cubic fourfold. Beauville - Donagi Construction.

Thm: Let X be a smooth cubic fourfold. then the Fano scheme of lines on X, denoted by $F_1(X)$, is a HK Mfd of type $K3^{[2]}$.

Pf: $V \cong \mathbb{C}^6$, $\underset{\dim = 8.}{Gr(2,V)} \hookrightarrow \mathbb{P}(\wedge^2 V) \cong \mathbb{P}^{14}$

take a generic 8-plane $L \subset \mathbb{P}^{14}$.

then $S = Gr(2,V) \cap L$ is a polarized K3 surface of degree 14.

On the other side, consider $\underset{\mathbb{P}^{14}}{\underset{\cong}{\mathbb{P}(\wedge^2 V^*)}} \supset \underset{\mathbb{P}^5}{\underset{\cong}{L^*}}$

$\Delta \subset \mathbb{P}(\wedge^2 V^*)$ consists of degenerated 2-forms.
Smooth cubic hypersurface

$X := \Delta \cap L^*$: smooth cubic fourfold.
(Pfaffian cubic fourfold).

Last time: $S^{[2]} \longrightarrow F_1(X)$.

$(x,y) \longmapsto \{$ forms in L^* vanishing on $x+y$ $\}$.

$(x,y) \longleftarrow\!\shortmid \ \ell < X$
find a $W < V$ 4-subspace.

x, y is the intersection pts of $Gr(2,W)$ with

$\mathbb{P}(\Lambda^2 W) \cap L$

So for smooth Pfaffian cubic fourfold X, we have $F_1(X)$ is a

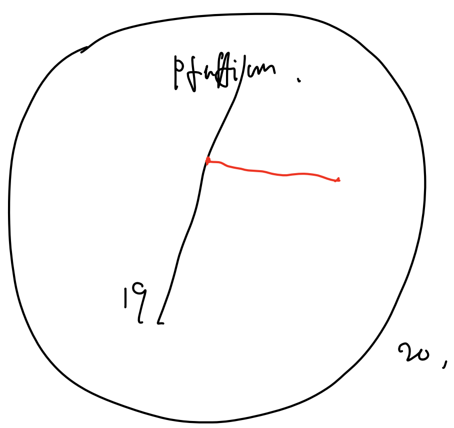
HK Mfd of type $K3^{[2]}$.

Moduli of smooth cubic fourfolds $= \mathbb{P}$(space of smooth cubic poly. in $x_1, \dots, x_6$)
quotient by $SL(6)$.

is connected [GIT quotient].

Any smooth cubic fourfold X is deformation equivalent to a

smooth Pfaffian cubic fourfold X_0.



$F_1(X)$ is deformation equivalent to $F_1(X_0)$ as complex manifold

$\Rightarrow$ $F_1(X)$ is still HK.

So $F_1(X)$ is a HK Mfd of type $K3^{[2]}$.

Moduli of cubic fourfolds $= \mathbb{P}$(cubic polynomials) $/\!/ SL(6)$.

Monomial of degree 3 in $x_1, \dots, x_6$.

$x_1^{\alpha_1} \cdots x_6^{\alpha_6}$ $\alpha_1 + \cdots + \alpha_6 = 3$, $\alpha_1, \cdots, \alpha_6 \geq 0$.

$$\binom{3+6-1}{6-1} = \binom{8}{5} = \binom{8}{3} = 8 \times 7 = 56.$$

$\dim SL(6) = 6 \times 6 - 1 = 35$.

Fact. (Mumford, etc). A smooth hypersurface of degree $d \geq 3$ and $\dim n \geq 2$, is stable w.r.t. $SL(n+2)$.

finite automorphism group, closed orbit.

$\dim$ (moduli of cubic fourfolds)

$= \dim (\mathbb{P}(\text{cubic polynomials})) - \dim (SL(6))$

$= (56-1) - (36-1) = 20$.

