is ≥ 2 and the group G is semisimple and adjoint (i.e., has trivial center) then one can show that a generic G-bundle on X has no non-trivial automorphisms. This means that the stack $\mathrm{Bun}_G(X)$ has a dense open set of "generic" bundles, which is a scheme (in fact, a smooth algebraic variety). Yet understanding the geometry of $\mathrm{Bun}_G(X)$ requires considering arbitrarily degenerate bundles, around which this stack is no longer a scheme and has a very complicated structure.

3.2 Classification of vector bundles on $\mathbb{P}^1$

Theorem 3.4. *Every rank 2 vector bundle E on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n) \oplus \mathcal{O}(m)$ for unique $n \geq m$.*

Proof. Uniqueness is clear, because we can recover

$$n = \max\{i : \mathrm{Hom}(\mathcal{O}(i), E) \neq 0\}$$

from E, and $m = \deg E - n$.

Existence of (n, m) is more interesting. We start with a simple lemma.

Lemma 3.5. *Let E be a vector bundle on $\mathbb{P}^1$.*

(i) There exists $m \in \mathbb{Z}$ such that $\mathrm{Hom}(\mathcal{O}(m), E) \neq 0$.

(ii) If $\varphi\colon \mathcal{O}(m) \to E$ is a nonzero homomorphism then for some $r \geq 0$ there exists a nonvanishing homomorphism $\widehat{\varphi}\colon \mathcal{O}(m + r) \to E$, with $r > 0$ unless φ is nonvanishing.

(iii) If $m \geq n - 1$ then any short exact sequence

$$0 \to \mathcal{O}(m) \to E \to \mathcal{O}(n) \to 0$$

splits.

Proof. (i) Let φ be a nonzero regular section of E over $\mathbb{A}^1$ (it exists since $E|_{\mathbb{A}^1}$ is trivial as noted above). Then φ extends to a rational section of E over $\mathbb{P}^1$ of some order m at ∞. Thus φ defines a regular map $\varphi\colon \mathcal{O}(m) \to E$.

(ii) If φ vanishes only at points $p_1, \ldots, p_s$ to orders $r_1, \ldots, r_s$, then it defines a nonvanishing map

$$\widehat{\varphi}\colon \mathcal{O}(m) \otimes \mathcal{O}(\sum_{i=1}^{s} r_i p_i) = \mathcal{O}(m + \sum_{i=1}^{s} r_i) \to E.$$

.

(iii) Such short exact sequences are classified by

$$\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(n), \mathcal{O}(m)) = H^1(\mathbb{P}^1, \mathcal{O}(m-n)) \cong H^0(\mathbb{P}^1, \mathcal{O}(n-m-2))^* \tag{3}$$

where $\cong$ is Serre duality using the canonical bundle $\mathcal{K} = \mathcal{O}(-2)$. Hence if $m \geq n-1$, this Ext group vanishes and thus the sequence splits. □

Now fix a rank 2 vector bundle E on $\mathbb{P}^1$. By Lemma 3.5(i),(ii) there is a nonvanishing map $\varphi\colon \mathcal{O}(m) \to E$ for some $m \in \mathbb{Z}$. Hence we obtain a short exact sequence

$$0 \to \mathcal{O}(m) \xrightarrow{\varphi} E \to \mathcal{O}(n) \to 0 \tag{4}$$

where $n = \deg E - m$. (The cokernel of φ is a line bundle since φ is nonvanishing, so it has the form $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$, and it is clear that $n = \deg E - m$.) Let R be the maximum of integers r such that $\mathrm{Hom}(\mathcal{O}(r), E) \neq 0$. It exists and satisfies $R \leq \max(m, n)$ by applying $\mathrm{Hom}(\mathcal{O}(r), -)$ to (4); namely, for $r > \max(m, n)$, both $\mathrm{Hom}(\mathcal{O}(r), \mathcal{O}(m))$ and $\mathrm{Hom}(\mathcal{O}(r), \mathcal{O}(n))$ vanish, so $\mathrm{Hom}(\mathcal{O}(r), E) = 0$ as well.

Recall that E is realized by the clutching function

$$g(z) = \begin{pmatrix} z^m & f(z) \\ 0 & z^n \end{pmatrix}$$

where $f(z)$ is a Laurent polynomial. We can modify g by $g \mapsto h_1 \circ g \circ h_2^{-1}$ where

$$h_1 = \begin{pmatrix} 1 & \varphi(z) \\ 0 & 1 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 1 & \psi(z^{-1}) \\ 0 & 1 \end{pmatrix},$$

and $\phi, \psi \in \mathbf{k}[u]$. Using such a modification, we can reduce uniquely to the case where f has no monomials z^s except where $m < s < n$. A map $\mathcal{O}(r) \to E$ is a section of the bundle $\mathcal{O}(-r) \otimes E$ with transition map

$$\begin{pmatrix} z^{m-r} & z^{-r} f(z) \\ 0 & z^{n-r} \end{pmatrix}.$$

Hence a regular section of this bundle is a pair $\binom{x_0(z)}{y_0(z)}, \binom{x_\infty(z^{-1})}{y_\infty(z^{-1})}$ of vector-valued polynomials, such that

$$\begin{pmatrix} z^r x_0(z) \\ z^r y_0(z) \end{pmatrix} = \begin{pmatrix} z^m & f(z) \\ 0 & z^n \end{pmatrix} \begin{pmatrix} x_\infty(z^{-1}) \\ y_\infty(z^{-1}) \end{pmatrix}.$$

In particular, the left hand sides $z^r y_0$ and $z^r x_0$ have no monomials of degree $< r$. So we just need to find x_∞ and y_∞ so that the right hand sides also have no monomials of degree $< r$. This means that $\deg y_\infty \leq n - r$ and $f(z) y_\infty(z)$ has no terms of degree between $m + 1$ and r. This gives $r - m$ homogeneous linear equations in $n - r + 1$ unknowns. Hence non-zero solutions definitely exist if $r - m < n - r + 1$, or, equivalently, $r \leq \frac{m+n}{2}$. So we have $R \geq \frac{m+n}{2}$. Moreover, any nonzero map $\mathcal{O}(R) \to E$ must be an embedding, otherwise R is not maximal by Lemma 3.5(ii). So there is a short exact sequence

$$0 \to \mathcal{O}(R) \to E \to \mathcal{O}(m + n - R) \to 0$$

Since $R \geq m + n - R$, this short exact sequence splits by Lemma 3.5(iii), as desired. □

Corollary 3.6 (Grothendieck, [Gr]). *A rank n vector bundle on $\mathbb{P}^1$ is uniquely of the form*

$$\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$$

for integers $m_1 \geq \cdots \geq m_n$.

Proof. The proof is by induction on the rank n. Let us prove the statement for rank $n+1$ assuming it is known for smaller ranks. Let m_0 be the maximal possible integer r for which $\mathrm{Hom}(\mathcal{O}(r), E) \neq 0$. By the induction hypothesis and Lemma 3.5(i),(ii), we have an extension

$$0 \to \mathcal{O}(m_0) \to E \to \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n) \to 0. \tag{5}$$

For $j = 1, \dots, n$, consider the subbundle $E_j \subset E$ which is the preimage of $\mathcal{O}(m_j)$. Then we have a short exact sequence

$$0 \to \mathcal{O}(m_0) \to E_j \to \mathcal{O}(m_j) \to 0,$$

and m_0 is the maximal possible integer r for which $\mathrm{Hom}(\mathcal{O}(r), E_j) \neq 0$. So by Theorem 3.4, $E_j = \mathcal{O}(m_0) \oplus \mathcal{O}(m_j)$ and $m_0 \geq m_j$. Since this is true for any $j = 1, \dots, n$, the sequence (5) splits by Lemma 3.5(iii) and $E \cong \mathcal{O}(m_0) \oplus \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$, as desired. □

Problem 4. Classify PGL_n-bundles and SL_n-bundles on $\mathbb{P}^1$.

Problem 5. (i) Let E be the rank 2 vector bundle on $\mathbb{P}^1$ given by the short exact sequence

$$0 \to \mathcal{O}(0) \to E \to \mathcal{O}(m) \to 0$$

for $m \geq 2$, with transition function (from U_∞ to U_0)

$$g(z) := \begin{pmatrix} 1 & f(z) \\ 0 & z^m \end{pmatrix}$$

for some polynomial $f(z) = a_1 z + \cdots + a_{m-1} z^{m-1}$. Given $\frac{m}{2} \leq n \leq m$, find the condition on $a_1, \dots, a_{m-1} \in \mathbb{C}$ such that $E \cong \mathcal{O}(k) \oplus \mathcal{O}(m-k)$ for some $\frac{m}{2} \leq k \leq n$.

(ii) For even m and $n = \frac{m}{2}$, find a polynomial $H(a_1, \dots, a_{2n-1})$ such that $E \cong \mathcal{O}(n) \oplus \mathcal{O}(n)$ iff $H(a_1, \dots, a_{2n-1}) \neq 0$.

Problem 6. Let $B \subset \mathrm{GL}_n$ be the subgroup of upper-triangular matrices. Show that any GL_n-bundle E on a smooth projective curve X admits a B-structure, i.e. is associated to a (non-unique) B-bundle.[6]

[6]This actually holds for any reductive group, see [DS] and references therein.

3.3 Bundles with level structure and local presentations of $\mathrm{Bun}_{\mathrm{PGL}_2}(\mathbb{P}^1)$

For $G = \mathrm{PGL}_2$, Theorem 3.4 (and the solution to Problem 4) implies that every G-bundle E on $\mathbb{P}^1$ is of the form $\mathcal{O}(n) \oplus \mathcal{O}(0)$ for a unique $n = N(E) \geq 0$. This allows us to describe in more detail the structure of the stack $\mathrm{Bun}_G(\mathbb{P}^1)$.

To do so, consider the nested sequence of open substacks $\mathrm{Bun}_G(\mathbb{P}^1)_n \subset \mathrm{Bun}_G(\mathbb{P}^1)$ of bundles E with $N(E) \leq n$; i.e., for a commutative algebra A, $\mathrm{Bun}_G(\mathbb{P}^1)_n(A)$ is the set of isomorphism classes of G-bundles on $\mathrm{Spec} A \times \mathbb{P}^1$ which at every geometric point of $\mathrm{Spec} A$ give a bundle on $\mathbb{P}^1$ with $N(E) \leq n$. We would like to present $\mathrm{Bun}_G(\mathbb{P}^1)_n$ as the quotient $[Y_n/H_n]$ of a smooth variety Y_n by an algebraic group H_n. There are, in fact, many different (albeit equivalent) ways to choose this presentation, which correspond to various ways to "rigidify" the classification problem for G-bundles, so that the automorphism groups of all bundles with $N(E) \leq n$ are killed.[7]

One convenient way to do so is to consider bundles E equipped with *level n structure* at 0, i.e., with a trivialization over the *n-th infinitesimal neighborhood* $\widehat{0}_n :=$ $\mathrm{Spec}(\mathbf{k}[z]/z^{n+1})$ of $0 \in \mathbb{P}^1$. If we present bundles E by trivializing them on U_0, U_∞ and writing down the clutching function as above, then such a trivialization of E over $\widehat{0}_n$ differs from its trivialization over U_0 restricted to $\widehat{0}_n$ by an element h of the group $H_n := G(\mathbf{k}[z]/z^{n+1})$. On the other hand, in Example 3.2 we saw that an automorphism of $\mathcal{O}(n) \oplus \mathcal{O}(0)$ for $n > 0$ is given in the chart U_0 by a matrix (a_{ij}) with $a_{22} = 1$, $a_{11} = \mathrm{const}$, $a_{21} = 0$ and $a_{12} = p(z)$, where p is a polynomial of degree n. On the other hand, if $n = 0$ then an automorphism is given by a constant matrix. This implies that if $N(E) \leq n$ then any non-trivial automorphism of E changes the element h, so $\mathrm{Aut}(E, h) = 1$. From this one can deduce that the stack $Y_n := \widetilde{\mathrm{Bun}}_G(\mathbb{P}^1)_n$ classifying G-bundles (E, h) on $\mathbb{P}^1$ with level n structure at 0 such that $N(E) \leq n$ is actually a smooth algebraic variety with an action of H_n (by changing h), and

$$\mathrm{Bun}_G(\mathbb{P}^1)_n = [Y_n/H_n].$$

Note that the action of H_n on Y_n is not free: the stabilizer of (E, h) is isomorphic to $\mathrm{Aut}(E)$.

More generally, one can define Y_n by taking h to be a collection of level n_i structures h_i at any given distinct points $z_i \in \mathbb{P}^1$, so that $\sum_i (n_i + 1) = n + 1$. In this case H_n will be $\prod_i G(\mathbf{k}[z]/z^{n_i+1})$. The above definition then corresponds to the case when all z_i coalesce at 0.

Now consider the set $\mathrm{Bun}_G(\mathbb{P}^1)(\mathbf{k}) \cong \mathbb{Z}_{\geq 0}$ (where the bijection sends E to $N(E)$), and let us ask what topology on $\mathbb{Z}_{\geq 0}$ is induced by the stack structure. First of all, note that $\mathrm{Bun}_G(\mathbb{P}^1)$ has two disjoint parts $\mathrm{Bun}_G(\mathbb{P}^1)^{\mathrm{even}}$, $\mathrm{Bun}_G(\mathbb{P}^1)^{\mathrm{odd}}$, corresponding

[7] Recall from Example 3.2 that if $N(E) > 0$ then $\mathrm{Aut}(E)$ has dimension $N(E) + 2$, so these automorphism groups get arbitrarily large. Thus for large n the group H_n must be large as well, as it must contain these automorphism groups for all E with $N(E) \leq n$.

to bundles of even and odd degree. Thus we have a decomposition of topological spaces $\mathbb{Z}_{\geq 0} = \mathbb{Z}_{\geq 0}^{\mathrm{even}} \sqcup \mathbb{Z}_{\geq 0}^{\mathrm{odd}}$. Also since the sets $\mathrm{Bun}_G(\mathbb{P}^1)_n$ are open, we see that the subsets

$$U_r^{\mathrm{even}} = \{0, 2, \ldots, 2r-2\} \subset \mathbb{Z}_{\geq 0}^{\mathrm{even}},\ U_r^{\mathrm{odd}} = \{1, 3, \ldots, 2r-1\} \subset \mathbb{Z}_{\geq 0}^{\mathrm{odd}}$$

are open for all $r \geq 0$.

Proposition 3.7. *$U_r^{\mathrm{even}}, U_r^{\mathrm{odd}}$ are the only proper open subsets of $\mathbb{Z}_{\geq 0}^{\mathrm{even}}, \mathbb{Z}_{\geq 0}^{\mathrm{odd}}$. In particular, the latter spaces are connected.*

Proof. It suffices to show that the closure of any $k \in \mathbb{Z}_{\geq 0}$ contains $k+2$. To this end, consider the space of short exact sequences

$$0 \to \mathcal{O}(0) \to E \to \mathcal{O}(k+2) \to 0;$$

as we have seen, it is the vector space $V_k = \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(k+2), \mathcal{O}(0)) = H^0(\mathbb{P}^1, \mathcal{O}(k))$ of dimension $k+1$. For $v \in V_k$ let E_v denote the corresponding bundle. It is clear that $E_0 \cong \mathcal{O}(k+2) \oplus \mathcal{O}(0)$, and it is easy to show that if $E_v = \mathcal{O}(k+2) \oplus \mathcal{O}(0)$ then $v = 0$. On the other hand, there is $w \in V_k, w \neq 0$ such that $E_w \cong \mathcal{O}(k+1) \oplus \mathcal{O}(1)$, which is isomorphic to $\mathcal{O}(k) \oplus \mathcal{O}(0)$ as a G-bundle (see Problem 5). Hence $E_{\lambda w} \cong \mathcal{O}(k) \oplus \mathcal{O}(0)$ as a G-bundle for any nonzero scalar λ. Thus the family of bundles $E_{\lambda w}$, $\lambda \in \mathbb{A}^1$ has

$$N(E_{\lambda w}) = \begin{cases} k+2,\ \lambda = 0 \\ k,\ \lambda \neq 0 \end{cases}$$

It follows that $k+2$ lies in the closure of k and we are done. □

In particular, it follows that $\mathrm{Bun}_{\mathrm{SL}_2}(\mathbb{P}^1)(\mathbf{k}) = \mathbb{Z}_{\geq 0}^{\mathrm{even}}$ with the above topology, and $\mathrm{Bun}_{\mathrm{SL}_2}(\mathbb{P}^1)$ is the nested union of open substacks $\mathrm{Bun}_{\mathrm{SL}_2}(\mathbb{P}^1)_n$, comprising bundles with $\mathrm{Hom}(\mathcal{O}(n+1), E) = 0$, which correspond to the open subsets U_{n+1}^+. Moreover,

$$\mathrm{Bun}_{\mathrm{SL}_2}(\mathbb{P}^1)_n \cong \mathrm{Bun}_{\mathrm{PGL}_2}(\mathbb{P}^1)_{2n}^+ = \widetilde{\mathrm{Bun}}_{\mathrm{PGL}_2}(\mathbb{P}^1)_{2n}^+ / H_{2n},$$

where $\widetilde{\mathrm{Bun}}_{\mathrm{PGL}_2}(\mathbb{P}^1)_{2n}^+$ is the variety classifying pairs (E, h) where E is a GL_2-bundle of degree $2n$ with $N(E) \leq 2n$ and h is a level structure on E at 0 of order $2n$.

In conclusion, let us give a concrete realization of the variety $\widetilde{\mathrm{Bun}}_{\mathrm{PGL}_2}(\mathbb{P}^1)_{2n}^+$, by embedding it into the Grassmannian $\mathrm{Gr}(2n, 4n+2)$. To this end, let $(E, h) \in \widetilde{\mathrm{Bun}}_{\mathrm{PGL}_2}(\mathbb{P}^1)_{2n}^+$ and consider the space of sections $H^0(\mathbb{P}^1, E \otimes \mathcal{O}(-1))$, which has dimension $2n$. For each $s \in H^0(\mathbb{P}^1, E \otimes \mathcal{O}(-1))$, the trivialization h gives rise to its $2n$-th Taylor approximation $\nu_h(s) = \binom{s_1}{s_2}$, where $s_j \in \mathbf{k}[z]/z^{2n+1}$. This defines a linear map

$$\nu_h \colon H^0(\mathbb{P}^1, E \otimes \mathcal{O}(-1)) \to \mathbf{k}^2[z]/z^{2n+1} \cong \mathbf{k}^{4n+2}.$$

It follows from Theorem 3.4 that ν_h is injective, so we obtain an H_{2n}-equivariant map

$$\nu\colon \widetilde{\mathrm{Bun}}_{\mathrm{PGL}_2}(\mathbb{P}^1)^+_{2n} \to \mathrm{Gr}(2n, \mathbf{k}^2[z]/z^{2n+1}) = \mathrm{Gr}(2n, 4n+2).$$

To describe its image, note that $\widetilde{\mathrm{Bun}}_{\mathrm{PGL}_2}(\mathbb{P}^1)^+_{2n}$ has $n+1$ H_{2n}-orbits, i.e., those of the bundles $E_j = \mathcal{O}(n+j) \oplus \mathcal{O}(n-j)$, $0 \le j \le n$, with the standard trivializations h^j at 0. For these bundles, we have

$$\nu(E_j, h^j) = V_j := \{\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \colon \deg(s_1) \le n+j-1,\ \deg(s_2) \le n-j-1\}.$$

Thus $\mathrm{Im}\,\nu = \sqcup_{0\le j\le n} H_{2n} \cdot V_j$. But it is easy to check that $\overline{H_{2n} \cdot V_0} \supset V_j$, so we get that

$$\mathrm{Im}\,\nu \subset \overline{H_{2n} \cdot V_0}$$

is an open dense H_{2n}-invariant subset.

One can also show that ν is injective; in fact it is an embedding. This gives rise to an explicit realization of $\widetilde{\mathrm{Bun}}_{\mathrm{PGL}_2}(\mathbb{P}^1)^+_{2n}$ as a locally closed subvariety of dimension $6n$ in $\mathrm{Gr}(2n, 4n+2)$, as desired.

Remark 3.8. One can do the same taking h to be a collection of level n_i structures h_i at any given distinct points $z_i \in \mathbb{P}^1$, so that $\sum_i (n_i + 1) = 2n+1$, as noted above. The least degenerate case is $n_i = 0$ for $i \in [1, 2n+1]$. In this case $H_{2n} = G^{2n+1}$ which acts naturally on $\mathrm{Gr}(2n, (\mathbf{k}^2)^{2n+1}) = \mathrm{Gr}(2n, 4n+2)$, and $\nu_h(s)$ is defined by taking the values of s at $z_1, \dots, z_{2n+1}$. In this case $Y_{2n} \subset \mathrm{Gr}(2n, 4n+2)$ is an open subset of $\overline{G^{2n+1} \cdot V_0}$, where $V_0 \subset (\mathbf{k}^2)^{2n+1}$ is the space of values of pairs of polynomials $(s_1(z), s_2(z))$ of degree $n-1$ at $z_1, \dots, z_{2n+1}$.

