Today: Beauville - Donagi construction.

cubic fourfolds $X \longleftrightarrow$ HK of deformation type $K3^{[2]}$.

$H^4(X, \mathbb{Z})$. $\longleftrightarrow$ BBF form.

1985. Beauville - Donagi

La variété des droites d'une hypersurface cubique de dimension 4

$X \subset \mathbb{P}^5$ hypersurface of degree 3.

called cubic fourfold

is the zero locus of a homogeneous polynomial with 6 variables

and of degree 3.

e.g. $x_1^3 + x_2^3 + \cdots + x_6^3 = 0$ Fermat cubic fourfold

Usually ask X to be smooth.

Consider $F_1(X) = \{ \ell \subset X \mid \ell \text{ is a } \underline{\text{line}} \text{ contained in } \mathbb{P}^5 \}$

$\ell \in Gr(2,6)$.

Fact: If X is a smooth cubic hypersurface

of dimension $n \geq 2$ (cubic n-fold),

then $F_1(X)$ is naturally a proj. smooth variety of dim $2n-4$.

e.g. $n=2$, X cubic surface, $F_1(X)$ is the set of 27 $\underline{\text{different}}$ points,

$\dim F_1(X) = 0$, smooth.

$n=3$. X cubic threefold, $F_1(X)$ is a smooth proj. surface of

general type. [is used by Clemens - Griffiths to show irrationality of smooth cubic threefolds].

In our situation, $n=4$, then $F_1(X)$ is a smooth proj. 4-fold.

Thm (Beauville - Donagi):

For a smooth cubic fourfold X, $F_1(X)$ is a HK mfd of type $K3^{[2]}$.

Let V be a complex vector space of dimension 6.

$$\begin{array}{ccc} Gr(2,V) & \xhookrightarrow{\text{Plücker embedding}} & \mathbb{P}(\Lambda^2 V). \\ \dim & & 14. \\ V_1 & \longmapsto & \Lambda^2 V_1 \end{array}$$

Choose a generic 8-plane $L \subset \mathbb{P}(\Lambda^2 V)$.

Let $S = Gr(2,V) \cap L$ is a simply connected smooth proj surface,

Fact: K_S trivial.

S is a K3 surface together with an embedding in $L \cong \mathbb{P}^8$.

S is a polarized K3 surface of degree $8\times 2-2 = 14$.

$V^* = \mathrm{Hom}(V, \mathbb{C})$. $\quad \underset{14}{\mathbb{P}(\Lambda^2 V)} \supset \underset{8}{L}$.

$$\mathbb{P}(\Lambda^2 V^*) \supset \underset{\substack{\cong \\ \mathbb{P}^5}}{L^*} := \{ [x] \mid x \in \Lambda^2 V^*, \ x \perp L \}.$$

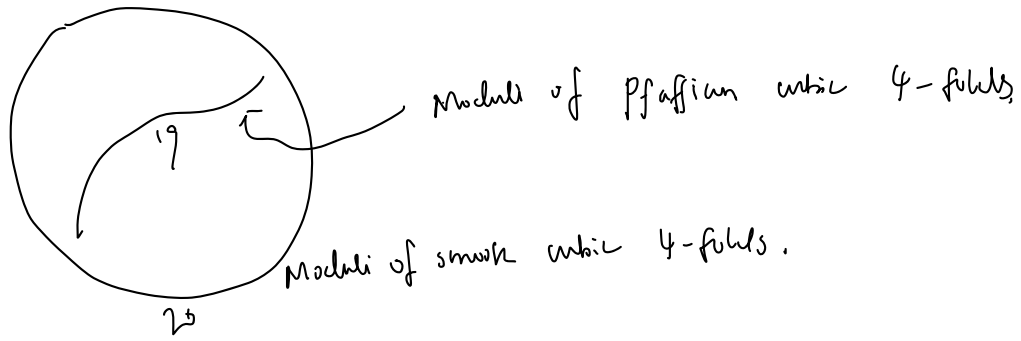
$\Delta \subset \mathbb{P}(\Lambda^2 V^*)$ is the set of degenerated 2-forms.

$\varphi \in \Lambda^2 V^*$ degenerated $\overset{\text{def}}{\Longleftrightarrow}$ $\Lambda^3 \varphi = 0$.

So Δ is a smooth hypersurface in $\mathbb{P}(\Lambda^2 V^*)$

Let $X := L^* \cap \Delta$, is a smooth cubic fourfold in L^*.

X is called a Pfaffian cubic fourfold.



$S = Gr(2, V) \cap L$. K3 with polarization of degree 14.

$X = L^* \cap \Delta$ cubic 4-fold.

Aim: construct an isomorphism

$$F_1(X) \xrightarrow{\cong} S^{[2]}.$$

First we construct a map $S^{[2]} \longrightarrow F_1(X)$.

$S^{[2]}$ = Blowup of the diagonal of the symmetric product $S^{(2)}$.

$S^{(2)} \ni \{(x,y) \text{ no order} \mid x, y \in S\}$.

take $x, y \in S$, $x \neq y$, then (x,y) represent a point in $S^{[2]}$,

$S = Gr(2, V) \cap L$, so x, y represent two 2-planes in V,

genericity of $L \Rightarrow x \cap y = 0$.

then x, y spans a 4-plane $x+y$ in V.

Now look at elements of L^* which vanishes on $\Lambda^2(x+y) \cong \mathbb{C}^6$
5. $\cap$
$\Lambda^2 V$

Since $x, y \in S \subset L$,

$\Rightarrow$ all elements of L^* vanish on $\Lambda^2 x, \Lambda^2 y \in \Lambda^2(x+y)$,

So the condition "vanishing on $\Lambda^2(x+y)$" exactly contributes

4 freedom-constraints.