Pfaffian cubic fourfold,

$$\mathbb{P}(\Lambda^2 V^*) \supset L^* \supset L^* \cap \Delta.$$
$$\mathbb{P}(\Lambda^2 V^*) \cong \mathbb{P}^{14}, \quad L^* \cong \mathbb{P}^5$$

$L \in Gr(6, 15)$ $\dim Gr(6, 15) = 6 \times 9 = 54$.

different L may give rise to isomorphic cubic fourfolds.

$L_1^* \cap \Delta \cong L_2^* \cap \Delta$ if and only if L_1, L_2 lie in one orbit of the action of $PGL(V)$.

$PGL(V) \curvearrowright \mathbb{P}(\Lambda^2 V)$, $\{L\}$.

$\dim PGL(V)$

$\shortparallel$

dim (Moduli of Pfaffian cubic fourfolds) $= 54 - 35 = 19$.

BBF form,

X HK of dim n,

$\exists\ q_X : H^2(X, \mathbb{Z}) \to \mathbb{Z}$, $C_X \in \mathbb{Q}^{>0}$, s.t.

$\forall\ \alpha \in H^2(X, \mathbb{C})$, $\int_X \alpha^n = C_X\, q_X(\alpha)^{\frac{n}{2}}$.

q_X: integral, non-divisible, nondegenerate,

Ex: $X = S^{[m]}$, $m \geq 2$.

$H^2(X, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta$, as lattices,

where on the left side one has q_X,

on the right side $H^2(S, \mathbb{Z})$ is the K3 lattice,

$\delta^2 = 2-2m$. $\delta \perp H^2(S, \mathbb{Z})$.

$$C_X = \frac{(2m)!}{m! \cdot 2^m}.$$

Ex.: T complex torus of dimension 2,

$X = K_m(T) =$ Ker of $T^{[m+1]} \to T$.

$H^2(K_m(T), \mathbb{Z}) \cong H^2(T, \mathbb{Z}) \oplus \mathbb{Z}\delta$. as lattices.

Left side: q_X

right side: $H^2(T, \mathbb{Z}) \cong U^3$, $\delta^2 = -2-2m$.

$$C_X = \frac{(2m)!\,(m+1)}{m!\cdot 2^m}.$$

Thm (Huybrechts, Demailly-Bouchson):

A HK mfd X is projective if and only if there is a line bundle L on X s.t. $q_X(c_1(L)) > 0$.

For $S = Gr(2,V) \cap L$ polarized K3 surface of degree 14.

A generic pol. K3 of degree 14 arises in this way.

Moduli of such S = Moduli of $S^{[2]}$, has dimension 19.

For smooth cubic fourfolds X,

$F_1(X)$ is a HK mfd of type $K3^{[2]}$.

$F_1(X)$ admits a polarization, i.e. an ample line bundle L with $[L] \in \mathrm{Pic}(F_1(X))$ primitive,

$L^2 = 6$. $(q(L) = 6)$

HK admits unobstructed universal deformation.

$$\begin{array}{ccc} Y \hookrightarrow & \mathcal{Y} & \\ & \downarrow & \\ & B & \longrightarrow \mathbb{P}H^2(Y,\mathbb{C}). \\ & b & \longmapsto [H^{2,0}(\mathcal{Y}_b)] \end{array}$$

image of B is an analytic open subset of $Z(q_Y)$

$\dim B = b_2(Y) - 2$

$b_2(K3^{[m]}) = 23$, for $m \geq 2$.