All the other cases are degenerations of this, when some of the points z_i coalesce, with the original setting corresponding to all the points z_i coalescing at 0.

Remark 3.9. Similar methods can be used to locally represent $\mathrm{Bun}_G(X)$ as a quotient for general X and G.

3.4 Principal G-bundles on $\mathbb{P}^1$

We will now explain how to generalize Grothendieck's theorem from GL_n to an arbitrary split connected reductive group G. For this, we first need to reformulate it.

Recall that rank n vector bundles are the same as GL_n-bundles, and GL_n is the group of invertible $n \times n$ matrices, so it contains a maximal torus T consisting of diagonal matrices with non-zero entries on the diagonal: $T = \mathrm{GL}_1^n$. Grothendieck's theorem says that every GL_n-bundle on $\mathbb{P}^1$ admits a T-structure, meaning that it is associated to some T-bundle E by sending E to $E \times_T \mathrm{GL}_n$. Recall that T-bundles on $\mathbb{P}^1$ correspond to n-tuples of integers $\mathbf{m} = (m_1, \dots, m_n)$: the T-bundle $E(\mathbf{m})$ corresponding to such a tuple is $\mathcal{O}(m_1)^\times \times \cdots \times \mathcal{O}(m_n)^\times$, where $\mathcal{O}(m)^\times$ is the bundle of nonzero vectors in $\mathcal{O}(m)$. This shows that the same GL_n-bundle can come from many different T-bundles, because the data of a T-bundle is sensitive to the ordering of the integers $m_1, \ldots, m_n$. Precisely, if $E(\mathbf{m}), E(\mathbf{n})$ are T-bundles on $\mathbb{P}^1$, then by Grothendieck's theorem

$$E(\mathbf{m}) \times_T \mathrm{GL}_n \cong E(\mathbf{n}) \times_T \mathrm{GL}_n$$

if and only if there exists a permutation $w \in S_n$ such that $\mathbf{m} = w\mathbf{n}$.

Now let G be a split connected reductive group, $T \subset G$ a maximal torus, $N(T) \subset G$ the normalizer of T, and $W := N(T)/T$ the Weyl group of (G, T). Then T-bundles on $\mathbb{P}^1$ are canonically parametrized by the cocharacter lattice $\mathbf{X}_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ (namely, the bundle $E(\mu)$ attached to the cocharacter μ has this cocharacter as its clutching map). Note that W acts naturally on T, hence on $\mathbf{X}_*(T)$.

Theorem 3.10 ([Gr]). *Any G-bundle on $\mathbb{P}^1$ is associated to a T-bundle E, and given two T-bundles $E(\mu)$ and $E(\nu)$, we have*

$$E(\mu) \times_T G \cong E(\nu) \times_T G$$

if and only if $\mu = w\nu$ for some element $w \in W$.

Proof idea. Reduce to the case of vector bundles by considering representations of G. For more detail, see [Gr, MS]. □

Thus Theorem 3.10 says that isomorphism classes of G-bundles on $\mathbb{P}^1$ are labeled by $\mathbf{X}_*(T)/W$.

Remark 3.11. The cocharacter lattice $\mathbf{X}_*(T)$ is the *character* or *weight lattice* $\Lambda^\vee$ of the *Langlands dual group* $G^\vee$. Thus Theorem 3.10 says that isomorphism classes of G-bundles on $\mathbb{P}^1$ are labeled by

$$\Lambda^\vee/W \cong \Lambda^\vee_+,$$

the set of dominant integral weights for $G^\vee$, which labels its irreducible representations. This is one of the simplest instances of *Langlands duality*.

4 The double quotient realization of $\mathrm{Bun}_G(X)$

4.1 The double quotient construction

As before, let X be a smooth irreducible projective curve and G a split connected reductive group over a field $\mathbf{k}$. In Section 3, we attached to this pair (X, G) the moduli stack $\mathrm{Bun}_G(X)$ of principal G-bundles on X. In general, $\mathrm{Bun}_G(X)$ is a very complicated object, but most of these complications will not be relevant for us. We defined $\mathrm{Bun}_G(X)$

via its functor of points, but now we would like to describe $\mathrm{Bun}_G(X)$ in a slightly more explicit way.

For simplicity, let's assume first that G is semisimple and $\mathbf{k}$ is algebraically closed. By Theorem 2.8, every $E \in \mathrm{Bun}_G(X)$ trivializes once any chosen point is removed from X. So pick a point $x \in X$. Cover X by two charts: a disk around x, and $X \setminus x$. In algebraic geometry, we do not have small disks, but we can take a *formal* disk D_x around x instead. To describe G-bundles on X using these two charts, it suffices to study the transition function on the intersection

$$(X \setminus x) \cap D_x = D_x^\times \tag{6}$$

where $D_x^\times$ is the *punctured formal disk*.

Let us explain the precise meaning of this equality. Let $\mathcal{O}_{(x)}$ be the local ring of the point x,

$$\mathbf{k}[D_x] = \mathcal{O}_x \coloneqq \varprojlim_{n\to\infty} \mathcal{O}_{(x)}/\mathfrak{m}_x^n,$$

where $\mathfrak{m}_x \subset \mathcal{O}_{(x)}$ is the maximal ideal, and

$$\mathbf{k}[D_x^\times] =: K_x$$

be the field of fractions of $\mathcal{O}_x$. More concretely, if t is a formal coordinate on X near x, then we obtain identifications[8]

$$\mathcal{O}_x \cong \mathbf{k}[[t]], \ \ K_x \cong \mathbf{k}((t)).$$

Also let $R_x \coloneqq \mathcal{O}(X \setminus x)$ be the ring of regular functions on the affine curve $X \setminus x$. Then we have an inclusion $R_x \hookrightarrow K_x$ defined by the Laurent series expansion at x, (it is easy to show that it does not depend on the choice of the formal coordinate), and algebraically equality (6) just means that

$$R_x \cdot \mathcal{O}_x = K_x.$$

Now, G-bundles E on X are defined by transition maps $g(z)$ from D_x to $X \setminus x$, or equivalently, elements $g \in G(K_x)$, up to $g \mapsto h_1 \circ g \circ h_2^{-1}$ where $h_1 \in G(R_x)$ and $h_2 \in G(\mathcal{O}_x)$. We have thus arrived at the following proposition.

Proposition 4.1. *$\mathrm{Bun}_G(X)(\mathbf{k}) = G(R_x)\backslash G(K_x)/G(\mathcal{O}_x)$.*

[8] In complex analysis, holomorphic functions on a disk are given by convergent Taylor series, and holomorphic functions on a punctured disk which are meromorphic at the puncture are given by convergent Laurent series which are finite in the negative direction. We make these series formal by removing the convergence assumption, and then they make sense over any field.

It is useful to take this double quotient in two steps. Namely, for any reductive G consider the one-sided quotient $\mathrm{Gr}_G := G(\mathbf{k}((t)))/G(\mathbf{k}[[t]])$, called the *affine Grassmannian.*[9] It is an *ind-variety* — infinite-dimensional, but a nested union of projective varieties of increasing dimension. We then obtain that for G semisimple, G-bundles on X correspond to orbits of $G(R_x)$ on Gr_G.

Example 4.2. If $G = \mathrm{GL}_n$ then geometric points of Gr_G are lattices $L \subset \mathbf{k}((t))^n$, i.e. finitely generated spanning $\mathbf{k}[[t]]$-submodules (which are necessarily free of rank n). Note that for any such L there exists N such that

$$t^{-N}\mathbf{k}[[t]]^n \subset L \subset t^N\mathbf{k}[[t]]^n. \tag{7}$$

Thus Gr_G is the nested union of the sets $\mathrm{Gr}_{G,N}$ of lattices satisfying (7). Note that each $\mathrm{Gr}_{G,N}$ can be realized as a closed subvariety of the Grassmannian $\mathrm{Gr}(\mathbf{k}[[t]]^n/t^{2N}) = \mathrm{Gr}(2nN)$ (disjoint union of Grassmannians of subspaces of all dimensions in $\mathbf{k}^{2nN}$) by sending L to $t^N L/t^{2N}\mathbf{k}[[t]]^n$ (namely, it is the locus of all subspaces which are invariant under multiplication by t), i.e., it has a natural structure of a projective variety. Thus Gr_G is a nested union of projective varieties.

We can generalize this construction by removing multiple points from X instead of just one. Namely, let $S \subset X$ be a non-empty finite subset, and take the two charts $U_1 := X \setminus S$ and $U_2 := \bigsqcup_{x\in S} D_x$. Then

$$U_1 \cap U_2 = \bigsqcup_{x\in S} D_x^\times,$$

and therefore, by the same reasoning as above,

$$\mathrm{Bun}_G(X)(\mathbf{k}) = G(R_S)\Big\backslash \prod_{s\in S} G(K_x)\Big/ \prod_{x\in S} G(\mathcal{O}_x), \tag{8}$$

where $R_S := \mathbf{k}[X \setminus S]$.

This, however, does not quite work for non-semisimple reductive groups G, e.g. $G = \mathrm{GL}_1 = \mathbb{G}_m$, since in this case there is no finite set S such that all G-bundles are trivialized on $X \setminus S$ (as explained at the end of Subsection 2.8). So, to generalize the above construction to such groups, we will remove *all* geometric points of X. This sounds like then there will be nothing left, but in fact this is not the case; the "Grothendieck generic point" still remains! Indeed, removing a single point in algebraic geometry means considering rational functions which are allowed to have a pole at that point. So, removing all points means considering rational functions which are

[9]Recall that the usual Grassmannian is the quotient of GL_n by a maximal parabolic subgroup. Similarly, Gr_G is the quotient of the affine Kac-Moody group $G(\mathbf{k}((t)))$ by the maximal parabolic subgroup $G(\mathbf{k}[[t]])$. This explains the origin of the term "affine Grassmannian".

allowed to have poles anywhere, i.e. simply all rational functions. We thus obtain the presentation

$$\mathrm{Bun}_G(X)(\mathbf{k}) = G(\mathbf{k}(X))\Big\backslash {\prod_{x\in X}}' G(K_x)\Big/ \prod_{x\in X} G(\mathcal{O}_x) \tag{9}$$

which is now valid for all reductive groups.

The prime in the product is a technical but important detail. It denotes the *restricted product* consisting of elements with only finitely many coordinates having poles, i.e. not lying in $G(\mathcal{O}_x)$. The restricted product arises because we are taking a colimit of (8) over *finite* sets S in order to obtain (9).

Finally, if $\mathbf{k}$ is not algebraically closed, we can perform the same construction using finite subsets $S \subset X(\overline{\mathbf{k}})$ which are Galois-invariant, where $\overline{\mathbf{k}} \supset \mathbf{k}$ is the algebraic closure of $\mathbf{k}$. Namely, let $\Gamma \coloneqq \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, then

$$\mathrm{Bun}_G(X)(\mathbf{k}) = G(\mathbf{k}(X))\Big\backslash {\prod_{x\in X(\overline{\mathbf{k}})/\Gamma}}' G(K_x)\Big/ \prod_{x\in X(\overline{\mathbf{k}})/\Gamma} G(\mathcal{O}_x).$$

For example, if $\mathbf{k}$ is finite, then all the valuations v of $K = \mathbf{k}(X)$ come from points $x \in X(\overline{\mathbf{k}})/\Gamma$, and the corresponding completions $K_v = K_x$ of K with respect to v are locally compact non-discrete topological fields (also called *local fields*). Such fields K are called *global fields*. We get

$$\mathrm{Bun}_G(X)(\mathbf{k}) = G(K)\Big\backslash G\Big({\prod_{v\in \mathrm{Val}(K)}}' K_v\Big)\Big/ G\Big(\prod_{v\in \mathrm{Val}(K)} \mathcal{O}_v\Big),$$

where $\mathrm{Val}(K)$ is the set of valuations of K. This is called an *arithmetic quotient*.

4.2 Analogy with number theory

Similar quotients arise in number theory. This is because while global fields of characteristic $p > 0$ all have the form $\mathbf{k}(X)$ for finite fields $\mathbf{k}$, in characteristic 0 they are *number fields*, i.e. finite extensions of $\mathbb{Q}$.

Definition 4.3. If K is a global field, the *ring of adèles* of K is

$$\mathbb{A} \coloneqq \mathbb{A}_K \coloneqq {\prod_{v\in \mathrm{Val}(K)}}' K_v.$$

Note that while for $K = \mathbf{k}(X)$, all valuations are non-archimedean (discrete), for a number field K, there are two kinds of valuations: archimedean (embed K into $\mathbb{C}$ and take the absolute value) and non-archimedean (discrete, or p-adic valuations). Rings of integers $\mathcal{O}_v \subset K_v$ make sense only for non-archimedean valuations. Let

$$\mathcal{O}_{\mathbb{A}} \coloneqq \prod_{v\in \mathrm{Val}_{n.a.}(K)} \mathcal{O}_v$$

where $\mathrm{Val}_{n.a.}(K)$ is the set of non-archimedean valuations of K. Then we can consider the *arithmetic quotient*

$$\mathcal{M} := G(K)\backslash G(\mathbb{A})/G(\mathcal{O}_{\mathbb{A}}).$$

This generalizes $\mathrm{Bun}_G(X)$, because if $K = \mathbf{k}(X)$, then $\mathcal{M} = \mathrm{Bun}_G(X)(\mathbf{k})$.

Example 4.4. Let $K = \mathbb{Q}$. Then there are p-adic (non-archimedean) valuations of K with respect to all primes p, with completion $\mathbb{Q}_p$, and also the usual archimedean valuation $x \mapsto |x|$ corresponding to $p = \infty$, with completion $\mathbb{R}$. Thus

$$\mathbb{A} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p$$
$$\mathcal{O}_{\mathbb{A}} = \prod_{p \text{ prime}} \mathbb{Z}_p,$$

and

$$\mathcal{M} = G(\mathbb{Q})\Big\backslash\Big(G(\mathbb{R}) \times \prod_{p \text{ prime}}{}' G(\mathbb{Q}_p)\Big)\Big/ \prod_{p \text{ prime}} G(\mathbb{Z}_p)$$

Moreover, by the weak approximation theorem ([PR], p.402),

$$G(\mathbb{Q}) \cdot \prod_{p \text{ prime}} G(\mathbb{Z}_p) = \prod_{p \text{ prime}}{}' G(\mathbb{Q}_p),$$

and

$$G(\mathbb{Q}) \cap \prod_{p \text{ prime}} G(\mathbb{Z}_p) = G(\mathbb{Z}),$$

hence

$$\mathcal{M} = G(\mathbb{Z})\backslash G(\mathbb{R}).$$

For instance, if $G = \mathrm{Sp}_{2n}$, then $\mathcal{M} = \mathrm{Sp}(2n, \mathbb{Z})\backslash \mathrm{Sp}(2n, \mathbb{R})$, and taking a quotient by $U(n)$ on the right gives the moduli space of n-dimensional abelian varieties

$$\mathcal{A}_n = \mathrm{Sp}(2n, \mathbb{Z})\backslash \mathrm{Sp}(2n, \mathbb{R})/U(n).$$

In particular, if $G = \mathrm{SL}_2 = \mathrm{Sp}_2$, then

$$\mathcal{A}_1 = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/U(1)$$

is the moduli space of elliptic curves, which is the home of classical modular forms.

Example 4.5. Let $G = \mathrm{GL}_1$ and $\mathbf{k}$ be any field. Then we obtain

$$\mathrm{Pic}(X)(\mathbf{k}) = \mathrm{GL}_1(\mathbf{k}(X))\backslash \mathrm{GL}_1(\mathbb{A})/\mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) = \mathbf{k}(X)^\times\backslash\mathbb{A}^\times/\mathcal{O}_{\mathbb{A}}^\times.$$

The quotient of $\mathbb{A}^\times/\mathcal{O}_{\mathbb{A}}^\times$ by $\mathbf{k}^\times$ is the group of divisors $\mathrm{Div}(X)$ on X, and $\mathbf{k}(X)^\times/\mathbf{k}^\times$ is its subgroup of principal divisors, $\mathrm{PDiv}(X)$. (Here, $\mathbf{k}^\times$ is the intersection $\mathbf{k}(X)^\times \cap \mathcal{O}_{\mathbb{A}}^\times$.) Thus we recover the classical definition of the Picard group of X:

$$\mathrm{Pic}(X)(\mathbf{k}) = \mathrm{Div}(X)/\mathrm{PDiv}(X).$$

5 Stable bundles and Higgs fields

5.1 Stable bundles

Hitchin systems are integrable hamiltonian systems, and hamiltonian systems live on symplectic manifolds (or varieties), so we need to come up with one. A natural way to create a symplectic variety in our setting is to consider the cotangent bundle $T^* \operatorname{Bun}_G(X)$. But, as explained in Section 3, $\operatorname{Bun}_G(X)$ is not a variety, or even a scheme. Rather, it is a stack, and the cotangent bundle of a stack is well-defined only as another stack. We will avoid these difficulties, however, and work only with a particular open set in $T^* \operatorname{Bun}_G(X)$ — the cotangent bundle $T^* \operatorname{Bun}_G^\circ(X)$ of the subset $\operatorname{Bun}_G^\circ(X)$ of suitably defined "generic" bundles, a smooth open subvariety of $\operatorname{Bun}_G(X)$ which is nonempty and dense if the genus of g of X is ≥ 2 (see Remark 3.3).

Assume $g \geq 2$ and first suppose that G is simple and adjoint. Then a "generic" bundle in $\operatorname{Bun}_G(X)$ has trivial automorphism group, so in the local presentation of $\operatorname{Bun}_G(X)$ as a quotient of an algebraic variety by a group, the group acts freely at such a bundle. The locus of "generic" bundles therefore forms a smooth algebraic variety $\operatorname{Bun}_G^\circ(X)$.

There are many ways to specify what "generic" means. We will use a *stability condition*.

For instance, let $G = \mathrm{PGL}_n$. Recall that G-bundles are rank n vector bundles modulo tensoring with line bundles. Recall also that if E is a vector bundle on X, then there are two integers attached to it: the degree $d(E)$ (given by the first Chern class), and the rank $r(E)$.

Definition 5.1. If $E \neq 0$ then the *slope* of E is

$$\mu(E) \coloneqq \frac{d(E)}{r(E)}.$$

We say E is *stable* if for every subbundle $0 \neq E' \subsetneq E$,

$$\mu(E') < \mu(E).$$

There is a more technical definition, due to Ramanathan, for other reductive groups G, which we will not state. We refer the reader to [R] for details.

It is easy to see that if L is a line bundle and E a vector bundle, then E is stable if and only if $E \otimes L$ is stable. This implies that stability is well-defined for PGL_n-bundles.

Theorem 5.2 ([R])**.** *Stable bundles have the trivial group of automorphisms, and form a smooth variety which is an open subset* $\operatorname{Bun}_G^\circ(X) \subset \operatorname{Bun}_G(X)$.

Remark 5.3. Let $\mathcal{M}_G^\circ(X) \coloneqq T^* \operatorname{Bun}_G^\circ(X)$. The Hitchin system will initially live on $\mathcal{M}_G^\circ(X)$, but actually there is a partial compactification $\mathcal{M}_G(X)$ of $\mathcal{M}_G^\circ(X)$ called the

Hitchin moduli space, which is still symplectic, to which the Hitchin system naturally extends (see [Hl]). For the simplest example of this, see Subsection 9.8. This sort of extension to a partial compactification is a common phenomenon in integrable systems.

For general semisimple G, not necessarily of adjoint type, there is a straightforward extension of this story. Namely, a bundle is "generic" if it is *regularly stable*, meaning that it is stable and its group of automorphisms reduces to the center $Z(G)$ (which is the smallest it can be). The resulting $\mathrm{Bun}_G^\circ(X)$ is still a stack with stabilizer $Z(G)$ at every point, but because this stabilizer is the same everywhere, we can rigidify like we did for $\mathrm{Pic}(X)$ and obtain a variety. In other words, we may ignore the stackiness and just consider the underlying variety.

5.2 Higgs fields

Before we go further, we should compute $\dim \mathrm{Bun}_G^\circ(X)$, or, equivalently since it is a smooth variety, the dimension $\dim T_E \mathrm{Bun}_G^\circ(X)$ of the tangent space at a point $E \in \mathrm{Bun}_G^\circ(X)$. This tangent space is just the deformation space of the bundle E, classifying its first-order deformations.