So elements in L^* vanishing on $\Lambda^2(x+y)$ form a line

$l \subset L^*$.

all elements in l are degenerated since they vanish on $\Lambda^2(x+y)$

So $l \subset X = L^* \cap \Delta$.

So we obtain a map $S^{[2]}$ - Exceptional divisor $\to F_1(X)$

A more detailed analysis implies: this map can extend to

$$S^{[2]} \longrightarrow F_1(X).$$

Next we construct $F_1(X) \longrightarrow S^{[2]}$.

take $l \subset X$,

l parametrizes a 2-dim'l v.sp. of degenerated 2-forms in

Lemma: If L is generic, then there exists $\Lambda^2 V^*$.

a 4-subspace $W \subset V$, s.t. any element in ℓ

vanishes restricted to W.

$\varphi \in \Lambda^2 V^*$ φ degenerated (i.e. $\Lambda^3 \varphi = 0$)

then we can take basis of V, s.t.

$$\varphi \sim \left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline & & \\ 0 & & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|cc|c} 0 & 1 & & & \\ -1 & 0 & & 0 & 0 \\ \hline & & 0 & 1 & \\ 0 & & -1 & 0 & 0 \\ \hline 0 & & & 0 & 0 \end{array}\right)$$

$\mathrm{Ker}(\varphi) := \{ v \in V \mid \varphi(v, w) = 0, \ \forall w \in V \}$.

$\dim \mathrm{Ker}(\varphi) = 2$ or 4.

We claim: If L is generically chosen, then any two different

elements $\varphi_1, \varphi_2 \in \ell$ must satisfy $\mathrm{Ker}(\varphi_1) \cap \mathrm{Ker}(\varphi_2) = 0$

Pf of the claim: If $\exists \ \varphi_1, \varphi_2 \in \ell$, $\dim(\mathrm{Ker}(\varphi_1) \cap \mathrm{Ker}(\varphi_2)) \geq 1$

take $v \in \mathrm{Ker}\,\varphi_1 \cap \mathrm{Ker}\,\varphi_2 - 0$

then $\forall \varphi \in \ell$, we have $\mathrm{Ker}(\varphi) \ni v$.

ℓ vanishes on $v \wedge V$: 5-dim'l subspace of $\Lambda^2 V$.

If fix v, $\dim(v \wedge V) = 5$,

a generic L^* contains exactly 1 point vanishing on $v \wedge V$.

So L with existence of $l \in L^*$ vanishing on $v \wedge V$ is not generic.

We can calculate for a fixed v, the dimension of the space of such L.

$(v \wedge V)^* \subset \mathbb{P}(\Lambda^2 V^*)$ $\quad L^* \cap (v \wedge V)^*$ dim ≥ 1.

$\quad 9 \qquad\qquad 14$

$$9 \times 5 = 45.$$

Generic, parametrizing space of L has dim $9 \times 6 = 54$.

v has freedom 5.

So such L form a space of dim at most $45 + 5 = 50$

So for generic L, $\nexists\, v$, s.t. $\exists\, l \in L^*$, with l vanishes on $v \wedge V$

The claim is proved.

We want to construct $F_1(X) \longrightarrow S^{[2]}$.

take $l \in F_1(X)$, If $\exists$ different $\varphi_1, \varphi_2 \in l$, s.t. $\mathrm{Ker}(\varphi_1)$, $\mathrm{Ker}(\varphi_2)$ have dimension 2,

by the claim: $\mathrm{Ker}(\varphi_1) \cap \mathrm{Ker}(\varphi_2) = 0$,

Let $W = \mathrm{Ker}(\varphi_1) + \mathrm{Ker}(\varphi_2) \subset V$.

dim $= 4$.

Claim: φ_1, φ_2 vanishes on W.

$\mathrm{Ker}(\varphi_2) = W_1$. enough to show φ_2 vanishes on W_1.

take basis, s.t.

$$\varphi_1 \sim \begin{array}{c} \left(\begin{array}{cc|cc|cc} 0 & 1 & & & & \\ -1 & 0 & & & & \\ \hline & & 0 & 1 & & \\ & & -1 & 0 & & \\ \hline & & & & & \\ & & & & & \end{array}\right) \\ \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \end{array}$$

$W_1 = \langle e_5, e_6 \rangle$.

If $\varphi_2(e_5 \wedge e_6) \neq 0$, then w.r.t. this basis, we have:

$$\varphi_2 \sim \left(\begin{array}{c|c} * & * \\ \hline * & \neq 0 \end{array}\right)$$

then

$\exists\ \varepsilon \in \mathbb{C}^\times$, s.t. $\varphi_1 + \varepsilon \varphi_2$ has nonzero determinant,

contradiction because $[\varphi_1 + \varepsilon \varphi_2] \in \ell < \Delta$

should be degenerated.

We now have $W \subset V$ with ℓ vanishes on $\Lambda^2 W \subset \Lambda^2 V$,

$\ell \in L^* < \mathbb{P}(\Lambda^2 V^*)$

Consider $\underset{4}{Gr(2, W)} \subset \underset{5}{\mathbb{P}(\Lambda^2 W)}$

smooth quadric hypersurface.

$$\underbrace{\mathbb{P}(\Lambda^2 W)}_{5} \cap \underset{8}{L} \subset \underset{14}{\mathbb{P}^2(\Lambda^2 V)}$$

dim = 1 since ℓ vanishes on $\Lambda^2 W$.

this line intersects $Gr(2, W)$ in 2 points

$\rightsquigarrow$ a point in $S^{[2]}$.

So we obtain $F_1(X) \longrightarrow S^{[2]}$. $\square$