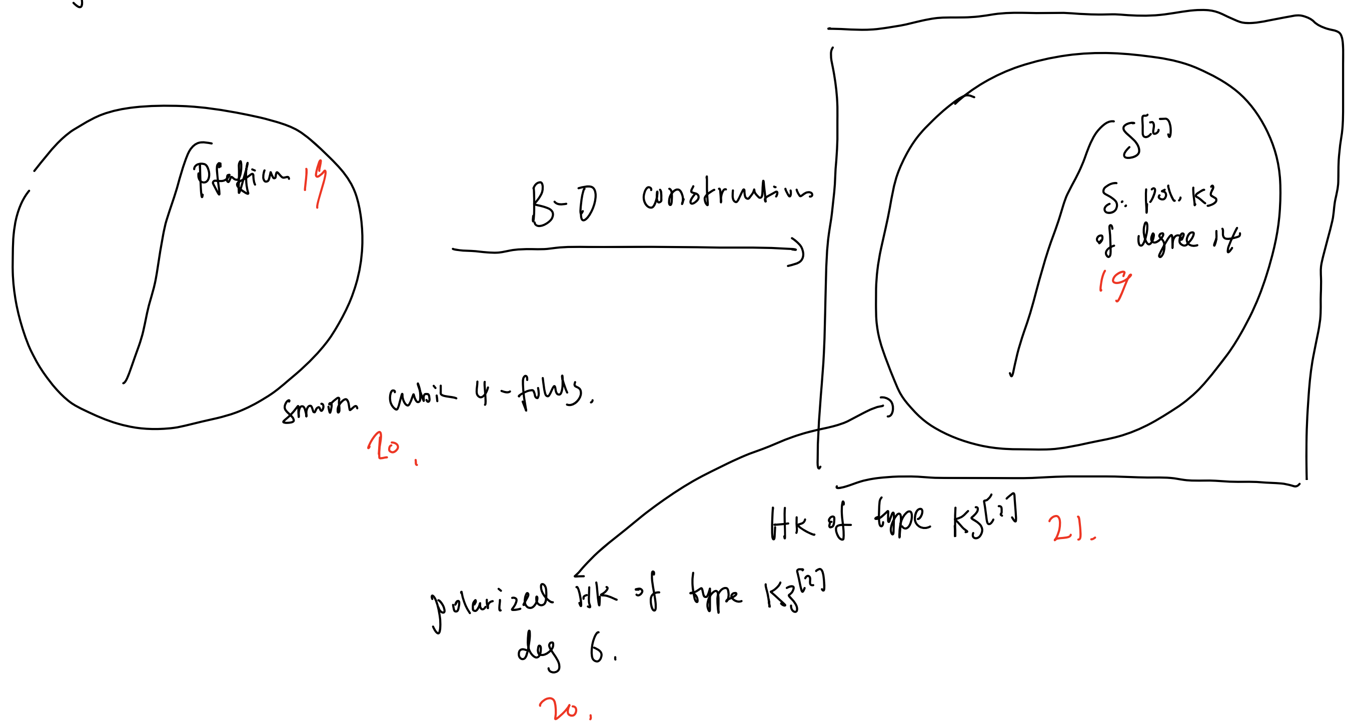
So $\dim B = 21$.

i.e. for Y of type $K3^{[m]}$, $m \geq 2$.

the universal deformation of Y has 21-dim'l base.

for smooth cubic fourfold X, $F_1(X)$ is polarized HK of type $K3^{[2]}$ and of degree 6.

Moduli of such polarized HK has dim. 20.



X: smooth cubic fourfold.

$$\left(H^2(F_1(X), \mathbb{Z}), q\right) \cong \Lambda_{K3} \oplus \langle -2 \rangle.$$

$$H^2(K3^{[2]}, \mathbb{Z}) \cong \Lambda_{K3} \oplus \langle -2 \rangle.$$

Abel Jacobi map: $\alpha: H^4(X,\mathbb{Z}) \longrightarrow H^2(F_1(X),\mathbb{Z})$.

$$\begin{tikzcd}
 & P \subset X \times F \arrow[dl, "\text{surj.}"'] \arrow[dr] & \\
X & & F
\end{tikzcd}$$

$P = \{(x,\ell) \mid x \in \ell\}$.

$\mathbb{P}^1$-bundle.

fiber genus 4 curve.

$x \in H^4(X,\mathbb{Z})$, pull back x to P, then integrate along fibers of $\begin{matrix} P \\ \downarrow \\ F \end{matrix}$ $\rightsquigarrow$ $\alpha(x) \in H^2(F_1(X),\mathbb{Z})$.