Let $\mathfrak{g}$ be the Lie algebra of G. For a G-bundle E on X, let $\operatorname{ad} E$ be the *adjoint bundle* of E, i.e., the vector bundle $E_{\mathfrak{g}}$ associated to E via the adjoint representation $\mathfrak{g}$ of G.

Problem 7. Show, using Čech 1-cocycles, that the tangent space to $\mathrm{Bun}_G^\circ(X)$ at E (i.e., the deformation space of E) is $H^1(X, \operatorname{ad} E)$.

Example 5.4. Let $G = \mathrm{GL}_n$. First order deformations of a vector bundle E are classified by $\mathrm{Ext}^1(E, E)$; this is unsurprising because, affine-locally, we are just deforming modules over the coordinate rings $\mathbf{k}[U]$, $U \subset X$. Since

$$\mathrm{Ext}^1(E, E) = \mathrm{Ext}^1(\mathcal{O}, E^* \otimes E) = H^1(X, \operatorname{ad} E),$$

this agrees with our claim that deformations of E are classified by $H^1(X, \operatorname{ad} E)$.

The invariant pairing on $\mathfrak{g}$ may be used to identify $(\operatorname{ad} E)^* \cong \operatorname{ad} E$. Using this identification and Serre duality, it follows that

$$\begin{aligned} T_E^* \mathrm{Bun}_G^\circ(X) = H^1(X, \operatorname{ad} E)^* &\cong H^0(X, \mathcal{K}_X \otimes (\operatorname{ad} E)^*) \\ &= H^0(X, \mathcal{K}_X \otimes \operatorname{ad} E). \end{aligned}$$

Elements of $H^0(X, \mathcal{K}_X \otimes \operatorname{ad} E)$ have a very concrete geometric interpretation: they are 1-forms on X with coefficients in $\operatorname{ad} E$. From physics, they have the name *Higgs fields* on E. The space of Higgs fields on E is usually denoted by $\Omega^1(X, \operatorname{ad} E)$. A pair (E, ϕ) where $\phi \in \Omega^1(X, \operatorname{ad} E)$ is called a *Higgs pair*.

5.3 The dimension of the variety of stable bundles

It remains to compute $\dim H^0(X, \mathcal{K}_X \otimes \mathrm{ad}\, E)$. To this end, we will use the following lemma.

Lemma 5.5. *Let V be a vector bundle on X of degree d and rank r. Then the Euler characteristic $\chi(X, V)$ equals $d - (g-1)r$.*

Proof. By the Riemann-Roch theorem, if L is a line bundle on X then

$$\chi(X, L) = \deg L - (g-1).$$

By Problem 6, V has a B-structure, i.e., a filtration $0 = V_0 \subset V_1 \subset \dots \subset V_r = V$ such that V_i/V_{i-1} are line bundles. Thus

$$\chi(X, V) = \sum_{i=1}^{r} \chi(X, V_i/V_{i-1}) = \sum_{i=1}^{r} (\deg(V_i/V_{i-1}) - (g-1)) = d - (g-1)r.$$

□

Now note that since the vector bundle $\mathrm{ad}E$ is self-dual, it has degree 0. Thus $\mathcal{K}_X \otimes \mathrm{ad}E$ has degree $2(g-1)\dim G$. So it follows from Lemma 5.5 that

$$\chi(X, \mathcal{K}_X \otimes \mathrm{ad}E) = (g-1)\dim G.$$

On the other hand, by Serre duality $\dim H^1(X, \mathcal{K}_X \otimes \mathrm{ad}\, E) = \dim H^0(X, \mathrm{ad}\, E)$, and $H^0(X, \mathrm{ad}\, E) = 0$ because stable bundles have no (infinitesimal) automorphisms.[10] Thus

$$\dim H^0(X, \mathcal{K}_X \otimes \mathrm{ad}E) = \dim H^0(X, \mathcal{K}_X \otimes \mathrm{ad}E) - \dim H^1(X, \mathcal{K}_X \otimes \mathrm{ad}\, E)$$
$$= \chi(X, \mathcal{K}_X \otimes \mathrm{ad}E) = (g-1)\dim G.$$

Hence we get

Proposition 5.6. *If G is semisimple then* $\dim \mathrm{Bun}_G^\circ(X) = (g-1)\dim G$.

For $G = \mathbb{G}_m$, we know that $\mathrm{Bun}_G(X) = \mathrm{Pic}(X)$ has dimension g. So for general reductive G, we have

Proposition 5.7.

$$\dim \mathrm{Bun}_G^\circ(X) = (g-1)\dim G + \dim Z(G).$$

Example 5.8. For $G = \mathrm{GL}_n$, we have

$$\dim \mathrm{Bun}_G^\circ(X) = (g-1)n^2 + 1.$$

[10] This is where we use that G is semisimple, because otherwise generic G-bundles still have infinitesimal automorphisms, and so $\dim H^1$ will not vanish in the Euler characteristic.

6 The classical Hitchin integrable system

6.1 The classical Hitchin system for GL_n, SL_n and PGL_n

We are now ready to introduce the main character of our story — the Hitchin integrable system. We will take $\mathbf{k} = \mathbb{C}$. As a warm-up, we begin with $G = \mathrm{SL}_n$ or PGL_n. In this case, as we've just shown,

$$\dim \mathrm{Bun}_G^\circ(X) = (n^2 - 1)(g - 1),$$

and elements in $\mathrm{Bun}_G^\circ(X)$ are pairs (E, ϕ) where E is a stable bundle and $\phi \in \Omega^1(X, \mathrm{End}\, E)$ is a Higgs field with trace zero.

Definition 6.1 (Hitchin system for $G = \mathrm{SL}_n$)**.** The *Hitchin base* is the vector space

$$\mathcal{B} = \mathcal{B}_{X,G} \coloneqq \bigoplus_{i=1}^{n-1} H^0(X, \mathcal{K}_X^{\otimes(i+1)}).$$

The *Hitchin map* is

$$p \colon T^* \mathrm{Bun}_G^\circ(X) \to \mathcal{B}, \quad p(E, \phi) \coloneqq (\mathrm{tr} \wedge^2 \phi, -\mathrm{tr} \wedge^3 \phi, \ldots, (-1)^n \mathrm{tr} \wedge^n \phi).$$

Remark 6.2. Note that the fibers of $\mathrm{ad}\, E$ are isomorphic to $\mathfrak{sl}_n$ non-canonically; namely, the isomorphism is well defined only up to inner automorphisms. However, since the functions $\mathrm{tr} \wedge^i A$ of a matrix A are conjugation-invariant, $\mathrm{tr} \wedge^i \phi$ is still well defined for any Higgs field ϕ.

By the Riemann–Roch theorem, $\dim H^0(X, \mathcal{K}_X^{\otimes(i+1)}) = (2i + 1)(g - 1)$, and so the dimension of the Hitchin base is

$$\dim \mathcal{B} = \sum_{i=1}^{n-1} (2i + 1)(g - 1) = (n^2 - 1)(g - 1) = \dim \mathrm{Bun}_G^\circ(X).$$

Theorem 6.3 (Hitchin, [H])**.** *The map p is generically a Lagrangian fibration*[11]*, i.e., defines an integrable system.*

This means that coordinate functions on $\mathcal{B}$, pulled back by p, are Poisson-commuting and functionally independent. Explicitly, if we choose a basis $b_1, \ldots, b_d \in \mathcal{B}$ (say, compatible with the direct sum decomposition) and write

$$p(E, \phi) = \sum_{j=1}^{d} H_j(E, \phi) b_j,$$

[11] A Lagrangian fibration is a fibration with Lagrangian fibers.

then $H_j(E,\phi)$ are the Poisson-commuting *classical Hitchin hamiltonians*, forming an integrable system.

We will prove Theorem 6.3 in Subsections 6.4,6.5 and 6.6.

Note that Theorem 6.3 extends verbatim to the equivalent case $G = \mathrm{GL}_n$. In this case, we just need to include a linear term into the Hitchin map. Namely, if we set

$$\mathcal{B} = \mathcal{B}_{X,G} := \bigoplus_{i=0}^{n-1} H^0(X, \mathcal{K}_X^{\otimes(i+1)})$$

and

$$p\colon T^*\operatorname{Bun}_G^\circ(X) \to \mathcal{B}, \quad p(E,\phi) := (-\operatorname{tr}\phi, \operatorname{tr}\wedge^2\phi, -\operatorname{tr}\wedge^3\phi, \dots, (-1)^n \operatorname{tr}\wedge^n\phi).$$

(the full collection of coefficients of the characteristic polynomial $\det(\lambda-\phi)$ of ϕ), then Theorem 6.3 with the same formulation (and proof) holds for GL_n.

Remark 6.4. Another (equivalent) choice for Hitchin hamiltonians is the collection of components of $(\operatorname{tr}\phi, \operatorname{tr}\phi^2, \dots, \operatorname{tr}\phi^n)$.

Example 6.5. If $G = \mathrm{GL}_1$ then the Hitchin hamiltonians (say, for degree zero bundles) are just the momenta $p_i := (\mathrm{tr}\phi)_i$, $i = 1, \dots, g$, on $T^*\mathrm{Jac}(X) = \mathrm{Jac}(X) \times H^0(\mathcal{K}_X)$.

6.2 Classical Hitchin system for general G

To generalize the classical Hitchin system to an arbitrary reductive group G, we first need to recall *Chevalley's theorem* for the Lie algebra $\mathfrak{g} := \mathrm{Lie}G$:

Theorem 6.6 (see [E1, Section 10])**.** *The algebra $\mathbb{C}[\mathfrak{g}]^G$ of G-invariant polynomials on $\mathfrak{g}$ is a polynomial algebra $\mathbb{C}[Q_1, \dots, Q_r]$, where $r := \operatorname{rank} G$ and Q_i are homogeneous polynomials.*

Let $d_i := \deg Q_i$, so that $d_1 \le d_2 \le \cdots \le d_r$. These numbers do not depend on the choice of Q_i and are called the *degrees* of G (or $\mathfrak{g}$), and it is clear that the number of i such that $d_i = 1$ is $\dim Z(G)$. Moreover, if $e := \sum_{i=1}^r e_i \in \mathfrak{g}$ is the regular nilpotent then $\operatorname{ad} e$ is a direct sum of Jordan blocks of sizes $2d_i - 1$, $1 \le i \le r$ ([E1, Lemma 17.1]). Thus

$$\sum_{i=1}^r (2d_i - 1) = \dim G. \tag{10}$$

Example 6.7. $\mathbb{C}[\mathfrak{gl}_n]^{\mathrm{GL}_n} = \mathbb{C}[Q_1, Q_2, \dots, Q_n]$, where $Q_m(A) := (-1)^m \operatorname{tr}\wedge^m A$. Thus the degrees of GL_n are $1, 2, \dots, n$. Similarly, $\mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} = \mathbb{C}[Q_2, \dots, Q_n]$. Thus the degrees of SL_n or PGL_n are $2, \dots, n$.

Let $Q \in \mathbb{C}[\mathfrak{g}]^G$ be homogeneous of degree m, and let (E, ϕ) be a Higgs pair. A conjugation-invariant function like Q may be evaluated fiberwise on $\phi \in \Omega^1(X, \operatorname{ad} E)$ to produce elements

$$Q(\phi) \in H^0(X, \mathcal{K}_X^{\otimes m}).$$

The Hitchin base should therefore be

$$\mathcal{B} = \mathcal{B}_{X,G} := \bigoplus_{i=1}^{r} H^0(X, \mathcal{K}_X^{\otimes d_i}).$$

Definition 6.8 (Hitchin system for general G)**.** The *Hitchin map* is

$$p \colon T^* \operatorname{Bun}_G^\circ(X) \to \mathcal{B}_{X,G}, \quad p(E, \phi) := (Q_1(\phi), \dots, Q_r(\phi)).$$

Recall that $\dim H^0(X, \mathcal{K}_X^{\otimes m})$ equals g if $m = 1$ and $(2m-1)(g-1)$ if $m > 1$. Hence

$$\dim \mathcal{B} = \sum_{i=1}^{r} (2d_i - 1)(g-1) + |\{i : d_i = 1\}| = (g-1) \dim G + \dim Z(G) = \dim \operatorname{Bun}_G^\circ(X).$$

Thus, we might hope that p is an integrable system. And this indeed turns out to be the case.

Theorem 6.9 (Hitchin [H], Faltings [Fa1], Ginzburg [Gi]; see a discussion in [BD1], 2.2.4 and 2.10)**.** *Theorem 6.3 holds for any connected reductive group G.*

The restriction of the Hitchin map to the locus where it is a Lagrangian fibration is called the *Hitchin fibration*.

Hitchin proved Theorem 6.9 for classical groups G, and then Faltings and later Ginzburg proved it for general G. The proof has two parts:

1. showing that the coordinates of $Q_j(\phi)$ are in involution (i.e., Poisson-commute);
2. showing that they are functionally independent.

Functional independence is equivalent to p being a dominant map, meaning that the image of p contains an open dense subset.

Below we will discuss part 1 for general G and part 2 for $G = \mathrm{GL}_n$.

Remark 6.10. It is known ([BD1]) that Hitchin hamiltonians are the only global regular functions on the cotangent bundle $T^*\operatorname{Bun}_G(X)$ to the stack $\operatorname{Bun}_G(X)$, and usually the same is true for $T^*\operatorname{Bun}_G^\circ(X)$ (as unstable bundles occur in codimension ≥ 2); for instance, this is obvious for $G = GL_1$ (Example 6.5). For this reason, we inevitably encounter the Hitchin system as soon as we start doing geometry or analysis on $\operatorname{Bun}_G(X)$.

6.3 Hamiltonian reduction

For the proof of Hitchin's theorem, we must first review the *Marsden–Weinstein symplectic (or hamiltonian) reduction* (see [MR] for more details).

Let Y be a manifold (or variety), and H be a Lie group (or algebraic group) acting on Y on the right. In this case, H acts by hamiltonian automorphisms on T^*Y, and so there is a moment map

$$\mu\colon T^*Y \to \mathfrak{h}^*$$

where $\mathfrak{h}$ is the Lie algebra of H. This is defined to be dual to the infinitesimal action map

$$a\colon \mathfrak{h} \to \mathrm{Vect}(Y) = \Gamma(Y, T_Y),$$

meaning that

$$\mu(x,p)(b) := \langle p, a(b)_x\rangle, \qquad \forall (x,p) \in T^*Y,\ b \in \mathfrak{h}.$$

Theorem 6.11 (Marsden–Weinstein reduction)**.** *If H acts freely on Y then 0 is a regular value of μ (so $\mu^{-1}(0)$ is smooth) and the quotient $\mu^{-1}(0)/H$ has a natural symplectic structure. Furthermore, there is a natural isomorphism of symplectic manifolds*

$$\mu^{-1}(0)/H \cong T^*(Y/H).$$

This can be used to construct integrable systems as follows. Suppose $\dim Y/H = n$, and $F_1, \ldots, F_n$ are H-invariant functions on T^*Y which are in involution ($\{F_i, F_j\} = 0$). Then they define functions $\overline{F}_i$ on the quotient $T^*(Y/H)$, by first restricting F_i to the zero fiber $\mu^{-1}(0) \subset T^*Y$ of the moment map, and then descending to the quotient by H. It is easy to check that $\{\overline{F}_i, \overline{F}_j\} = 0$. There is already the right number of them to form an integrable system; if in addition they are functionally independent, then $\overline{F}_1, \ldots, \overline{F}_n$ form an integrable system on $T^*(Y/H)$.[12]

Remark 6.12. In general, especially when there is no effective way of writing the functions explicitly, it is difficult to check whether a set of functions are in involution. But in this method of constructing integrable systems, sometimes the functions F_i are in involution on T^*Y for silly reasons. This happens, for instance, when Y is a vector space and the F_i all depend only on the momentum coordinates on $T^*Y = Y \times Y^*$, or, more generally, if Y is a Lie group and F_i depend only on momenta on $T^*Y = Y \times Y^*$ and are conjugation invariant. In fact, this is exactly what is going to happen in the case of Hitchin systems.

[12]Note however that there are too few functions to form an integrable system on T^*Y, so it was necessary to descend to the quotient to obtain an integrable system.

6.4 Proof for Hitchin's theorem, part 1

The proof is based on an infinite dimensional version of hamiltonian reduction, which involves additional subtleties. While this can be done with full rigor, here we will ignore these subtleties and pretend that we are in the finite dimensional case.

Our job is to show that the coordinates of $Q_j(\phi)$ Poisson-commute. Without loss of generality assume that G is semisimple, and let $x \in X$. Then by Proposition 4.1

$$\mathrm{Bun}_G(X) = G(R)\backslash G(K)/G(\mathcal{O}),$$

where $R \coloneqq R_x,\ K \coloneqq K_x,\ \mathcal{O} \coloneqq \mathcal{O}_x$. Denote the preimage of $\mathrm{Bun}_G^\circ(X)$ in $G(K)$ by $G^\circ(K)$. Then the group $G(R) \times G(\mathcal{O})$ acts on $G^\circ(K)$ with stabilizer $Z(G)$. So we may express $T^* \mathrm{Bun}_G^\circ(X)$ as the hamiltonian reduction of $T^*G^\circ(K)$ by the free action of $(G(R) \times G(\mathcal{O}))/Z(G)$.

We will now construct some $(G(R)\times G(\mathcal{O}))/Z(G)$-invariant functions on $T^* \mathrm{Bun}_G^\circ(X)$ which manifestly Poisson commute, and then descend them to the hamiltonian reduction to get the Hitchin hamiltonians. To this end, let us trivialize the cotangent bundle of $T^*G^\circ(K)$ by left (or right) translations. Thus we obtain an isomorphism $T^*G^\circ(K) \cong G^\circ(K)\times\mathfrak{g}(K)^*$. Observe that the Lie algebra $\mathfrak{g}(K)$ carries a $G(K)$-invariant pairing, given by

$$\langle a(t), b(t)\rangle_{\mathfrak{g}(K)} \coloneqq \mathrm{Res}_{t=0}\langle a(t), b(t)\rangle_{\mathfrak{g}}\, dt,$$

where t is a formal coordinate near x. This allows us to identify $\mathfrak{g}(K)^*$ with $\mathfrak{g}((t))\, dt$. Hence we get an isomorphism

$$T^*G^\circ(K) \cong G^\circ(K) \times \mathfrak{g}((t))\, dt.$$

Thus points of $T^*G^\circ(K)$ can be viewed as pairs $(g, \widetilde{\phi})$, where $g \in G^\circ(K)$ and $\widetilde{\phi}$ is a Higgs field on $D_x^\times$. So, as before, we can apply invariant functions $\{Q_i\}_{i=1}^r$ on $\mathfrak{g}$ to $\widetilde{\phi}$ to get the functions

$$H_{i,n} \coloneqq c_n(Q_i(\widetilde{\phi})), \quad 1 \le i \le r,\ n \in \mathbb{Z}$$

(the n-th Laurent coefficient of $Q_i(\widetilde{\phi})$) on $T^*G^\circ(K)$.

Observe that by construction the functions $H_{i,n}$ depend only on the momentum coordinates on $T^*G^\circ(K)$, and that the Poisson bracket of momenta on the cotangent bundle to a Lie group coincides with the commutator of the corresponding Lie algebra elements. Since in addition the $H_{i,n}$ are invariant, we obtain that

$$\{H_{i,n}, H_{j,m}\} = 0.$$

Now, it is easy to check that the following square commutes, with $H_i := Q_i(\widetilde{\phi})$:

$$\begin{array}{ccc} T^*G^\circ(K) & \xrightarrow{(H_1,\dots,H_r)} & \bigoplus_{i=1}^r \mathbf{k}((t))\,(dt)^{d_i} \\ \uparrow & & \uparrow \\ \mu^{-1}(0) & & \\ \downarrow & & \\ T^*\operatorname{Bun}_G^\circ(X) & \xrightarrow{\;p\;} & \bigoplus_i H^0(X, \mathcal{K}_X^{\otimes d_i}) \end{array} \tag{11}$$

where the vertical arrow on the right is just the Taylor expansion at x of global differentials on X of degree d_i. Thus, since the Taylor expansion map is injective, Poisson-commutativity for the coordinates of the Hitchin map p follows from the Poisson-commutativity of $H_{i,n}$, as desired.

Remark 6.13. We see that the hamiltonians $H_{i,n}$ descend to 0 for $n < 0$ and to $c_n(Q_i(\phi))$ for $n \geq 0$.

6.5 Spectral curves

Now we would like to prove functional independence of the Hitchin hamiltonians for $G = \mathrm{GL}_n$ (the case $n = 1$ is trivial, so we will assume that $n \geq 2$). This will use an important technique which appears across the field of integrable systems: the theory of *spectral curves.*

Take a point

$$b = (b_1, \dots, b_n) \in \mathcal{B} = \bigoplus_{i=0}^{n-1} H^0(X, \mathcal{K}_X^{\otimes(i+1)}).$$

Consider the polynomial

$$\lambda^n + b_1\lambda^{n-1} + \cdots + b_n =: \prod_{i=1}^{n}(\lambda - \lambda_i).$$

Since $b_m \in H^0(X, \mathcal{K}_X^{\otimes m})$, the quantities λ_i are all 1-forms on X (more precisely, they are branches of a multivalued 1-form). So

$$\{\lambda_1(x), \dots, \lambda_n(x)\} \subset T_x^*X.$$

Varying $x \in X$ produces a closed subset $C_b \subset T^*X$ associated to b - the graph, or Riemann surface, of the multi-valued 1-form $\lambda(x)$ defined by the equation

$$\lambda^n + b_1\lambda^{n-1} + \cdots + b_n = 0.$$

In fact, it is clear that C_b is a projective algebraic curve inside T^*X defined by the equation

$$y^n + b_1(x)y^{n-1} + \cdots + b_n(x) = 0,$$

where y is the coordinate along the cotangent fibers, and that the natural projection $\pi\colon C_b \to X$ has degree n.

Definition 6.14. C_b is called the *spectral curve* of b and the map π is called the *spectral cover.*

Now let $(E, \phi) \in T^* \operatorname{Bun}_G^\circ(X)$. Applying p produces $p(E, \phi) = (b_1, \ldots, b_n)$ where the b_i are just the coefficients of the characteristic polynomial of ϕ. In other words,

$$\lambda^n + b_1\lambda^{n-1} + \cdots + b_n = \det(\lambda - \phi),$$

and the $\lambda_i(x)$ are just the eigenvalues of $\phi(x)$, for $x \in X$. So the curve $C(E, \phi) \coloneqq C_{p(E,\phi)}$ is traced out in T^*X by the spectrum of $\phi(x)$ as $x \in X$ varies, which explains the term "spectral curve". (It is important to keep in mind that the eigenvalues are 1-forms, so they naturally live in T^*X).

Theorem 6.15 ([H])**.** *The spectral curve C_b is smooth and irreducible for generic $b \in \mathcal{B}$.*

Proof. Note that smoothness and irreducibility of C_b are open conditions with respect to b. Thus it suffices to show that there exists at least one b for which C_b is smooth and irreducible.

Lemma 6.16. *(i) If L is a line bundle on X of degree $d \geq 2g$ then a generic section of L has only simple zeros.*

(ii) A generic section of $\mathcal{K}_X^{\otimes n}$ has only simple zeros.

Proof. (i) The Riemann-Roch theorem implies that if M is a line bundle on X of degree $m \geq 2g - 2$ then $\dim H^0(X, M) = m - g + 1$ unless $m = 2g - 2$, $M = \mathcal{K}_X$, in which case $\dim H^0(X, M) = m - g + 2 = g$. This means that the variety Y of sections of L with a double zero has dimension $d - g$. Indeed, such a section is determined by the position x of the double zero on X and a section of the line bundle $M \coloneqq L \otimes \mathcal{O}(-2x)$, and $H^0(X, L \otimes \mathcal{O}(-2x))$ has dimension $d - g - 1$ unless $L \otimes \mathcal{K}_X^{-1} = \mathcal{O}(2x)$. But this can only happen for finitely many x and in this case $H^0(X, L \otimes \mathcal{O}(-2x)) = d - g$. Thus $Y \neq H^0(X, L) \cong \mathbb{C}^{d-g+1}$, which implies the statement.

(ii) follows from (i) and the fact that the degree of $\mathcal{K}_X^{\otimes n}$ is $2n(g - 1)$, which is $\geq 2g$ for $g \geq 2$. □

By Lemma 6.16(ii), a generic section s of $\mathcal{K}_X^{\otimes n}$ has only simple zeros. Thus for $b \coloneqq (0, \ldots, 0, s)$, the spectral curve C_b given by the equation $y^n = s(x)$ in T^*X is smooth and irreducible, as desired. □

Theorem 6.17 ([H]). *The genus of the spectral curve for generic b is given by the formula*

$$g(C_b) = n^2(g-1)+1.$$

Proof. Since $g(C_b)$ is deformation-invariant, we can compute it at the point

$$b_1 = b_2 = \cdots = b_{n-1} = 0,\ b_n = s,$$

where s is a section of $\mathcal{K}_X^{\otimes n}$ with simple zeros, whose existence is guaranteed by Lemma 6.16(ii). Then the degree n map $\pi\colon C_b \to X$ is unramified except at the zeros of the section s. So $g(C_b)$ may be computed by the Riemann–Hurwitz formula. Namely, since $s \in H^0(X, \mathcal{K}_X^{\otimes n})$ is a section of a degree $2n(g-1)$ bundle, it has $2n(g-1)$ zeros. Thus the Riemann–Hurwitz formula for the Euler characteristic of C_b gives

$$\chi(C_b) = (\chi(X) - 2n(g-1))\, n + 2n(g-1) = -n^2(2g-2).$$

(as $\chi(X) = 2-2g$). Thus

$$g(C_b) = \frac{2-\chi(C_b)}{2} = n^2(g-1)+1. \qquad \square$$

6.6 Proof for Hitchin's theorem, part 2, for $G = \mathrm{GL}_n$

Let us now proceed with part 2 of the proof of Hitchin's theorem. If we have a Higgs pair (E, ϕ) such that the spectral curve of $C(E, \phi)$ is C, then there is an *eigenline bundle* L_ϕ on C. Namely, the fiber of L_ϕ at a point $\lambda \in C$ lying over $x \in X$ is the eigenline of $\phi(x)$ with eigenvalue λ, in the generic situation where all the eigenvalues are distinct. By analyzing the ramification points, it is easy to show that L_ϕ extends naturally to the whole C.

The vector bundle E is then reconstructed from C and L_ϕ as the pushforward of L_ϕ:

$$E \cong \pi_* L_\phi$$

(algebraically, this means that Zariski locally on X we regard the $\mathcal{O}_C$-module L_ϕ as an $\mathcal{O}_X$-module using the map $\pi^*\colon \mathcal{O}_X \to \mathcal{O}_C$). This is because the fiber E_x is the direct sum $\bigoplus_{\lambda\in\pi^{-1}(x)}(L_\phi)_\lambda$ when all the eigenvalues are distinct.

Moreover, we can also recover ϕ from L_ϕ, because the action of ϕ on E is just multiplication by the cotangent coordinate on L_ϕ.

The line bundle L_ϕ on C_b has some degree d which is independent of b since it is a topological invariant. (One can compute d explicitly if desired, see [H], but we do not need it here.) Then

$$L_\phi \in \mathrm{Pic}_d(C_b) \cong \mathrm{Jac}(C_b)$$

and so $p^{-1}(b) \subset T^* \operatorname{Bun}_G^\circ(X)$ gets identified with a subset of $\operatorname{Jac}(C_b)$. Thus for generic b,

$$\dim p^{-1}(b) \leq n^2(g-1)+1$$

(the dimension of $\operatorname{Bun}_G^\circ(X)$). So it follows from part 1 that p is generically a Lagrangian fibration, as desired.

Remark 6.18. We see that for $G = \mathrm{GL}_n$ a generic fiber $p^{-1}(b)$ of the Hitchin map p is an open set in an abelian variety - the Jacobian of the spectral curve C_b. Similarly, for $G = \mathrm{SL}_n$, $p^{-1}(b)$ is generically an open set in the kernel of $\pi_\bullet\colon \operatorname{Jac}(C_b) \to \operatorname{Jac}(X)$, where $\pi_\bullet$ is the *norm map* of Jacobians induced by π (this kernel is also an abelian variety but not a Jacobian, in general; it is called the *Prym variety* of the map π). The missing points of these abelian varieties are added back when one extends p to the Hitchin moduli space $\mathcal{M}_G(X)$ (see Remark 5.3), as the extended Hitchin map $p : \mathcal{M}_G(X) \to \mathcal{B}$ is proper ([H]). This is consistent with the Liouville-Arnold theorem ([A]), which says that fibers of a Lagrangian fibration (i.e., common level sets of an integrable system), when compact and connected, are Lagrangian tori, and the Hamiltonian dynamics is a linear flow on each of them.

7 Classical Hitchin systems for G-bundles with parabolic structures and twisted classical Hitchin systems

7.1 Principal bundles with parabolic structures

Let G be a connected reductive group. Recall that a closed subgroup $P \subset G$ is called *parabolic* if it contains a Borel subgroup, or, equivalently, if the quotient G/P is a projective variety (called the *partial flag variety* corresponding to P). For a subdiagram D of the Dynkin diagram D_G of G, there is a parabolic subgroup $P(D) \subset G$, the connected subgroup whose Lie algebra is generated by the Chevalley generators e_i, h_i, $i \in D_G$ and f_i, $i \in D$, and every parabolic subgroup is conjugate to $P(D)$ for a unique D. Borel subgroups B are the smallest parabolics — those conjugate to $P(\emptyset)$ — and then $G/P = G/B$ is the (full) *flag variety*. For instance, if $G = \mathrm{GL}_n$, then subdiagrams of the Dynkin diagram correspond to compositions $n = n_1 + \cdots + n_r$, and the corresponding parabolic subgroup $P(n_1, \ldots, n_r)$ consists of upper block-diagonal matrices with blocks of size $n_1, \ldots, n_r$. The partial flag variety G/P is then the variety $\mathcal{F}_{n_1,\ldots,n_r}(\mathbf{k}^n)$ of partial flags $0 = V_0 \subset V_1 \subset \ldots \subset V_r = \mathbf{k}^n$ such that $\dim V_i/V_{i-1} = n_i$ for all i (note that this description depends only on $n_1, \ldots, n_r$, and not the choice of P). The smallest parabolics are the Borel subgroups, conjugate to the group $P(1, \ldots, 1)$ of upper triangular matrices, and then G/B is the variety $\mathcal{F}(n) = \mathcal{F}_{1,\ldots,1}(\mathbf{k}^n)$ of full flags in $\mathbf{k}^n$.

Now let X be a smooth irreducible projective curve, $t_1, \ldots, t_N \in X$, and $P_1, \ldots, P_N \subset G$ be parabolic subgroups. Denote by $\operatorname{Bun}_G(X, t_1, \ldots, t_N, P_1, \ldots, P_N)$ the moduli stack of G-bundles on X with P_i-structures at t_i for $i = 1, \ldots, N$. Points of this stack are often called *parabolic bundles*.

It is clear that we have a fibration

$$\theta\colon \operatorname{Bun}_G(X, t_1, \ldots, t_N, P_1, \ldots, P_N) \to \operatorname{Bun}_G(X)$$

(forgetting the parabolic structures) with fiber $G/P_1 \times \cdots \times G/P_N$.

Example 7.1. Let $\mathcal{E}$ be a GL_n-bundle on X and E denote the associated rank n vector bundle. Canonically, the fiber of $\mathcal{E}$ at a point $x \in X$ is $\mathcal{E}_x = \{\text{bases in } E_x\}$. In particular, if $P = P(n_1, \ldots, n_r)$ then P is the stabilizer of a partial flag

$$0 \subset V_1 \subset \cdots \subset V_r = \mathbf{k}^n$$

where the quotients V_i/V_{i-1} are vector spaces of dimensions n_i, for $i = 1, \ldots, r$. So a P-structure on E at x is the set of bases compatible with this flag, i.e. such that there is an nested sequence of subsets of the basis which are bases of the V_i. Thus, choosing a P-structure on E at x is equivalent to fixing a partial flag in E_x of type $(n_1, \ldots, n_r)$.

It follows that the fiber of θ at E is canonically $\theta^{-1}(E) = \prod_{j=1}^N \mathcal{F}_{n_{j1},\ldots,n_{jr_j}}(E_{t_j})$, where $(n_{j1}, \ldots, n_{jr_j})$ is the type of P_j.

The basic example we will consider is $G = \mathrm{GL}_2$ and $P_i = B$ is the Borel subgroup. A B-structure at t_i is then a choice of a line $\ell_i \subset E_{t_i}$, so the fiber of θ over E in this case is $\theta^{-1}(E) = \prod_{j=1}^N \mathbb{P}E_{t_j}$.

One reason to consider parabolic bundles is that it gives rise to interesting problems already for $g = 0$ and $g = 1$, which leads to nice explicit formulas. Indeed, without parabolic structures, there are no stable bundles for $g < 2$; all bundles have non-trivial automorphism groups, even if G is adjoint. However in presence of parabolic structures, automorphisms must preserve it, so the automorphism group shrinks. In particular, for a sufficient number of marked points there will be a lot of bundles with trivial automorphism group.

For example, if $N \geq 3$ and G is adjoint, then a generic G-bundle on $\mathbb{P}^1$ with parabolic structures has trivial automorphism group. For example, consider $G = \mathrm{PGL}_2$ and let E be the trivial G-bundle on X. Then $\operatorname{Aut}(E) = \mathrm{PGL}_2$. But we have parabolic structures $\ell_1, \ldots, \ell_N$, where $\ell_i \in \mathbb{P}E_{t_i} = \mathbb{P}^1$ for each i. So the set $\operatorname{Bun}_G^{\mathrm{triv}}(X, t_1, \ldots, t_N, P_1, \ldots, P_N)$ of parabolic structures on the trivial bundle is just $[(\mathbb{P}^1)^N / \mathrm{PGL}_2]$. This is still stacky, because when the N points $\ell_1, \ldots, \ell_N$ in $\mathbb{P}^1$ coincide, there is still a non-trivial automorphism group. But we can consider the smaller open set given by points $(\ell_1, \ldots, \ell_N) \in (\mathbb{P}^1)^N$ such that $\ell_{N-2}, \ell_{N-1}, \ell_N$ are distinct. It is well-known that PGL_2 acts simply 3-transitively on $\mathbb{P}^1$, i.e. given three distinct points on $\mathbb{P}^1$, there is a unique element in PGL_2 which sends them to $(0, 1, \infty)$. Hence the open set parameterizing N-tuples of points with the last three points distinct is

$$(\mathbb{P}^1)^{N-3} \subset [(\mathbb{P}^1)^N / \mathrm{PGL}_2].$$

Note that it is a smooth projective variety.

In general,

$$\mathrm{Bun}_G^{\mathrm{triv}}(X, t_1, \dots, t_N, P_1, \dots, P_N) \cong \left(\prod_i G/P_i\right)/G$$

where the G-action is diagonal.

Remark 7.2. Like before, the Hitchin system for parabolic bundles should be defined on the cotangent bundle of the variety of *stable parabolic bundles.* There are many notions of stability for bundles with parabolic structures, depending on parameters called *weights* (see [MSe]), but for us it will not matter which notion we use, since it does not matter much on which open dense subvariety on the stack of bundles we initially define the system. In any case, in the examples we consider, the trivial bundle with generic parabolic structures will be stable.

7.2 Classical Hitchin systems with parabolic structures

Recall that we have constructed the Hitchin system by realizing $\mathrm{Bun}_G(X)$ as a double quotient of the loop group and then descending invariant functions from the cotangent bundle to the loop group to the double quotient. We may do the same when there are parabolic structures. Namely, recall that

$$\mathrm{Bun}_G(X) = G(R_{t_1,\dots,t_N})\Big\backslash \prod_i G(K_{t_i})\Big/ \prod_i G(\mathcal{O}_{t_i}),$$

where $R_{t_1,\dots,t_N} := \mathbf{k}[X \setminus \{t_1, \dots, t_N\}]$, and parabolic structures are local at each of the t_i, so we should modify the right quotient. Let $\mathrm{ev}\colon G(\mathbf{k}[[t]]) \to G$, $g(t) \mapsto g(0)$ be the evaluation function at $t = 0$, and let $\widetilde{P}_i := \mathrm{ev}^{-1}(P_i)$. In other words, $\widetilde{P}_i$ consists of Taylor series whose constant term lies in $P_i \subset G$. Then

$$\mathrm{Bun}_G(X, t_1, \dots, t_N, P_1, \dots, P_N) = G(R_{t_1,\dots,t_N})\Big\backslash \prod_i G(K_{t_i})\Big/ \prod_i \widetilde{P}_i.$$

The discrepancy between $\mathrm{Bun}_G(X, t_1, \dots, t_N, P_1, \dots, P_N)$ and $\mathrm{Bun}_G(X)$ is therefore $\prod_i G/P_i$, exactly as stated earlier.

Now consider the space $T^*(\prod_{j=1}^N G(K_{t_j}))$. This is the set of $(g_1, \dots, g_n, \widetilde{\phi}_1, \dots, \widetilde{\phi}_n)$ where $g_j \in G(K_{t_j})$ and $\widetilde{\phi}_j$ are Higgs fields on $D_{t_j}^\times$. So we may take the usual hamiltonians

$$H_{i,j,n} := c_n(Q_i(\widetilde{\phi}_j))$$

on this space (using some formal coordinates z_j near t_j) and do the same reduction as before but now with respect to the subgroup $G(R_{t_1,\dots,t_N}) \times \prod_i \widetilde{P}_i$. The result is an integrable system on $T^*\,\mathrm{Bun}_G^\circ(X, t_1, \dots, t_N, P_1, \dots, P_N)$. Points of this space are pairs

(E, ϕ) where E is a bundle with parabolic structures, and $\phi \in \Omega^1(X \setminus \{t_1, \ldots, t_N\}, \operatorname{ad} E)$ is a Higgs field *with tame singularities and nilpotent residues.* Namely, it is easy to check that the condition at the singulatities is that ϕ has at most first-order poles only at the points $t_1, \ldots, t_N$, and the residue $\operatorname{Res}_{t_i} \phi$ *strictly preserves* the flag $\{V_i\}$ specified by the parabolic structure at t_i. Here, "strictly" means that $\operatorname{Res}_{t_i} \phi$ lies in the unipotent radical of the Lie algebra $\mathfrak{p}_i := \operatorname{Lie} P_i$. For instance, in the $G = \mathrm{GL}_n$ case, this means that the residue preserves the flag and acts by 0 on the associated graded of $\mathbf{k}^n$ under the flag filtration.

7.3 Garnier system

Let us compute the classical Hitchin system for PGL_2 in genus $g = 0$. For this purpose, we will assume for convenience that $t_1, \ldots, t_N \in \mathbb{A}^1 \subset \mathbb{P}^1$ and that the parabolic structure at t_j is given by $y_j \in \mathbb{A}^1 \subset \mathbb{P}^1 = \mathrm{PGL}_2 / B$.

Let z be a coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$. The Higgs field ϕ is a 1-form with simple poles at t_j, valued in $\mathfrak{sl}_2$. So

$$\phi = \sum_{j=1}^{N} \frac{A_j}{z - t_j} \, dz, \qquad A_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \in \mathfrak{sl}_2,$$

satisfying the following conditions. First, ϕ must be regular at $\infty \in \mathbb{P}^1$ because there is no marking/puncture there. This is the case if and only if

$$\sum_{j=1}^{N} A_j = 0. \tag{12}$$

Second, $A_j \begin{pmatrix} y_j \\ 1 \end{pmatrix} = 0$ (in particular, since A_j has trace 0, it must be nilpotent). This is the condition

$$\begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \begin{pmatrix} y_j \\ 1 \end{pmatrix} = 0,$$

which says that $a_j = c_j y_j$ and $b_j = -a_j y_j$, so $b_j = -c_j y_j^2$. Hence

$$A_i = c_i \begin{pmatrix} y_i & -y_i^2 \\ 1 & -y_i \end{pmatrix}. \tag{13}$$

One may check that $c_j = p_j$ are the momentum coordinates, i.e., the symplectic form is $\omega = \sum_j dp_j \wedge dy_j$.

Let us now compute the Hitchin hamiltonians. The generating function for them is

$$H(z) = \frac{1}{2} \operatorname{tr} \phi^2 = \frac{1}{2} \operatorname{tr} \left(\sum_{i,j} \frac{A_i}{z - t_i} \frac{A_j}{z - t_j} \right)$$

(we drop the factor $(dz)^2$ for brevity). The $i = j$ terms drop out, because A_i is nilpotent and thus $A_i^2 = 0$. Thus we get

$$H(z) = \sum_{i<j} \frac{\operatorname{tr} A_i A_j}{(z - t_i)(z - t_j)}.$$

Using the identity

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right),$$

this can be rewritten as

$$H(z) = \sum_{i\neq j} \frac{\operatorname{tr} A_i A_j}{(t_i - t_j)(z - t_i)}.$$

It remains to compute the trace. Using (13), we have

$$\operatorname{tr} A_i A_j = p_i p_j \operatorname{tr} \begin{pmatrix} y_i & -y_i^2 \\ 1 & -y_i \end{pmatrix} \begin{pmatrix} y_j & -y_j^2 \\ 1 & -y_j \end{pmatrix} = -p_i p_j (y_i - y_j)^2.$$

Finally, let's take residues:

$$G_i \coloneqq \operatorname{Res}_{t_i} H(z) = \sum_{j\neq i} \frac{p_i p_j (y_i - y_j)^2}{t_j - t_i}.$$

These functions are called the *Garnier hamiltonians*. By construction, we have

$$\{G_i, G_j\} = 0.$$

on $T^*\mathbb{C}^N$, which can also be checked directly.