Thm (Beauville - Donagi):

① For hyperplane class $h \in H^{1,1}(X,\mathbb{Z})$

$\alpha(h^2) \in H^2(F_1(X),\mathbb{Z})$ is ample and $q(\alpha(h^2)) = 6$.

② Let $H^4(X,\mathbb{Z})_0 = (h^2)^\perp$,

$H^2(F_1(X),\mathbb{Z})_0 = (\alpha(h^2))^\perp$ w.r.t. q.

then $\alpha: H^4(X,\mathbb{Z})_0 \to (H^2(F_1(X),\mathbb{Z})_0, -q)$

is an isomorphism of lattices.

$\alpha: H^{3,1}(X,\mathbb{C}) \xrightarrow{\cong} H^{2,0}(F_1(X),\mathbb{C})$

X Pfaffian $\quad F_1(X) \cong S^{[2]}$

$H^2(F_1(X),\mathbb{Z}) = H^2(S^{[2]},\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta$.

S: polarized of degree 14. take $l \in H^{1,1}(S,\mathbb{Z})$, $l^2 = 14$.

Claim: $\alpha(h^2) = 2l - 5\delta$.

$q(2l-5\delta) = 4\times 14 + 25\times(-2) = 6$.

the norm of $\alpha(h^2)$ does not change as X deforms.

For any smooth cubic fourfold X, $\alpha(h^2)$ is an ample class of $F(X)$ of degree 6.

$$H^4(X,\mathbb{Z})_0 \xrightarrow{\sim} H^2(F(X),\mathbb{Z})_0(-1).$$

$$(h^2)^\perp \qquad\qquad (\alpha(h^2))^\perp$$

$H^2(F(X),\mathbb{Z})_0(-1) \cong$ the orthogonal complement of $2l-5\delta$ in $(\Lambda_{K3} \oplus \mathbb{Z}\delta)(-1)$.

this is an even lattice of signature $(20,2)$, discriminant 3

$$\Rightarrow \quad \underbrace{H^4(X,\mathbb{Z})_0}_{\text{is even}} \cong \underset{(16,0)}{E_8^2} \oplus \underset{(2,2)}{U^2} \oplus \underset{(2,0)}{A_2}.$$

We can try to do this directly:

$X \subset \mathbb{P}^5$ smooth cubic fourfold,

$e(X) \rightsquigarrow b_4(X) = 23$.

Residue formula $\rightsquigarrow$ $H^4(X) = H^{3,1}(X) \oplus H^{2,2} \oplus H^{1,3}$.

1   21   1

$X = Z(F)$ F: cubic polynomial.

$H^{3,1}$ is generated by $\left[\dfrac{dx_1 \wedge \cdots \wedge dx_6}{F^2}\right] \in H^4(X, \mathbb{C})$.

$\dim H^{3,1} = 1$. $\dim H^{1,3} = 1$.

$H^{2,2} \leftrightarrow \left[\dfrac{G\, dx_1 \wedge \cdots \wedge dx_6}{F^3}\right]$.

$\varphi: H^4(X, \mathbb{Z}) \times H^4(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$,

Hodge-Riemann: $\varphi(h^2, h^2) = 3$, $\varphi / H^{2,2}(X, \mathbb{R})$ positive,

$\varphi(w, \bar{w}) < 0$. $w \in H^{3,1} - 0$

φ has signature $(21, 2)$,

φ is unimodular, odd.

$$\varphi \sim \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}$$

$(h^2)^\perp =: H^4(X, \mathbb{Z})_0$

We do not know this must be even

so we can not conclude $H^4(X, \mathbb{Z})_0 = E_8^2 \oplus U^2 \oplus A_2$.

So it seems B-D's calculation offers non-trivial information to determine the structure of $H^4(X, \mathbb{Z})_0$.

Pfaffian cubic fourfold is rational. (19-dim'l family of rat'l cubic fourfold)

Conjecture: a generic cubic fourfold is irrational.

but not a single cubic fourfold is known to be irrational yet.