Unfortunately, the Garnier hamiltonians do not quite define an integrable system on $T^*\mathbb{C}^N$, since they are functionally (in fact, linearly) dependent. Namely, by (12) and (13) we have

$$\sum_i p_i = \sum_i p_i y_i = \sum_i p_i y_i^2 = 0, \tag{14}$$

so

$$\sum_i G_i = \sum_i t_i G_i = \sum_i t_i^2 G_i = 0;$$

in fact, there are only $N-3$ independent hamiltonians among the G_i. However, by (14), (y, p) belongs to $\mu^{-1}(0) \subset T^*\mathbb{C}^N$, where $\mu\colon T^*\mathbb{C}^N \to \mathfrak{sl}_2^*$ is the moment map for the PGL_2-action. Thus $\{G_i\}$ define $N-3$ independent Poisson-commuting hamiltonians on the hamiltonian reduction $\mathcal{M}_N \coloneqq \mu^{-1}(0)/\mathrm{PGL}_2$, which has dimension $2(N-3)$. Thus we obtained an integrable system on this variety, called the *Garnier system*.

Problem 8. Find the spectral curve of the Garnier system and compute its genus. Compute the genus explicitly for $N = 4$. (Hint: let a, b be coprime polynomials of x of degrees n_a and n_b, and with simple roots. What is the genus of the normalization of the affine curve $y^2 = a(z)/b(z)$?)

7.4 Twisted classical Hitchin systems

It turns out that another added benefit of Hitchin systems for bundles with parabolic structures is that they have a twisted generalization, which allows us to produce more general integrable systems depending on many parameters. To introduce them, we first need to define a more general version of hamiltonian reduction called *hamiltonian reduction along an orbit* (see e.g. [E2], Subsection 1.4).

Let M be a symplectic variety, with hamiltonian action by a group H. Let

$$\mu\colon M \to \mathfrak{h}^*$$

be a moment map. Previously, we considered the hamiltonian reduction $\mu^{-1}(0)/H$, but more generally, we may consider

$$\mu^{-1}(O)/H$$

for any *coadjoint H-orbit* $O \subset \mathfrak{h}^*$. If the H-action is sufficiently nice, this quotient also has a canonical symplectic structure, and we can run the same construction of integrable systems as before: if F_i are H-invariant functions in involution on M, then they descend to functions $\overline{F}_i$ on $\mu^{-1}(O)/H$, which are also in involution.

In particular, returning to our setting, recall that the group $G(R_{t_1,\dots,t_N}) \times \prod_i G(\mathcal{O}_{t_i})$ acts on $\prod_i G(K_{t_i})$, and let $\mathbf{K}$ denote the kernel of the evaluation map $\prod_i G(\mathcal{O}_{t_i}) \to G^N$ at $(t_1,\dots,t_N)$. We reduce first by $\mathbf{K}$, after which there is a residual action of G^N, and then for a coadjoint orbit $O \subset (\mathfrak{g}^*)^N$, we can descend the Hitchin hamiltonians to $\mu^{-1}(O)/G^N$. The result is called a *twisted Hitchin system*. In particular, the case of parabolic structures arises from specific choices of O (so-called *Richardson nilpotent orbits*).

7.5 Twisted Garnier system

As before, points in $\mu^{-1}(O)/G^N$ are Higgs pairs (E,ϕ) where ϕ must satisfy some conditions. To illustrate, take $G = \mathrm{PGL}_2$. At t_i, take the coadjoint orbit O_i in $\mathfrak{sl}_2^* \cong \mathfrak{sl}_2$ of regular elements with eigenvalues $\pm\lambda_i$, and let $O \coloneqq \prod_i O_i$. Then generically E is a trivial bundle with parabolic structures $\ell_i \in \mathbb{P}^1$ at t_i, and ϕ has simple poles at each t_i with

$$\mathrm{Res}_{t_i} \phi|_{\ell_i} = \lambda_i \cdot \mathrm{id}$$

(The usual story with parabolic structures corresponds to the case $\lambda_i = 0$.)

As before, writing the Higgs field as

$$\phi = \sum_i \frac{A_i}{z - t_i}\, dz,$$

we get

$$A_i \begin{pmatrix} y_i \\ 1 \end{pmatrix} = \lambda_i \begin{pmatrix} y_i \\ 1 \end{pmatrix}.$$

Solving this equation, we obtain

$$A_i = \begin{pmatrix} -\lambda_i + p_i y_i & 2\lambda_i y_i - p_i y_i^2 \\ p_i & \lambda_i - p_i y_i \end{pmatrix}.$$

The trace of $A_i A_j$ then becomes

$$\operatorname{tr}(A_i A_j) = -(y_i - y_j)^2 p_i p_j + 2(\lambda_i p_j - \lambda_j p_i)(y_i - y_j) + 2\lambda_i \lambda_j.$$

The resulting hamiltonians

$$G_i(\lambda_1, \ldots, \lambda_N) = \sum_{j \neq i} \frac{(y_i - y_j)^2 p_i p_j - 2(\lambda_i p_j - \lambda_j p_i)(y_i - y_j) - 2\lambda_i \lambda_j}{t_j - t_i}$$

define the *deformed* or *twisted* Garnier system. The ordinary Garnier system is a specialization of this, when $\lambda_i = 0$.

7.6 Classical elliptic Calogero-Moser system

Another known integrable system which is a special case of the (twisted) Hitchin system is the *elliptic Calogero-Moser system*. Let us explain how it arises in this way.

Let X be an elliptic curve with zero denoted $0 \in X$, and consider a generic vector bundle of degree 0 and rank n on X. Line bundles of degree 0 on X are all of the form

$$L_q = \mathcal{O}(q) \otimes \mathcal{O}(0)^{-1}$$

for a point $q \in X$, with a meromorphic section given by $\frac{\theta(z-q)}{\theta(z)}$, where $\theta(z)$ is the Jacobi theta-function of X. Atiyah ([At]) showed that a generic rank n vector bundle of degree zero on X has the form

$$E = L_{q_1} \oplus \cdots \oplus L_{q_n},$$

say with $q_i \neq q_j$. Consider $G = \mathrm{GL}_n$, put one puncture at 0, and perform the twisted reduction procedure for the orbit $O_c = \langle cT \rangle \subset \mathfrak{sl}_n^* \cong \mathfrak{sl}_n$ at this puncture, where

$$T \coloneqq 1 - nE_{nn} = \operatorname{diag}(1, \ldots, 1, 1-n).$$

As $c \to 0$, this orbit degenerates into the closure of the orbit O_0 of a rank 1 nilpotent matrix, corresponding to the parabolic subgroup P with blocks of size $(n-1) \times (n-1)$ and 1×1, i.e. $G/P = \mathbb{P}^{n-1}$.

So we will consider the twisted Hitchin system for $G = \mathrm{PGL}_n$ with a P-structure at 0. Note that the group $\mathrm{Aut}(E) = (\mathbb{C}^\times)^{n-1}$ acts on $\mathbb{P}^{n-1}$ with an open orbit generated by the vector $(1, 1, \ldots, 1)$, so generically we may assume that the P-structure at 0 is defined by this vector.

The components ϕ_{ij} of the Higgs field ϕ are sections of $L_{q_i} \otimes L_{q_j}^{-1}$ with at most a first-order pole at $z = 0$. Hence, for $i \neq j$

$$\phi_{ij} = a_{ij} \frac{\theta'(0)\theta(z + q_i - q_j)}{\theta(z)\theta(q_i - q_j)},$$

$a_{ij} \in \mathbb{C}$, and $\phi_{ii} = p_i$ are the momenta, i.e. the symplectic form is $\omega = \sum_j dp_j \wedge dq_j$.

What is the condition for the matrix $A = (a_{ij})$? Since we are considering the twisted Hitchin system corresponding to the orbit O_c, A should act on the vector $(1, \ldots, 1)$ with eigenvalue $(1 - n)c$:

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = (1 - n)c \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

and be conjugate to T. This means that all off-diagonal entries a_{ij} are equal to some constant c, so

$$\phi_{ij} = c \frac{\theta'(0)\theta(z + q_i - q_j)}{\theta(z)\theta(q_i - q_j)}, \ i \neq j; \ \phi_{ii} = p_i.$$

This ϕ is (up to conjugation) *Krichever's Lax matrix* for the *elliptic Calogero–Moser (CM) system*, [Kr2].

Thus the generating function for the quadratic Hitchin hamiltonians is

$$\operatorname{tr} \phi^2 = \sum_i p_i^2 - c^2 \sum_{j \neq i} \frac{\theta'(0)^2 \theta(z - q_i + q_j)\theta(z - q_j + q_i)}{\theta(z)^2 \theta(q_i - q_j)^2}.$$

The second summand on the right hand side is $c^2(\wp(z) - \wp(q_i - q_j))$, where $\wp(z)$ is the Weierstrass function, since both functions have the same zeros and poles and the same coefficient of $\frac{1}{z^2}$. Thus the constant term of $\operatorname{tr} \phi^2$ has the form

$$H_2 \coloneqq CT(\operatorname{tr} \phi^2) = \operatorname{tr} \phi^2 - n(n-1)c^2 \wp(z) = \sum_i p_i^2 - c^2 \sum_{i \neq j} \wp(q_i - q_j). \tag{15}$$

This is exactly the *quadratic hamiltonian in the elliptic CM system.*[13] The constant terms of the traces $\operatorname{tr} \phi^k$, $1 \leq k \leq n$ then yield the first integrals of the elliptic CM system. They have the form

$$H_k = \sum_i p_i^k + \text{lower order terms with respect to } p_i;$$

for example, $H_1 = \sum_i p_i$ and H_2 is given by (15). Thus H_k are independent (as so are their leading terms, the power sum functions $\sum_i p_i^k$), which establishes complete integrability of the elliptic CM system.

[13] Note that c is not an essential parameter here since it can be rescaled by renormalizing time. Thus physically there are just two distinct cases, corresponding to positive or negative c^2 (attracting vs. repelling potential).

Exercise 7.3. Show that for $N = 4$ the Garnier system is equivalent to the elliptic CM flow for 2 particles. What is a geometric reason for it? Solve the Hamilton equation of the flow.

Exercise 7.4. Calculate the first integral H_3 for the elliptic CM system for $c = 1$. Namely, show that it has the form

$$H_3 = \sum_{i=1}^{N} p_i^3 + \sum_{i=1}^{N} a_i(q_1, \ldots, q_N)p_i + b(q_1, \ldots, q_N)$$

and compute the functions a_i and b.

Exercise 7.5. Let $L = \partial_z^2 - 2\sum_{i=1}^{N} \wp(z - q_i)$. Show that L commutes with a third-order differential operator if and only if $(q_1, \ldots, q_N)$ is a critical point of the Calogero–Moser potential $\sum_{j>i} \wp(q_i - q_j)$.

8 Quantization of Hitchin systems

8.1 Quantization of symplectic manifolds

What is a quantum integrable system, and what does it mean to quantize a classical integrable system? This is not even the most basic question, since classical integrable systems live on symplectic manifolds, so we should first say what it means to quantize a symplectic manifold. This story can be told for smooth manifolds, complex analytic manifolds, or algebraic varieties over any field.

Let M be a symplectic variety. We will pretend that M is affine so that there is no need to think about sheaves. Then the algebra $\mathcal{O}(M)$ of regular functions on M is a *Poisson algebra*, meaning that there is a Lie bracket $\{-, -\}$ on $\mathcal{O}(M)$ which is a derivation in the first (hence second) argument. Such Lie bracket is called a *Poisson bracket*. In classical mechanics, M is the phase space, and $\mathcal{O}(M)$ is the algebra of observables. Quantization means that observables are replaced by operators which may no longer commute. More precisely, we make the following definition.

Definition 8.1. A *quantization* of $\mathcal{O}(M)$ over $\mathbf{k}[\hbar]$ or $\mathbf{k}[[\hbar]]$ is an associative algebra $(A, *)$ over this ring equipped with an algebra isomorphism $\kappa \colon A/\hbar A \cong \mathcal{O}(M)$ so that A is a flat deformation of $A/\hbar A$ and

$$\lim_{\hbar \to 0} \frac{f * g - g * f}{\hbar} = \{f, g\}.$$

Here "flat deformation" means that $A \cong \mathcal{O}(M)[\hbar]$ as a $\mathbf{k}[\hbar]$-module (respectively $A \cong \mathcal{O}(M)[[\hbar]]$ as a $\mathbf{k}[[\hbar]]$-module) compatibly with κ.

Recall that if $A = \cup_{i\geq 0} F_i A$ is a $\mathbb{Z}_+$-filtered $\mathbf{k}$-algebra then the associated graded algebra $\mathrm{gr}(A)$ is

$$\mathrm{gr}(A) = \bigoplus_{i\geq 0} \mathrm{gr}(A)_i, \quad \mathrm{gr}(A)_i := F_i A / F_{i-1} A.$$

For $a \in F_i A$, let $p_i(a)$ be its image in $\mathrm{gr}(A)_i$. If $\mathrm{gr}(A)$ is commutative then it carries a Poisson bracket of degree -1 such that if $a \in F_i A$ and $b \in F_j A$ then

$$\{p_i(a), p_j(b)\} = p_{i+j-1}([a, b]).$$

In this case A is said to be a *filtered quantization* of $(\mathrm{gr}(A), \{,\})$.

Recall that the *Rees algebra* of A is

$$\mathrm{Rees}(A) = \sum_{i\geq 0} \hbar^i F_i(A) \subset A[\hbar].$$

It is easy to see that $\mathrm{Rees}(A)$ is a free $\mathbf{k}[\hbar]$-module, $\mathrm{Rees}(A)/\hbar\mathrm{Rees}(A) = \mathrm{gr}(A)$, $\mathrm{Rees}(A)/(\hbar - \hbar_0)\mathrm{Rees}(A) \cong A$ for any $\hbar_0 \in \mathbf{k}^\times$, and that if $\mathrm{gr}(A)$ is commutative then $\mathrm{Rees}(A)$ is a quantization of $(\mathrm{gr}(A), \{,\})$ over $\mathbf{k}[\hbar]$.

Example 8.2. Suppose $M = T^*Y$ for a smooth affine variety Y over $\mathbf{k}$ ($\mathrm{char}(\mathbf{k}) = 0$). Let $\mathcal{D}(Y)$ be the algebra of differential operators on Y, i.e., operators on $\mathcal{O}(Y)$ which in local coordinates look like

$$D = \sum_\alpha f_\alpha(x) \partial^\alpha, \tag{16}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}$ is a multi-index and $\partial^\alpha := \prod_i \partial_i^{\alpha_i}$. We have a filtration $\mathcal{D}(Y) = \cup_{N\geq 0} \mathcal{D}_{\leq N}(Y)$, where $\mathcal{D}_{\leq N}(Y)$ is the space of operators of order $\leq N$, i.e., those given by (16) such that all terms have $\sum_i \alpha_i \leq N$. Then $\mathcal{D}(Y)$ is a filtered quantization of $\mathcal{O}(T^*Y)$ with filtration by order, so the algebra of *quasiclassical differential operators*

$$\mathcal{D}_\hbar(Y) := \mathrm{Rees}(\mathcal{D}(Y))$$

is a quantization of T^*Y over $\mathbf{k}[\hbar]$.

It is easy to see that $\mathcal{D}_\hbar(Y)$ is generated over $\mathbf{k}[\hbar]$ by $\mathcal{O}(Y)$ and elements $\hbar v$ where v is a vector field on Y. For instance, if $Y = \mathbf{k}^n$ then $\mathcal{D}_\hbar(Y)$ is generated over $\mathbf{k}[\hbar]$ by x_i and $\widehat{p}_i := \hbar\partial_i$, with defining relations the *Heisenberg uncertainty relations*

$$[\widehat{p}_i, x_j] = \hbar\delta_{ij}, \quad [\widehat{p}_i, \widehat{p}_j] = [x_i, x_j] = 0.$$

Exercise 8.3. (i) (Grothendieck's definition of differential operators) Show that an operator $D\colon \mathcal{O}(Y) \to \mathcal{O}(Y)$ is a differential operator of order $\leq N$ if and only if for any $f_1, \dots, f_{N+1} \in \mathcal{O}(Y)$ one has

$$\mathrm{ad}(f_1) \cdots \mathrm{ad}(f_{N+1})(D) = 0,$$

where f_i denotes the operator of multiplying by f_i.

(ii) Let Y be a smooth affine algebraic variety over $\mathbb{C}$. Show that the algebra of differential operators $\mathcal{D}(Y)$ on Y is generated by $\mathcal{O}(Y)$ and elements ∇_v attached $\mathcal{O}(Y)$-linearly to vector fields v on Y with defining relations

$$[\nabla_v, f] = v(f),\ f \in \mathcal{O}(Y); \quad [\nabla_v, \nabla_u] = \nabla_{[v,u]},$$

and the filtration by order is defined by setting $\deg \mathcal{O}(Y) = 0$ and $\deg(\nabla_v) = 1$.

8.2 Quantization of classical integrable systems

Let $n \coloneqq \dim Y$. Recall that a classical integrable system on T^*Y consists of functionally (or, in the algebraic case, algebraically) independent functions $H_1, \dots, H_n$ on a dense open subset of T^*Y such that $\{H_i, H_j\} = 0$. These functions define a map

$$p\colon T^*Y \to \mathbb{A}^n$$

whose pullback $p^*\colon \mathcal{O}(\mathbb{A}^n) = \mathbf{k}[T_1, \dots, T_n] \to \mathcal{O}(T^*Y)$ sends T_i to H_i and is therefore an inclusion of a Poisson-commutative subalgebra $\mathbf{k}[H_1, \dots, H_n] \hookrightarrow T^*Y$. Then it is easy to prove the following lemma.

Lemma 8.4. *Any regular function on a dense open set of T^*Y which Poisson-commutes with all H_i is functionally (respectively, algebraically) dependent on them.*

Thus if Y is a smooth algebraic variety, then classical integrable systems on T^*Y correspond to Poisson-commutative polynomial subalgebras in $\mathcal{O}(T^*Y)$ with $\dim Y$ generators. This motivates the following definition.

Definition 8.5. A *quantization* of a classical integrable system $\{H_1, \dots, H_n\}$ on Y (also called a *quasiclassical quantum integrable system* quantizing $\{H_1, \dots, H_n\}$) is a non-commutative algebra A quantizing $\mathcal{O}(T^*Y)$, with an injection

$$\mathbf{k}[T_1, \dots, T_n] \hookrightarrow A, \qquad T_i \mapsto \widehat{H}_i$$

such that $\widehat{H}_i$ maps to H_i in $A/\hbar A = \mathcal{O}(T^*Y)$.

In particular, if $A = \mathcal{D}_\hbar(Y)$, then we can specialize $\hbar$ to a nonzero numerical value and obtain a commuting system of operators $\widehat{H}_1, \dots, \widehat{H}_n$ in $\mathcal{D}(Y)$. In general, such a system is called a *quantum integrable system* in $\mathcal{D}(Y)$, provided that these operators are algebraically independent. We thus see that any quantization of a classical integrable system on T^*Y in $\mathcal{D}_\hbar(Y)$ specializes to a family of quantum integrable systems in $\mathcal{D}(Y)$ parametrized by (generic) $\hbar \in \mathbf{k}^\times$.

Example 8.6. Let $Y = \mathbb{A}^1$, and $H_1 = H = p^2 + U(x)$. This system can be quantized by setting $\widehat{H} = \hbar^2\partial^2 + U(x)$.

Remark 8.7. Note that a *single* quantum integrable system in $\mathcal{D}(Y)$ need not have a classical limit, i.e., need not be a specialization of a quasiclassical integrable system (=a member of a 1-parameter family of integrable systems as above). A vivid example of this is the deformed rational Calogero-Moser system, see e.g. [FV].

The following non-trivial theorem is a quantum analog of Lemma 8.4.

Theorem 8.8 (Makar–Limanov, [ML])**.** *Let $B_1, \dots, B_n$ be a commuting algebraically independent family of elements of $\mathcal{D}(Y)$. If $B \in \mathcal{D}(Y)$ and $[B, B_i] = 0$ for all i then B is algebraically dependent on $B_1, \dots, B_n$.*

Thus a quantum integrable system is a maximal (up to algebraic extensions) commutative subalgebra in $\mathcal{D}(Y)$, and a quasiclassical quantum integrable system is one in $\mathcal{D}_\hbar(Y)$. This quasiclassical system quantizes a given classical system if it converges to it when $\hbar \to 0$ and $\widehat{p}_i \mapsto p_i$.

This gives rise to a naive quantization procedure: just replace all instances of p_j with $\hbar\partial_j$.[14] But this is not a good thing to do in general due to ordering issues: unlike p_i, the partial derivatives ∂_i do not commute with coordinates, so there is ambiguity as to whether, say, $x_i p_i$ should be replaced by $\hbar x_i \partial_i$ or $\hbar \partial_i x_i = \hbar x_i \partial_i + \hbar$. However this naive procedure does sometimes work, e.g. for the twisted Garnier system from Subsection 7.5.

Example 8.9. Recall the twisted Garnier system, given by the hamiltonians

$$G_i = \sum_{j \neq i} \frac{(x_i - x_j)^2 p_i p_j - 2(x_i - x_j)(\lambda_i p_j - \lambda_j p_i) - 2\lambda_i \lambda_j}{t_j - t_i}$$

where x_i and p_j are the standard coordinates and momenta. The naive quantization procedure produces

$$\widehat{G}_i \coloneqq \hbar^2 \sum_{j \neq i} \frac{(x_i - x_j)^2 \partial_i \partial_j - 2(x_i - x_j)(\frac{\lambda_i}{\hbar}\partial_j - \frac{\lambda_j}{\hbar}\partial_i) - 2\frac{\lambda_i}{\hbar}\frac{\lambda_j}{\hbar}}{t_j - t_i}.$$

It is convenient to introduce $\Lambda_i \coloneqq 2\lambda_i/\hbar$, so that this can be rewritten as

$$\widehat{G}_i = \hbar^2 \sum_{j \neq i} \frac{(x_i - x_j)^2 \partial_i \partial_j - (x_i - x_j)(\Lambda_i \partial_j - \Lambda_j \partial_i) - \frac{1}{2}\Lambda_i \Lambda_j}{t_j - t_i}.$$

There is actually a more insightful way to write this formula using Lie theory, which in particular allows us to see without computation that the operators $\widehat{G}_i$ commute.

[14] In actual quantum mechanics one replaces p_j with $-i\hbar\partial_j$, but since our considerations are purely algebraic, we drop the factor $-i$.

Namely, let $\mathfrak{sl}_2 = \langle e, f, h \rangle$, and recall that there is an action of $U(\mathfrak{sl}_2)$ by differential operators on $\mathbb{A}^1$ given by

$$f \mapsto -\partial_x, \qquad h \mapsto 2x\partial_x + \Lambda, \qquad e \mapsto x^2\partial_x + \Lambda x. \tag{17}$$

The Casimir tensor is the unique element $\Omega \in (\mathfrak{sl}_2 \otimes \mathfrak{sl}_2)^{\mathfrak{sl}_2}$ up to scaling, and is given by the formula

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h.$$

Then we have

$$\widehat{G}_i = \hbar^2 \sum_{j \neq i} \frac{\Omega_{i,j}}{t_i - t_j}.$$

Now note that this formula in fact makes sense for any simple finite dimensional Lie algebra $\mathfrak{g}$, with Ω being the Casimir tensor (unique up to scaling nonzero element in $(S^2\mathfrak{g})^{\mathfrak{g}}$. Moreover, it is obvious that

$$[\Omega_{12}, \Omega_{13} + \Omega_{23}] = 0,$$

and therefore

$$[\widehat{G}_i, \widehat{G}_j] = 0.$$

Thus we obtain a collection of commuting elements in $U(\mathfrak{g})^{\otimes n}$ called the *Gaudin hamiltonians* for the Lie algebra $\mathfrak{g}$.[15] It is easy to check that these quantize second-order Hitchin hamiltonians for $\mathfrak{g}$ on $\mathbb{P}^1$ with parabolic structures.

If we pick representations $V_1, \ldots, V_n$ of $\mathfrak{g}$, then we get commuting operators

$$\widehat{G}_i \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$$

which also commute with $\mathfrak{g}$ and therefore act on $(V_1 \otimes \cdots \otimes V_n)^{\mathfrak{g}}$ and more generally on $\mathrm{Hom}_{\mathfrak{g}}(V, V_1 \otimes \cdots \otimes V_n)$ for any representation V of $\mathfrak{g}$. This produces many interesting families of commuting operators.

However, for simple Lie algebras $\mathfrak{g}$ of rank > 1, this does not immediately produce an integrable system, because we are missing higher-order operators; for example, for $\mathfrak{g} = \mathfrak{sl}_n$ with $n > 2$ we are missing the components of $\mathrm{tr} \wedge^j \phi$ with $j > 2$. We might hope that our naive quantization procedure could help here, but the following example shows that unfortunately this is not quite the case.

Example 8.10 (Elliptic Calogero–Moser system)**.** Recall that the classical elliptic Calogero–Moser system has hamiltonian $H_2 = \sum_i p_i^2 - \sum_{j \neq i} \wp(q_i - q_j)$. So naively the quantized hamiltonian should be

$$\widehat{H}_2 \coloneqq \hbar^2 \left(\sum_i \partial_i^2 - \frac{1}{\hbar^2} \sum_{j \neq i} \wp(q_i - q_j) \right).$$

[15]Usually the factor $\hbar^2$ is dropped.

Theorem 8.11. *$\widehat{H}_2$ defines a quasiclassical quantum integrable system $\mathbb{C}[\widehat{H}_1, \ldots, \widehat{H}_n]$, which is the centralizer of $\widehat{H}_2$ in the algebra of quasiclassical differential operators in n variables.*[16]

While this is encouraging, Theorem 8.11 (integrability of the quantum CM system) unfortunately does not follow from the integrability of the classical CM system. For instance, according to Exercise 7.4, we have a classical integral

$$H_3 = \sum_i p_i^3 + \sum_i a_i(\mathbf{q}) p_i + b(\mathbf{q})$$

for some functions $a_i(\mathbf{q})$ and $b(\mathbf{q})$. And already in the middle term we have the ambiguity in ordering: in the quantized hamiltonian $\widehat{H}_3$, do we put $\partial_i a_i(q)$ or $a_i(q)\partial_i$, or something else? Clearly, we need a more systematic approach!

In fact there is no uniform way to quantize an arbitrary integrable system. In each specific case one usually needs to go back to the definition of the classical system and see if one can quantize the way in which it is obtained. This is exactly what we'll do in the case of Hitchin systems.

Exercise 8.12. (i) For $N = 4$ the Gaudin system for $\mathfrak{sl}_2$ reduces to a second order differential operator L in 1 variable with 4 singularities. Compute this operator after sending $(t_1, t_2, t_3, t_4) \mapsto (0, 1, \infty, t)$. (Hint: you can get the general shape of L by using that it has 4 regular singularities.)

(ii) Show that for $\Lambda_i = -1$ one obtains (up to adding a constant) the *Lamé operator* (with parameter $-\frac{1}{2}$)

$$L = \partial \circ x(x-1)(x-t) \circ \partial + x.$$

(iii) Let E be the elliptic curve $y^2 = x(x-1)(x-t)$. Then the function x defines a double cover $E \to \mathbb{P}^1$ branched at $(0, 1, \infty, t)$. Let us lift the operator L from (i) to this double cover. Show that the lift $\widetilde{L}$ is the *Darboux operator*

$$\widetilde{L} = \partial_z^2 - \sum_{i=0}^{3} \frac{\Lambda_i(\Lambda_i + 2)}{4} \wp(z + \varepsilon_i; \tau)$$

for $\varepsilon_1 = 0$, $\varepsilon_2 = \frac{1}{2}$, $\varepsilon_3 = \frac{\tau}{2}$, and $\varepsilon_4 = \frac{1+\tau}{2}$.

[16] The trigonometric or rational quantum Calogero–Moser systems can be obtained as limits from the elliptic one.

8.3 Quantum hamiltonian reduction

Recall our construction of the classical Hitchin system on $\mathrm{Bun}_G^\circ(X)$. There were two steps.

1. Represent $\mathrm{Bun}_G(X)$ as a double quotient, e.g. $G(R)\backslash G(K)/G(\mathcal{O})$.
2. Construct some commuting hamiltonians on $T^*G(K)$ which are invariant under the left and right actions of $G(K)$, and then descend them to $\mathrm{Bun}_G^\circ(X)$ using hamiltonian reduction by $G(R) \times G(\mathcal{O})$.

To retrace our steps, we must discuss a quantized version of hamiltonian reduction (along a choice of coadjoint orbit).

Classically, a group H acts in a hamiltonian manner on a symplectic manifold M, with moment map $\mu\colon M \to \mathfrak{h}^*$, and we define the hamiltonian reduction $\mu^{-1}(0)/H$ which is a symplectic manifold. Note that μ can be viewed as a Poisson homomorphism between Poisson algebras $\mu\colon S(\mathfrak{h}) = \mathcal{O}(\mathfrak{h}^*) \to \mathcal{O}(M)$. In the quantum setting, we therefore must consider a group H acting on an algebra A, and the natural way to quantize μ is to ask for an algebra homomorphism

$$\mu\colon U(\mathfrak{h}) \to A.$$

This is the input data that one must supply. The classical condition that the moment map is H-equivariant and dual to the action map becomes the condition that μ is H-equivariant and

$$z \cdot a = [\mu(z), a], \qquad \forall z \in \mathfrak{h}.$$

Finally, classically, we considered the quotient M/H, for which

$$\mathcal{O}(M/H) = \mathcal{O}(M)^H \subset \mathcal{O}(M)$$

is a Poisson subalgebra, so a natural way to quantize the locus $\mu^{-1}(0)/H \subset M/H$ cut out by the equation $\mu(m) = 0$ is given by the following definition.

Definition 8.13. The *quantum hamiltonian reduction* of A by H is the algebra

$$\overline{A} \coloneqq A^H/(A\mu(\mathfrak{h}))^H.$$

Note that $A\mu(\mathfrak{h}) \subset A$ is only a left ideal, but one can check that, after taking H-invariants, $(A\mu(\mathfrak{h}))^H \subset A^H$ is a two-sided ideal.

Note that if H is reductive and acts on A locally finitely, then the operation of quotienting by $A\mu(\mathfrak{h})$ commutes with the operation of taking H-invariants, so

$$\overline{A} = (A/A\mu(\mathfrak{h}))^H.$$

To replace 0 with a coadjoint orbit $O \subset \mathfrak{h}^*$, the equation $\mu(m) = 0$ is replaced by $\mu(m) \in O$. Thus in the quantum setting, we need to find a two-sided ideal $I \subset U(\mathfrak{h})$ which quantizes the orbit O, in the sense that $U(\mathfrak{h})/I$ is a quantization of O. Then the quantum hamiltonian reduction is

$$\overline{A} = A^H/(A\mu(I))^H,$$

which coincides with $(A/A\mu(I))^H$ in the reductive case.

When $O = 0$, the ideal I is the augmentation ideal, i.e. the kernel of $U(\mathfrak{h}) \to \mathbb{C}$, so that $A\mu(I) = A\mu(\mathfrak{h})$ and we recover the previous case.

More generally, given a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and a character $\lambda\colon \mathfrak{p} \to \mathbb{C}$, we have the two-sided ideal $I_\lambda \subset U(\mathfrak{g})$ which is the annihilator of the parabolic Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \lambda$. The quantum hamiltonian reduction using this ideal corresponds to the classical hamiltonian reduction along the orbit O_λ of λ extended arbitrarily to an element of $\mathfrak{g}^*$ (note that all such extensions are conjugate), which occurs in the construction of twisted Hitchin systems.

8.4 The quantum anomaly

Let us now implement this for the Hitchin system (first without parabolic structures). We should take the (huge!) algebra $A := \mathcal{D}(G(K))$ of differential operators on $G(K)$, and the quantum Hitchin system should be obtained from some 2-sided-invariant differential operators.

This is an infinite-dimensional group and there is actually quite a bit of technical trouble in talking about differential operators on such a group. But here we will ignore such issues and pretend that $L := G(K)$ is an ordinary Lie group; a fully rigorous treatment can be found in [BD1].

What are 2-sided-invariant differential operators on a Lie group L? Left-invariant differential operators are well-known to be identified with the universal enveloping algebra $U(\mathfrak{l})$ of the Lie algebra $\mathfrak{l}$ of L. Therefore 2-sided-invariant differential operators are identified with

$$U(\mathfrak{l})^L = Z(U(\mathfrak{l})),$$

the center of $U(\mathfrak{l})$.

And here we are in for an unpleasant surprise. In our case, $\mathfrak{l} = \mathfrak{g}((t))$, but for G semisimple, unfortunately the center of $U(\mathfrak{l})$ is trivial. (Technically, since $\mathfrak{l}$ is infinite-dimensional, one should take a suitable completion of the universal enveloping algebra, but, even so, the center is still trivial.) Thus our first attempt to quantize the Hitchin system fails.

We can explain what went wrong. Classically, recall that $H_2 = \frac{1}{2}(\phi, \phi)$, and the Higgs field ϕ has the form $\phi(z)\,\frac{dz}{z}$ with $\phi \in \mathfrak{g}((t))$. Let

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n z^{-n}.$$

Also pick an orthonormal basis $\{a_i\}$ of $\mathfrak{g}$ and write

$$\phi_m =: \sum_i \phi_m^i a_i.$$

Then we have (dropping $(dz)^2$ for brevity)

$$H_2 = \frac{1}{2}\sum_n z^{-n-2}\sum_m (\phi_m, \phi_{n-m}) = \sum_n T_n z^{-n-2},$$

where

$$T_n := H_{2,-n-2} = \frac{1}{2}\sum_{m,i} \phi_m^i \phi_{n-m}^i$$

We now want to generalize these formulas to the quantum case so that $[\phi_\ell^j, H_2] = 0$ for all ℓ, i.e.,

$$[\phi_\ell^j, T_n] = 0 \tag{18}$$

for all ℓ, n, with

$$[\phi_m^i, \phi_\ell^j] = \sum_r C_{ij}^r \phi_{m+\ell}^r,$$

where C_{ij}^r are the structure constants of the commutator in $\mathfrak{g}$ in the basis $\{a_i\}$. But in the definition of $H_{2,n}$ we have an infinite sum, so to make it meaningful, say, on highest-weight representations of $\mathfrak{g}((t))$, we need *normal-ordering*:

$$T_n = \frac{1}{2}\sum_{m,i} :\phi_m^i \phi_{n-m}^i:\,,$$

where the normally ordered product $:\phi_m^i \phi_\ell^i:$ is defined by the formula

$$:\phi_m^i \phi_\ell^i: \; = \begin{cases} \phi_m^i \phi_\ell^i \text{ if } m \le \ell \\ \phi_\ell^i \phi_m^i \text{ if } m > \ell \end{cases}$$

But then commutation relation (18) will not hold, as seen from the following exercise (physicists call this phenomenon *quantum anomaly*).

Exercise 8.14. Show that if the inner product $(,)$ in the definition of H_2 is normalized so that long roots of $\mathfrak{g}$ have squared length 2 then

$$[\phi_\ell^j, T_i] = \ell h^\vee \phi_{\ell+n}^j,$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, defined by the formula

$$\mathrm{tr}(\mathrm{ad}(a)^2) = 2h^\vee(a,a),\;\; a \in \mathfrak{g}.$$

8.5 Twisted differential operators

In fact, we were doomed to fail from the start, since Beilinson and Drinfeld showed ([BD1]) that every globally defined differential operator on $\mathrm{Bun}_G(X)$ is a scalar. However, we can recall something from physics to save the day: differential operators on a manifold are *not* the most natural quantization of functions on its cotangent bundle. Recall from quantum mechanics that classical observables on $M = T^*Y$ should quantize to operators on $L^2(Y)$. But to define $L^2(Y)$, one must fix a measure on Y, and there is no natural choice for the measure in general. The solution is to take $L^2(Y, \Omega^{1/2})$, where Ω is the bundle of densities, and the L^2-norm is given by

$$\|f(y)|dy|^{1/2}\|^2 := \int_Y |f(y)|^2 \, |dy|.$$

Thus, from this point of view, the most natural quantization of functions on the cotangent bundle is the algebra of *twisted* differential operators $\mathcal{D}(Y, \mathcal{K}_Y^{1/2})$ acting on a square root of the canonical bundle on Y. Remarkably, this algebra is independent of the choice of this square root, and moreover makes sense even when such a square root does not exist at all.

Definition 8.15. Let Y be a smooth variety and L be a line bundle on Y. An *L-twisted differential operator* on Y is a differential operator acting on local sections of L.

Let $\mathcal{D}(Y, L^{\otimes n})$ be the algebra of $L^{\otimes n}$-twisted differential operators. Similarly to Exercise 8.3(ii), it is generated by $\mathcal{O}(Y)$ and elements ∇_v attached $\mathcal{O}(Y)$-linearly to vector fields v on Y, with relations that deform the relations in Exercise 8.3(ii). Namely, if one locally picks a connection on L with curvature denoted ω, the only relation that changes is $[\nabla_v, \nabla_u] = \nabla_{[v,u]}$ for vector fields v, u, which is replaced by

$$[\nabla_v, \nabla_u] = \nabla_{[v,u]} + n\omega(v, u).$$

So it makes sense to take any $n \in \mathbb{C}$ (in particular, $n = \frac{1}{2}$, as noted above).

Exercise 8.16. For $\lambda \in \mathbb{C}$, let U_λ be the quotient of $U(\mathfrak{sl}_2)$ by the relation $C = \frac{1}{2}\lambda(\lambda+2)$, where $C = ef + fe + \frac{1}{2}h^2$ is the Casimir element. Show that U_λ is isomorphic to the algebra of twisted differential operators $\mathcal{D}(\mathbb{P}^1, \mathcal{O}(1)^{\otimes\lambda})$. This is the simplest instance of the *Beilinson-Bernstein localization*, [BB]. (Hint: use the representation of $\mathfrak{sl}_2$ given by formula (17)).

8.6 Twisted differential operators on $\mathrm{Bun}_G(X)$

Motivated by this, we replace $\mathcal{D}(\mathrm{Bun}_G(X))$ with the algebra $\mathcal{D}(\mathrm{Bun}_G(X), \mathcal{K}^{1/2})$ of differential operators on the line bundle $\mathcal{K}^{1/2}$ (which in this case actually exists, see [BD1]). An important feature of this line bundle is that it can be obtained by descending a line bundle from $G(K) = G((t))$ to the double quotient. Namely, for brevity let G be simply connected and simple and $\mathbf{k} = \mathbb{C}$. Then there exists the *Kac–Moody central extension*

$$1 \to \mathbb{C}^\times \to \widehat{G} \to G((t)) \to 1$$

which gives rise to a $\mathbb{C}^\times$-bundle (i.e., a line bundle) $\mathcal{L}$ on $G((t))$ with

$$c_1(\mathcal{L}) \in H^2(G((t)), \mathbb{Z}) \cong H^3(G, \mathbb{Z}) \cong \mathbb{Z}$$

being the element 1. The Lie algebra of the resulting *Kac-Moody group* $\widehat{G}$ is the *affine Kac–Moody algebra*

$$\widehat{\mathfrak{g}} := \mathfrak{g}((t)) \oplus \mathbb{C}\mathrm{K}$$

with commutator given by

$$[a(t), b(t)] := [a, b](t) + \mathrm{Res}_{t=0}(da(t), b(t))\mathrm{K}.$$

where the inner product on $\mathfrak{g}$ is normalized so that long roots have squared length 2.

Theorem 8.17 ([BD1]). *The bundle $\mathcal{K}_{\mathrm{Bun}_G(X)}$ is the descendant of the line bundle $\mathcal{L}^{-2h^\vee}$ on $\mathrm{Bun}_G(X)$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.*

So we need to work with two-sided-invariant elements in $\mathcal{D}(G((t)), \mathcal{L}^{-h^\vee})$, which is the center of the quotient

$$\widehat{U}(\widehat{\mathfrak{g}})/\langle \mathrm{K} = -h^\vee \rangle$$

where $\widehat{U}(\widehat{\mathfrak{g}})$ denotes a suitable completion of the universal enveloping algebra $U(\widehat{\mathfrak{g}})$.

8.7 The Sugawara construction

The central element K of the algebra $\widehat{U}(\widehat{\mathfrak{g}})$ can be specialized to a numerical value k called the *level*. This produces the algebra $\widehat{U}_k(\widehat{\mathfrak{g}})$, the completed enveloping algebra at level k. Remarkably, it turns out that the algebra $\widehat{U}_k(\widehat{\mathfrak{g}})$ for $k \in \mathbb{C}$ has a nontrivial center if and only if $k = -h^\vee$ (the so-called *critical level*). In fact, some of these central elements are not difficult to construct.

Namely, let us keep the notation of Subsection 8.4, but now view T_n as elements of $\widehat{U}(\widehat{\mathfrak{g}})$ (so that they specialize to the elements T_n of Subsection 8.4 when K is set to 0).

Let W be the (formal) *Witt algebra*, i.e., the Lie algebra of vector fields on $D^\times$, $\mathrm{W} = \mathbb{C}((t))\partial_t$, with bracket

$$[f\partial_t, g\partial_t] = (fg' - gf')\partial_t.$$

Then W has a (topological) basis $L_n := -t^{n+1}\partial_t, n \in \mathbb{Z}$, such that

$$[L_n, L_m] = (n - m)L_{m+n}.$$

It is well known (see [Ka]) that W has a unique non-trivial 1-dimensional central extension called the (formal) *Virasoro algebra* and denoted Vir. It has a (topological) basis consisting of $L_n, n \in \mathbb{Z}$ and a central element C such that

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{n^3-n}{12}\delta_{n+m,0}C.$$

If V is a representation of Vir with C acting as $c \in \mathbb{C}$, then we say that V has *central charge* c.

Generalizing the computation of Exercise 8.14 to the case of nonzero level, we obtain the following theorem.

Theorem 8.18 (Sugawara construction, see [Ka]). *One has*

$$[\phi_\ell, T_n] = (\mathrm{K} + h^\vee)\ell\phi_{n+\ell},$$
$$[T_n, T_m] = (\mathrm{K} + h^\vee)(n-m)T_{m+n} + \frac{n^3-n}{12}\mathrm{K}(\mathrm{K} + h^\vee)\dim(\mathfrak{g})\delta_{n+m,0}.$$

Hence, when K *is specialized to* $k \neq -h^\vee$*, the elements*

$$L_n := \frac{T_n}{k+h^\vee}$$

define a representation of the Virasoro algebra with central charge $c := \frac{k\dim\mathfrak{g}}{k+h^\vee}$ *compatible with* ϕ_ℓ*:*

$$[\phi_\ell, L_n] = \ell\phi_{n+\ell},$$
$$[L_n, L_m] = (n-m)L_{m+n} + \frac{n^3-n}{12}\frac{k\dim\mathfrak{g}}{k+h^\vee}\delta_{n+m,0}.$$

On the other hand, when $k = -h^\vee$ *(the critical level) then* T_n *are central elements.*

The elements T_n are called the *Sugawara elements*.

8.8 $\mathfrak{sl}_2$-opers on $D^\times$

Note that at the non-critical level $k \neq -h^\vee$ the commutation relations for T_n can be written as

$$[L_n, T_m] = (n-m)T_{m+n} + \frac{n^3-n}{12}K\dim(\mathfrak{g})\delta_{n+m,0}.$$

In particular, this has a well defined specialization at the critical level, to an action of W on $\mathbb{C}[T_n, n \in \mathbb{Z}]$:

$$L_n \circ T_m = (n-m)T_{m+n} - \frac{n^3-n}{12}h^\vee\dim(\mathfrak{g})\delta_{n+m,0}.$$

This action deforms the action in the classical case given by

$$L_n \circ T_m = (n-m)T_{m+n},$$

which expresses the fact that $T_n \in \mathrm{W}$ are linear functions on the space W^* of quadratic differentials $\mathbb{C}((t))(dt)^2$ on $D^\times$: for $\eta = \eta(t)(dt)^2 \in \mathbb{C}((t))(dt)^2$,

$$T_n(\eta) = -\mathrm{Res}(t^{n+1}\partial_t \cdot \eta) = -\mathrm{Res}(t^{n+1}\eta(t)dt).$$

In the quantum case, however, the additional summand means that T_n are rather *affine linear* functions on the hyperplane $\mathrm{Vir}_1^* \subset \mathrm{Vir}^*$ defined by the equation $C = 1$ (or, equivalently, $C = c$ for any nonzero value of c). So we may ask for the geometric meaning of elements of Vir_1^*; this is a certain deformation of the notion of a quadratic differential on $D^\times$. This question is answered by the following exercise.

Exercise 8.19. (i) Show that the coadjoint representation Vir^* of the Virasoro algebra Vir is isomorphic as a W-module (hence as an $\mathrm{Aut}(D)$-module) to the space of differential operators

$$L = \alpha\partial_t^2 + u(t) \colon \mathcal{K}^{-1/2} \to \mathcal{K}^{3/2}$$

on $D^\times$, for $\alpha \in \mathbb{C}$, i.e.,

$$L(f(t)(dt)^{-1/2}) = (\alpha f''(t) + u(t)f(t))(dt)^{3/2},$$

where $f, u \in \mathbb{C}((t))$. In particular, Vir_1^* is identified with the space of *Hill operators*

$$L = \partial_t^2 + u(t) \colon \mathcal{K}^{-1/2} \to \mathcal{K}^{3/2}.$$

(ii) Part (i) implies that the space of Hill operators is invariant under the group $\mathrm{Aut}(D)$ of formal changes of the variable t. Give another proof of this fact by giving a coordinate-free definition of a Hill operator; namely, interpret Hill operators as formally self-adjoint second-order differential operators $\mathcal{K}^{-1/2} \to \mathcal{K}^{3/2}$ with symbol 1.[17]

(iii) Compute how L transforms under changes $t \mapsto s(t)$ of the formal coordinate t. Hint: you should see the function

$$D(s) = \tfrac{s'''}{s'} - \tfrac{3}{2}(\tfrac{s''}{s'})^2$$

called the *Schwarzian derivative* of s.

Definition 8.20 ([BD1, BD2])**.** A Hill operator $L = \partial_t^2 + u(t) \colon \mathcal{K}^{-1/2} \to \mathcal{K}^{3/2}$, where $u \in \mathbb{C}((t))$, is called an $\mathfrak{sl}_2$*-oper* on $D^\times$.

[17] If L is a differential operator $\mathcal{K}^a \to \mathcal{K}^b$ on $D^\times$ then one can canonically define the formal adjoint $L^* \colon \mathcal{K}^{1-b} \to \mathcal{K}^{1-a}$ with respect to the pairing $(f,g) = \mathrm{Res}(fg)$ between $\mathcal{K}^b$ and $\mathcal{K}^{1-b}$, which is a differential operator of the same order. So if $a+b=1$, it makes sense to say that L is self-adjoint: $L = L^*$. Also if the order of L is n then we can canonically define its symbol $\sigma(L)$, which is a section of $\mathcal{K}^{b-a-n}$. So if $b-a=n$ then the symbol is a function (element of $\mathbb{C}((t))$).

This is a special case of the general notion of a $\mathfrak{g}$-oper for a finite dimensional simple Lie algebra $\mathfrak{g}$ defined in [BD1]. The terminology is motivated by the fact that the problem of solving the differential equation $L\psi = 0$ reduces to the problem of integrating the SL_2-connection

$$\nabla = \partial + \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}.$$

Thus we see that the subalgebra Z_{Sug} of the center Z of $\widehat{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ generated (topologically) by the Sugawara elements T_n may be interpreted as the algebra of polynomial functions on the space $\mathrm{Op}_{\mathfrak{sl}_2}(D^\times) \coloneqq \mathrm{Vir}_1^*$ of $\mathfrak{sl}_2$-opers on $D^\times$.

Exercise 8.21. Show that for a function $s = s(t)$ (C^3 on an interval or holomorphic on a disk), one has $D(s) = 0$ if and only if s is a Möbius transformation $s(t) = \frac{at+b}{ct+d}$. Use this to show that under Möbius transformations $t \mapsto \frac{at+b}{ct+d}$ (with b being a formal variable), $\mathfrak{sl}_2$-opers transform as quadratic differentials. (This is a reflection of the fact that the 2-cocycle $\frac{n^3-n}{12}\delta_{n+m,0}$ defining the Virasoro algebra vanishes for $n = -1, 0, 1$, i.e., on the Möbius Lie subalgebra $\langle L_{-1}, L_0, L_1\rangle$ of the Witt algebra W).

8.9 $\mathfrak{sl}_2$-opers on curves

Now let X be a smooth algebraic curve over $\mathbb{C}$. By analogy with the previous subsection we make the following definition.

Definition 8.22. An $\mathfrak{sl}_2$-oper on X is a differential operator $\mathcal{K}_X^{-1/2} \to \mathcal{K}_X^{3/2}$ which in local coordinates looks like $\partial_t^2 + u(t)$ where u is a regular function.

Similarly to Exercise 8.19, we may alternatively define an $\mathfrak{sl}_2$-oper as a formally self-adjoint second order differential operator $\mathcal{K}_X^{-1/2} \to \mathcal{K}_X^{3/2}$ with symbol 1. As this definition is independent of the choice of coordinates, so is the original one.

Exercise 8.23. Let X be a smooth irreducible projective curve over $\mathbb{C}$ of genus $g \geq 2$ and fix a spin structure on X (i.e., a square root $\mathcal{K}_X^{1/2}$; these choices form a torsor over the group $\mathrm{Jac}_{X,2} \cong (\mathbb{Z}/2)^{2g}$ of points of order 2 on $\mathrm{Jac}(X)$).

(i) Show that X admits a unique rank 2 vector bundle E_X, called the *oper bundle*, such that there exists a non-split short exact sequence

$$0 \to \mathcal{K}_X^{1/2} \to E_X \to \mathcal{K}_X^{-1/2} \to 0,$$

and this sequence is unique up to scaling the arrows.

(ii) Show that $\mathfrak{sl}_2$-opers on X are in natural bijection with connections on E_X.

(iii) Show that $\mathfrak{sl}_2$-opers on X exist and form a torsor over $H^0(X, \mathcal{K}_X^{\otimes 2})$ (i.e. an affine space of dimension $3g-3$).

(iv) Compute $\mathfrak{sl}_2$-opers explicitly on a genus 2 curve using the hyperelliptic realization.

Exercise 8.24. (i) Let X be a smooth irreducible projective curve over $\mathbb{C}$, viewed as a Riemann surface. Use the uniformization theorem to give an open cover of X with transition maps being Möbius transformations with real coefficients. Use this cover to identify the affine space $\mathrm{Op}_{\mathfrak{sl}_2}(X)$ with the vector space of quadratic differentials $H^0(X, \mathcal{K}_X^{\otimes 2})$. In particular, this introduces an origin in $\mathrm{Op}_{\mathfrak{sl}_2}(X)$, denoted L_{un} and called the *uniformization oper* (a term motivated by part (ii)).

(ii) Let $J\colon \mathbb{C}_+ \to X$ be a uniformization map, and let $f = J^{-1}$ be the multivalued inverse function. Show that $f(z) = \frac{\psi_1(z)}{\psi_2(z)}$, where $\{\psi_1, \psi_2\}$ is a basis of the space of holomorphic solutions of the differential equation $L_{\mathrm{un}}\psi = 0$ near some point $x \in X$ with Wronskian 1 in which the monodromy matrices of this equation have real entries. Show that both uniformization maps and such bases (up to sign of ψ_j) form $\mathrm{PSL}_2(\mathbb{R})$-torsors, and the above correspondence is an isomorphism between these torsors.

(iii) Let

$$\beta(z, \overline{z}) := \mathrm{Im}(\psi_1(z)\overline{\psi_2}(z)) = \frac{\psi_1(z)\overline{\psi_2}(z) - \psi_2(z)\overline{\psi_1}(z)}{2i}.$$

Show that β is a positive $-1/2$-density on X independent on the choice of the basis $\{\psi_1, \psi_2\}$.

(iv) Recall that in a Riemann surface, conformal metrics are in natural bijection with positive densities (in local coordinates, $\rho(dx^2+dy^2)$ corresponds to $\rho\, dx\, dy$). Show that (upon suitable normalization) β^{-2} is the positive density on X corresponding to the Poincaré metric (with Gaussian curvature -1).

Exercise 8.25 ([Fa2, Go]; see also [EFK1])**.** This is a generalization of Exercise 8.24.

(i) An $\mathfrak{sl}_2$-oper L on X is said to have *real monodromy* if there exists a basis $\{\psi_1, \psi_2\}$ of holomorphic solutions of the equation $L\psi = 0$ with Wronskian 1 in which its monodromy matrices are real. Show that this is equivalent to the existence of a non-zero single-valued real C^∞ solution β of the equation $L\beta = 0$, and such a solution, when exists, is unique up to scaling.

(ii) Let L be an $\mathfrak{sl}_2$-oper on X with real monodromy, and β be the corresponding real solution of the differential equation $L\beta = 0$. Show that the system of equations $\beta = 0$, $d\beta = 0$ has no solutions on X, so the zero set $Z(L)$ of β is a smooth 1-dimensional real submanifold on X (a collection of non-intersecting simple closed smooth curves).

(iii) Show that the density β^{-2} defines a complete Poincaré metric on $X \setminus Z(L)$ with logarithmic singularities on $Z(L)$ (i.e., near a point of $Z(L)$ it is isomorphic to the Poincaré metric on the upper half plane near the origin).

(iv) A *real projective structure* on X is an equivalence class of atlases of charts with transition maps being real Möbius transformations, i.e., elements of $\mathrm{PSL}_2(\mathbb{R})$ (two atlases are equivalent if they have a common refinement). Show that $\mathfrak{sl}_2$-opers on X with real monodromy correspond to equivalence classes of real projective structures on X.

8.10 Quantization of the Hitchin system for $G = \mathrm{SL}_2$

We can now use the Sugawara elements T_n to construct quantizations of quadratic Hitchin hamiltonians, as outlined at the end of Subsection 8.6. Namely, according to Theorem 8.17, T_n descend to second order twisted differential operators $\overline{T}_n \in \mathcal{D}(\mathrm{Bun}^\circ_G(X), \mathcal{K}^{1/2})$. By considering the semiclassical limit, it is easy to see that $\{\overline{T}_n\}$ span a vector space of dimension $3g-3$ and are algebraically independent. So for $G = \mathrm{SL}_2$ they provide a quantization of the Hitchin system.

Let us consider the case $G = \mathrm{SL}_2$ in more detail. Recall that in this case T_n can be interpreted as functions on the space $\mathrm{Op}_{\mathfrak{sl}_2}(D^\times)$. Namely, given $L \in \mathrm{Op}_{\mathfrak{sl}_2}(D^\times)$, $L = \partial_t^2 + u(t)$, where $u(t) = \sum_{n\in\mathbb{Z}} u_n t^n$, we have $T_n(L) = u_{-n-2}$. It is clear that the same should apply to $\overline{T}_n$, except that now $L \in \mathrm{Op}_{\mathfrak{sl}_2}(X)$ is an oper on X; this is just the quantum analog of the commutative diagram (11). Namely, similarly to Remark 6.13, for $n > -1$ the element T_n descends to 0, while for $n \le -2$ it descends to the Taylor coefficient u_{-n-2} of the oper L on X.

Moreover, as in the classical case, this story extends straightforwardly to the ramified case, with parabolic structures and, more generally, twistings (see Section 6), using the notion of quantum hamiltonian reduction with respect to an ideal in $U(\mathfrak{g})$ (cf. Subsection 8.3).

Exercise 8.26. For $\lambda \in \mathbb{C}$ let $I_\lambda \subset U(\mathfrak{sl}_2)$ be the ideal generated by $C - \frac{\lambda(\lambda+2)}{4}$, where $C = ef + fe + \frac{h^2}{2}$ is the Casimir element. Show that the twisted quantum Hitchin system for $X = \mathbb{P}^1$ with marked points $t_1, \dots, t_N$ and twistings by the ideals I_{λ_j} at t_j is the Gaudin system with parameters λ_j.

Finally, in view of Remark 6.10, by considering the semiclassical limit, we obtain the following theorem in the unramified case.

Theorem 8.27 ([BD1])**.** *Let A be the algebra of twisted differential operators on* $\mathrm{Bun}_G(X)$ *for $G = \mathrm{SL}_2$ generated by the elements $\overline{T}_n$. Then*

(i) A is a polynomial algebra in $3g-3$ generators, and $\mathrm{Spec} A$ *is naturally identified with the affine space of opers* $\mathrm{Op}_{\mathfrak{sl}_2}(X)$*;*

(ii) A coincides with the algebra of all global twisted differential operators $\mathcal{D}(\mathrm{Bun}_G(X), \mathcal{K}^{1/2})$.

Thus we see that quantization of the Hitchin system is given by a canonical map $\mathbb{C}[\mathrm{Op}_{\mathfrak{sl}_2}(X)] \to \mathcal{D}(\mathrm{Bun}_G(X), \mathcal{K}^{1/2})$, which happens to be an isomorphism. This shows that the affine space $\mathrm{Op}_{\mathfrak{sl}_2}(X)$ plays the role of quantization of the Hitchin base $\mathcal{B}$, which in the SL_2 case is the space of quadratic differentials on X.

Remark 8.28. In the ramified case the spectrum of the Hitchin system should be interpreted as the space of opers with appropriate singularities. We will not discuss this here and refer the reader to [Fr2, BD2].

8.11 The Feigin–Frenkel theorem and quantization of Hitchin systems in higher rank

We would now like to generalize quantization of Hitchin systems to simple groups G of rank $r > 1$. And here we encounter a serious difficulty: the elements $\overline{T}_n$ still span a space of dimension $3g - 3$, so no longer suffice for a quantum integrable system. To complete this collection of operators to an integrable system using our approach, we must construct more central elements of $\widehat{U}(\widehat{\mathfrak{g}})/\langle \mathrm{K} = -h^\vee \rangle$, independent of the T_n. Remarkably, the needed elements do exist, but the proof of their existence is quite non-trivial. This is the celebrated theorem of Feigin and Frenkel, which is one of the most fundamental facts about affine Lie algebras.

Theorem 8.29 (Feigin–Frenkel, [FF])**.** *The center of $\widehat{U}(\widehat{\mathfrak{g}})/\langle \mathrm{K} = -h^\vee \rangle$ is generated (topologically) by the elements $\widehat{H}_{i,n}$, $i = 1, \dots, r, n \in \mathbb{Z}$, which quantize the classical hamiltonians $H_{i,n}$, and such that $\widehat{H}_{2,n} = T_{-n-2}$.*

The difficult part of the theorem is to show that every $H_{i,n}$ for $i > 2$ admits a quantization $\widehat{H}_{i,n}$; the fact that these quantizations generate the center is then proved by a deformation argument.[18]

Using this result, Beilinson and Drinfeld proved the following theorem in the unramified case.

Theorem 8.30 (Beilinson–Drinfeld, [BD1])**.** *Let A be the algebra of twisted differential operators generated by the descendants $\widehat{\widehat{H}}_{i,n}$ of the elements $\widehat{H}_{i,n}$. Then*

(i) A is a polynomial algebra in $(g-1)\dim\mathfrak{g}$ generators, which provides a quantization of the quantum Hitchin system for $\mathfrak{g}$.

(ii) A coincides with the entire algebra $\mathcal{D}(\mathrm{Bun}_G(X), \mathcal{K}^{1/2})$.

[18] For classical groups, there are explicit formulas for $H_{i,n}$, but for exceptional groups of type E and F no nice formulas are known.

In the ramified case (with parabolic structures or, more generally, twistings), the story is similar — the quantum Hitchin hamiltonians are still constructed as descendants of $\widehat{H}_{i,n}$, although part (ii) of Theorem 8.30 no longer holds.

Exercise 8.31 (cf. [E3])**.** Construct a quantization of the Calogero–Moser system as a special case of the quantum twisted Hitchin system. Namely, quantize the procedure of Subsection 7.6.

8.12 Opers in higher rank and Langlands duality

Finally, it remains to extend to higher rank the notion of an oper, i.e., generalize the statement that A is the algebra of regular functions on the space of opers. This can be done as follows.

Let G be an adjoint simple group with Lie algebra $\mathfrak{g}$, and $B \subset G$ a Borel subgroup with Lie algebra $\mathfrak{b}$. Fix a principal $\mathfrak{sl}_2$-triple $e, h, f \in \mathfrak{g}$ such that $e, h \in \mathfrak{b}$.

Definition 8.32 ([BD1, BD2])**.** A **$\mathfrak{g}$-oper** on X is a triple (E, E_B, ∇), where E is a G-bundle on X, E_B is a B-reduction of E, and ∇ is a connection on E which has the form

$$\nabla = d + (f + b(t))dt,\ \ b \in \mathfrak{b}[[t]]$$

for any trivialization of E_B (and hence E) on the formal neighborhood of any point x of X (where t is a formal coordinate at x).[19]

In this definition, X could be any smooth curve, a formal disk, or a punctured formal disk. But if X is a projective curve then there is another, equivalent definition of an oper. Namely, let $\rho\colon SL_2 \to G$ be the homomorphism corresponding to the triple e, h, f. Then we can consider the associated G-bundle $E_{X,\rho}$ to the SL_2 oper bundle E_X of Exercise 8.23 via ρ.

Proposition 8.33 ([BD1, BD2])**.** *A $\mathfrak{g}$-oper is the same thing as a connection on the bundle $E_{X,\rho}$. In other words, the underlying bundle of a $\mathfrak{g}$-oper is always isomorphic to $E_{X,\rho}$, and any connection on $E_{X,\rho}$ admits a unique oper structure (i.e., a B-structure satisfying Definition 8.32).*

Exercise 8.34. Show that $\mathfrak{sl}_n$-opers on X bijectively correspond to differential operators $L\colon \mathcal{K}_X^{\frac{-n+1}{2}} \to \mathcal{K}_X^{\frac{n+1}{2}}$ such that in local coordinates

$$L = \partial_t^n + a_2(t)\partial_t^{n-2} + \cdots + a_n(t).$$

Theorem 8.35 ([BD1])**.** *(i) $\mathfrak{g}$-opers form an affine space $\mathrm{Op}_{\mathfrak{g}}(X)$ whose underlying vector space is the Hitchin base $\mathcal{B}_{X,G}$.*

[19] This definition doesn't actually depend on the choice of the principal $\mathfrak{sl}_2$-triple since all principal $\mathfrak{sl}_2$-triples $e, h, f \in \mathfrak{g}$ such that $e, h \in \mathfrak{b}$ are conjugate by B.

(ii) The algebra A of Theorem 8.30 is naturally isomorphic to $\mathbb{C}[\mathrm{Op}_{\mathfrak{g}^\vee}(X)]$, where $\mathfrak{g}^\vee$ is the Langlands dual Lie algebra of $\mathfrak{g}$ (i.e., the Lie algebra with the dual root system to the root system of $\mathfrak{g}$).

Remark 8.36. 1. Note that $\mathcal{B}_{X,G}$ is canonically isomorphic to $\mathcal{B}_{X,G^\vee}$.

2. In fact, a more precise version of the Feigin-Frenkel theorem provides a natural identification of the center of $\widehat{U}(\widehat{\mathfrak{g}})/\langle K = -h^\vee\rangle$ with the algebra of regular functions on the space $\mathrm{Op}_{\mathfrak{g}^\vee}(D^\times)$. Using this identification, the procedure of descending central elements to Hitchin hamiltonians corresponds to the inclusion $\mathrm{Op}_{\mathfrak{g}^\vee}(X) \hookrightarrow \mathrm{Op}_{\mathfrak{g}^\vee}(D^\times)$.

Thus Theorem 8.35 should be viewed as an instance of *Langlands duality*. It is, in fact, a starting point of the *geometric Langlands program*. But this is already beyond the scope of this paper.

9 Solutions of problems

9.1 Problem 1

(i) This follows since $\mathrm{Pic}(X \setminus 0) \cong \mathrm{Pic}(X)/\langle \mathcal{O}(0)\rangle = \mathrm{Pic}_0(X)$ and the class of L in $\mathrm{Pic}_0(X)$ is nontrivial.

(ii) We realize X as $\mathbb{C}/\langle 1, \tau\rangle$, where $\mathrm{Im}\,\tau > 0$. Let $\theta(z) := \theta(z, \tau)$ be the theta-function of X, which is an entire function with simple zeros on the period lattice of X. This function is periodic with period 1 and

$$\theta(z + \tau) = -e^{-2\pi i z}\theta(z).$$

Thus $\frac{\theta'}{\theta}$ is periodic with period 1 and $\frac{\theta'}{\theta}(z + \tau) = \frac{\theta'}{\theta}(z) - 2\pi i$. Hence for $a \notin \langle 1, \tau\rangle$, the *Lame–Hermite function*

$$H(z, a) := e^{a\frac{\theta'}{\theta}(z)}\frac{\theta(z - a)}{\theta(z)}$$

is doubly-periodic, i.e. is a holomorphic function on $X \setminus 0$. It has a simple zero at a and no other zeros and poles, but it has an essential singularity at 0: $H(z) \sim Cz^{-1}e^{\frac{a}{z}}$, $z \to 0$. Thus H may be viewed as a non-vanishing holomorphic section of the analytic line bundle $\mathcal{O}(a)^\vee$ over $X \setminus 0$. Hence $\mathcal{O}(a)$ is trivial on $X \setminus 0$. But the given line bundle L is of the form $L = \mathcal{O}(a) \otimes \mathcal{O}(0)^\vee$ for some $0 \neq a \in X$, so we are done.[20]

[20]The Lamé–Hermite function arises as the Baker–Akhiezer function for elliptic curve in the theory of integrable systems, see [Kr1].

9.2 Problem 2

Pick $r > 0$ such that $A(z)$ and $A^{-1}(z)$ are regular on the circle $|z| = r$ (this can be done since these functions have countably many poles, while the set of choices for r is uncountable). By rescaling z, we may assume without loss of generality that $r = 1$.

Consider the elliptic curve $X := \mathbb{C}^\times/q^{\mathbb{Z}}$. Cover X by two open sets:

$$U_1 = \{z \in \mathbb{C} : |q| < |z| < 1\},\ U_2 = \{z \in \mathbb{C} : 1 - \varepsilon < |z| < 1 + \varepsilon\},$$

for sufficiently small $\varepsilon > 0$. The intersection $U_1 \cap U_2$ has two connected components $W_\pm$, where

$$W_+ := \{z \in \mathbb{C} : 1 < |z| < 1 + \varepsilon\},\ W_- := \{z \in \mathbb{C} : 1 - \varepsilon < |z| < 1\}.$$

Define a holomorphic vector bundle E on X by the transition function $g(z)$ which equals 1 on W_- and $A(z)$ on W_+ (this is well defined since for small enough ε, the matrix functions A, A^{-1} are regular on W_+). It is easy to see that then vector solutions of the difference equation $f(qz) = A(z)f(z)$ correspond to meromorphic sections of E. By the GAGA theorem, E has an algebraic structure, hence trivializes algebraically on some open set $X \setminus \{x_1, \ldots, x_m\}$ by the Hilbert theorem 90. Thus E has a basis of meromorphic sections $f_1, \ldots, f_n$ with no poles outside $x_1, \ldots, x_m$. Arranging them into a matrix $f = (f_1, \ldots, f_n)$, we obtain the desired solution.

9.3 Problem 3

Let $n := \deg \mathcal{L}$. Without loss of generality, we may assume $n > 0$; otherwise, replace $\mathcal{L}$ with $\mathcal{L}^\vee$. Then $\mathcal{L} = \mathcal{O}(x_1 + \cdots + x_n)$ for some points $x_1, \ldots, x_n \in X$. Let $y \in X$ be such that $x_1 + \cdots + x_n = ny$. We can construct such a y by lifting from $X = \mathbb{C}/\Gamma$ to $\mathbb{C}$, taking y to be the average of the x_i, and then projecting back to $\mathbb{C}/\Gamma$. Then $\mathcal{L} = \mathcal{O}(y)^{\otimes n}$, so $\mathcal{L}$ is trivialized by deleting y.

The same is not true on a genus 2 curve X purely by dimension counting: the Jacobian has dimension two, but the choice of a point $x \in X$ is only one dimension's worth of freedom.

9.4 Problem 4

By Corollary 3.6, GL_n-bundles on $\mathbb{P}^1$ have the form $\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$ for unique $m_1 \geq m_2 \geq \cdots \geq m_n$, $m_i \in \mathbb{Z}$. Since PGL_n-bundles are equivalence classes of GL_n-bundles under tensor product with line bundles, and line bundles on $\mathbb{P}^1$ are all of the form $\mathcal{O}(k)$ for some $k \in \mathbb{Z}$, it follows that PGL_n-bundles are classified by the differences $\ell_i := m_i - m_n$, which form a sequence of integers $\ell_1 \geq \cdots \geq \ell_{n-1} \geq 0$.

Similarly, SL_n-bundles are GL_n-bundles with trivial determinant, so they have the form $\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$ for unique $m_1 \geq m_2 \geq \cdots \geq m_n$, $m_i \in \mathbb{Z}$, $\sum_{i=1}^n m_i = 0$.

9.5 Problem 5

(i) By Theorem 3.4, $E \cong \mathcal{O}(k) \oplus \mathcal{O}(m-k)$ for unique integer $\frac{m}{2} \leq k \leq m$. It follows that for an integer $\frac{m}{2} \leq n \leq m$

$$E \cong \mathcal{O}(k) \oplus \mathcal{O}(m-k),\ \frac{m}{2} \leq k \leq n \iff \operatorname{Hom}(\mathcal{O}(n+1), E) = 0.$$

In other words, we want to find $f(z)$ such that $E \otimes \mathcal{O}(-n-1)$ has no non-zero section. The transition function for this bundle is

$$\begin{pmatrix} z^{-n-1} & z^{-n-1} f(z) \\ 0 & z^{m-n-1} \end{pmatrix}$$

and a section is a pair $\left(\begin{pmatrix} x_0(z) \\ y_0(z) \end{pmatrix}, \begin{pmatrix} x_\infty(z^{-1}) \\ y_\infty(z^{-1}) \end{pmatrix}\right)$ of vector-valued polynomials such that

$$\begin{pmatrix} x_0(z) \\ y_0(z) \end{pmatrix} = \begin{pmatrix} z^{-n-1} & z^{-n-1} f(z) \\ 0 & z^{m-n-1} \end{pmatrix} \begin{pmatrix} x_\infty(z^{-1}) \\ y_\infty(z^{-1}) \end{pmatrix}.$$

Let's try to construct such a section. The equation for the second entry implies

$$y_\infty(z^{-1}) = c_0 + c_{-1} z^{-1} + \cdots + c_{n-m+1} z^{n-m+1}$$

for some scalars $\{c_{-i}\}_{i=0}^{m-n-1}$. Plugging this into the first equation, we get

$$z^{-n-1} x_\infty(z^{-1}) + z^{-n-1}(a_1 z + \cdots + a_{m-1} z^{m-1})(c_0 + \cdots + c_{n-m+1} z^{n-m+1})$$

must be a polynomial. We can use the freedom to choose $x_\infty(z^{-1})$ to cancel all terms of degree $< -n$. The remaining terms of degree in $[-n, -1]$ must vanish. This is the condition that the equations

$$\begin{aligned} a_1 c_0 + a_2 c_{-1} + \cdots + a_{m-n} c_{n-m+1} &= 0 \\ a_2 c_0 + a_3 c_{-1} + \cdots + a_{m-n+1} c_{n-m+1} &= 0 \\ &\vdots \\ a_n c_0 + a_{n+1} c_{-1} + \cdots + a_{m-1} c_{n-m+1} &= 0 \end{aligned}$$

have a non-zero solution $(c_0, \ldots, c_{n-m+1})$, i.e. that the matrix

$$A \coloneqq \begin{pmatrix} a_1 & a_2 & \cdots & a_{m-n} \\ a_2 & a_3 & \cdots & a_{m-n+1} \\ & & \cdots & \\ a_n & a_{n+1} & \cdots & a_{m-1} \end{pmatrix}$$

has rank $< m-n$. So the desired condition on $a_1, \ldots, a_{m-1}$ is that the matrix A has full rank (equal to $m-n$).

(ii) In the special case $m = 2n$ we get that $E \cong \mathcal{O}(n) \oplus \mathcal{O}(n)$ iff

$$H(a_1, \ldots, a_{2n-1}) \coloneqq \det \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ & & \ddots & \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{pmatrix} \neq 0.$$

This determinant is called the *Hankel determinant*.

Remark 9.1. An isomorphism $E \cong \mathcal{O}(n) \oplus \mathcal{O}(n)$ is given by polynomials $g_0(z)$ and $g_\infty(z^{-1})$ valued in GL_2 such that

$$\begin{pmatrix} 1 & f(z) \\ 0 & z^{2n} \end{pmatrix} = g_0(z) \begin{pmatrix} z^n & 0 \\ 0 & z^n \end{pmatrix} g_\infty(z^{-1})^{-1}.$$

This can be rewritten as

$$\begin{pmatrix} z^{-n} & z^{-n} f(z) \\ 0 & z^n \end{pmatrix} = g_0(z) g_\infty(z^{-1})^{-1}.$$

Finding such $g_0(z)$ and $g_\infty(z^{-1})$ is a *Birkhoff factorization* problem. Birkhoff factorizations exist only for sufficiently generic operators, and part (ii) of the problem asks to find the appropriate condition on $f(z)$ so that this Birkhoff factorization exists.

9.6 Problem 6

Recall that GL_n-bundles E are equivalently vector bundles. A B-structure on the vector bundle E is equivalently a filtration by subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that E_{i+1}/E_i are line bundles. We construct such a filtration by induction on the rank n. By Lemma 3.5, there is a line subbundle $L \subset E$, i.e., a short exact sequence

$$0 \to L \to E \to E' \to 0.$$

Since $\operatorname{rank} E' < \operatorname{rank} E$, by the induction hypothesis, E' has a filtration by subbundles

$$0 = E'_0 \subset E'_1 \subset \cdots \subset E'_{n-1} = E'.$$

Let E_{i+1} be the preimage of E'_i in E and $E_1 = L$. This gives the desired filtration of E.

9.7 Problem 7

Pick a finite cover $X = \bigcup_{i \in I} U_i$ such that the principal G-bundle E is trivialized on each U_i. (By Theorem 2.6, we may take a Zariski cover.) Recall that E is determined by transition functions, which are regular functions

$$g_{ij} \colon U_i \cap U_j \to G$$

satisfying $g_{ii} = \mathrm{id}$, $g_{ij} \circ g_{ji} = \mathrm{id}$, and the 1-cocycle condition $g_{ij} \circ g_{jk} \circ g_{ki} = \mathrm{id}$. Therefore, an infinitesimal deformation of E is given by the modification

$$g_{ij} \mapsto \widetilde{g}_{ij} \coloneqq g_{ij} \cdot \exp(\varepsilon \xi_{ij})$$

where $\varepsilon^2 = 0$, for a choice of a regular function $\xi_{ij} \colon U_i \cap U_j \to \mathfrak{g}$ for each $i, j \in I$. The conditions that $\{\widetilde{g}_{ij}\}_{i,j \in I}$ is still a set of valid transition functions, namely that $\widetilde{g}_{ii} = \mathrm{id}$, $\widetilde{g}_{ij} \circ \widetilde{g}_{ji} = \mathrm{id}$ and $\widetilde{g}_{ij} \circ \widetilde{g}_{jk} \circ \widetilde{g}_{ki} = \mathrm{id}$, hold if and only if

$$\xi_{ii} = 0, \; g_{ij} \circ \xi_{ij} \circ g_{ij}^{-1} = -\xi_{ji},$$
$$g_{ij} \circ \xi_{ij} \circ g_{ij}^{-1} + g_{ik} \circ \xi_{jk} \circ g_{ik}^{-1} + \xi_{ki} = 0.$$

These equations say precisely that the element

$$(\xi_{ij})_{i,j \in I} \in \bigoplus_{i,j \in I} H^0(U_i \cap U_j, \mathrm{ad}\, E)$$

lies in the kernel of the Čech differential. Similarly, recall that two sets $\{g_{ij}\}_{i,j \in I}$ and $\{g'_{ij}\}_{i,j \in I}$ of transition functions are equivalent (i.e., define isomorphic bundles) if and only if there exist regular functions $\{h_i \colon U_i \to G\}_{i \in I}$ such that $g'_{ij} = h_i \circ g_{ij} \circ h_j^{-1}$. A similar reasoning shows that the deformed transition functions defined by two different $(\xi_{ij})_{i,j \in I}$ and $(\xi'_{ij})_{i,j \in I}$ are equivalent if and only if they differ by the image, under the Čech differential, of an element

$$(\eta_i)_{i \in I} \in \bigoplus_{i \in I} H^0(U_i, \mathrm{ad}\, E).$$

Putting it all together, we find that $T_E \operatorname{Bun}_G^\circ(X)$ is the cohomology at the middle term of the Čech complex

$$\bigoplus_{i \in I} H^0(U_i, \mathrm{ad}\, E) \to \bigoplus_{i,j \in I} H^0(U_i \cap U_j, \mathrm{ad}\, E) \to \bigoplus_{i,j,k \in I} H^0(U_i \cap U_j \cap U_k, \mathrm{ad}\, E),$$

which by definition is $H^1(X, \mathrm{ad}\, E)$.

9.8 Problem 8

The Garnier system is the Hitchin system for parabolic PGL_2-bundles on $\mathbb{P}^1$ with marked points $t_1, \ldots, t_N \in \mathbb{C}$, parabolic structures $y_1, \ldots, y_N$, and Higgs field

$$\phi = \sum_{i=1}^{N} \frac{\begin{pmatrix} p_i y_i & -p_i y_i^2 \\ p_i & -p_i y_i \end{pmatrix}}{z - t_i} dz.$$

Thus the determinant $\det \phi$ is a rational function of z with poles only at $t_1, \ldots, t_N$ (we drop the factor $(dz)^2$ for brevity). Moreover, since the residues of ϕ are nilpotent, these poles are simple. Thus $\det \phi = \frac{a(z)}{b(z)}$ where $b(z) := (z - t_1) \cdots (z - t_N)$, hence $\deg b = N$. Also since ϕ is regular at infinity, the corresponding matrix function vanishes there to second order, implying that $\det \phi$, viewed as a function, generically vanishes to order 4 at infinity. It follows that $\deg a = N - 4$.

The spectral curve is generically the normalization of the hyperelliptic curve C defined by the equation $y^2 = \frac{a(z)}{b(z)}$. Replacing y by $y/b(z)$, we obtain the equation

$$y^2 = a(z)b(z).$$

Thus if $\deg a = n_a$ and $\deg b = n_b$ then the genus of C is $\frac{1}{2}(n_a + n_b) - 1$. So in our case the genus of C equals $\frac{1}{2}(2N - 4) - 1 = N - 3$, the dimension of the Garnier system, as expected (note that in genus 0, the connected components of the moduli spaces of PGL_2 and GL_2-bundles are the same, so the dimension of the moduli space of GL_2-bundles equals the genus of the spectral curve).

For $N = 4$, we get the equation

$$y^2 = (z - t_1)(z - t_2)(z - t_3)(z - t_4),$$

which gives an elliptic curve. If we make a Möbius transformation sending (t_1, t_2, t_3, t_4) to $(0, 1, \infty, t)$ then the equation of the spectral curve takes the form

$$y^2 = z(z - 1)(z - t).$$

Remark 9.2. We see that in the case of four parabolic points on $\mathbb{P}^1$, the phase space of the Hitchin system is $\mathcal{M}^\circ := T^*(\mathbb{A}^1 \setminus \{0, 1, t\}) = (\mathbb{A}^1 \setminus \{0, 1, t\}) \times \mathbb{A}^1$, and the Hamiltonian is

$$H = p^2 x(x - 1)(x - t).$$

Thus the generic level curves of H are elliptic curves $p^2 x(x - 1)(x - t) = C$, which are missing the four branch points (i.e., points of order 2). As indicated in Subsection 5.1, this means that we should expect a partial symplectic compactification $\mathcal{M}$ of $\mathcal{M}^\circ$ in which these points are present, and that the Hitchin map $p \colon \mathcal{M}^\circ \to \mathcal{B} = \mathbb{A}^1$ extends to $p\colon \mathcal{M} \to \mathcal{B} = \mathbb{A}^1$, which is now proper, with generic fibers being complete elliptic curves.

The variety $\mathcal{M}$ can be constructed as follows (see [Hl], Subsection 4.2). Let E be the elliptic curve which is a double cover of $\mathbb{P}^1$ branching at points $0, 1, t, \infty$. Let $\mathcal{M}_* := (E \times \mathbb{A}^1)/(\pm 1)$. This is an orbifold with four A_1 singularities corresponding to points of order 2 on E. Let $\mathcal{M}$ be the blow-up of $\mathcal{M}_*$ at these singularities. Then $\mathcal{M}$ contains four exceptional divisors D_0, D_1, D_t, D_∞ isomorphic to $\mathbb{P}^1$, and $\mathcal{M} \setminus \sqcup_{i=1}^4 D_i = \mathcal{M}^\circ$. Also the map p clearly extends to a proper map $p\colon \mathcal{M} \to \mathbb{A}^1$ given by the second projection to $\mathbb{A}^1/(\pm 1) \cong \mathbb{A}^1$. The space $\mathcal{M}$ is the simplest example of the *Hitchin moduli space*, which is a natural home of the classical Hitchin system. It is obtained from the original phase space $\mathcal{M}^\circ$ by partial (smooth) compactification – attaching four disjoint copies of $\mathbb{P}^1$.

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