Lecture 22 30 May 2022

Recall: Γ Recurrent $\Longleftrightarrow$ $D(\Gamma) = D_0(\Gamma)$

Comments on the proofs

Royden decomposition for transient Γ

$$D(\Gamma) = D_0(\Gamma) \oplus HD(\Gamma)$$

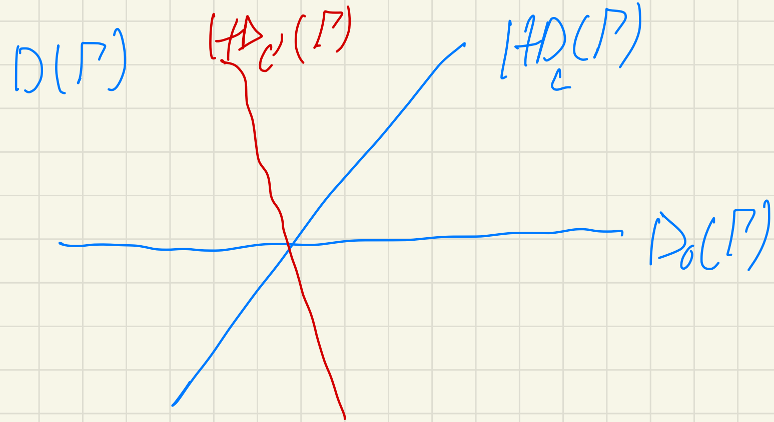
$$\Gamma \in O_{HD} \Longleftrightarrow \left(D(\Gamma) = D_0(\Gamma) \oplus \{\text{const}\}. \right) \quad (\#)$$

$\rightsquigarrow$ $\psi : \Gamma_1 \to \Gamma_2$

$\Gamma_1 \sim \Gamma_2$ recurrent ? $\Longleftrightarrow$ Study $D(\Gamma_i)$, $D_0(\Gamma_i)$ for $i = 1, 2$

$\Gamma_1, \Gamma_2 \in O_{HD}$? $\Longleftrightarrow$ Study Royden decomposition

Is it in the form of $(\#)$?



$\exists K > 0$ s.t.

$$\frac{c'}{K} \le c \le Kc'$$

Consider Γ: $\quad \psi^* D(\Gamma') = \left(\psi^* D(\Gamma') \cap D_0(\Gamma)\right) \oplus \left(\psi^* D(\Gamma') \cap HD(\Gamma)\right)$

$\psi \uparrow$? isom. $\qquad \psi \uparrow$ well-defined? ✓ iso? true, shall prove surjective. $\qquad \psi \cancel{\uparrow}$

Γ': $\quad D(\Gamma') = D_0(\Gamma') \qquad HD(\Gamma')$

Claim: Suppose $f = f' \circ \psi$. Then $f \equiv 0 \Rightarrow f' \equiv 0$.

It is true since ψ is surjective.

Γ' recurrent $\Rightarrow$ $e' \in D_0(\Gamma')$ $\Rightarrow e = \Psi^*(e') \in \Psi^* D(\Gamma') \cap D_0(\Gamma) \subset D_0(\Gamma)$

$\Rightarrow$ Γ recurrent

Next goal: $\Gamma \in \mathcal{O}_{HD} \Rightarrow \Gamma' \in \mathcal{O}_{HD}$,

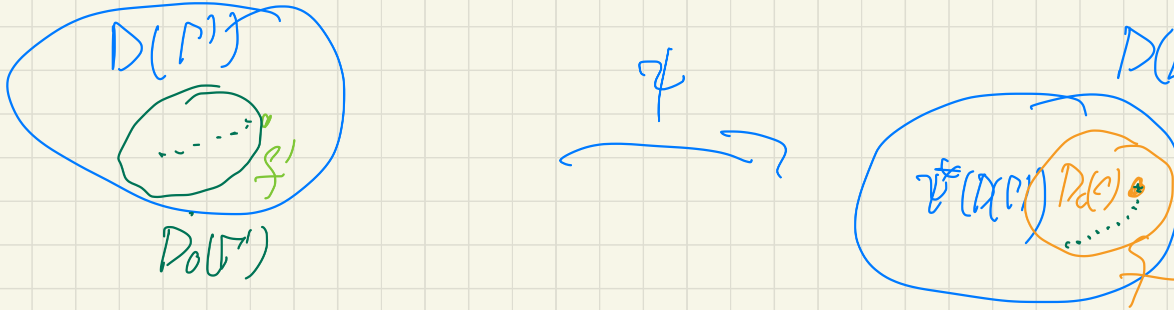
Claim: $f \in \Psi^* D(\Gamma') \cap D_0(\Gamma) \Leftrightarrow f' \in D_0(\Gamma')$,

Know already: $f' \in D_0(\Gamma') \Rightarrow f \in \Psi^* D(\Gamma') \cap D_0(\Gamma)$

Problem: If $f \in D_0(\Gamma) \Rightarrow \exists\ f_n$ with finite support

s.t. $\|f - f_n\| \to 0$ as $n \to \infty$

However, if $f \in \Psi^*(D(\Gamma'))$, $\not\Rightarrow$ $f_n \in \Psi^*(D(\Gamma'))$



Problem: We don't know $f' \in D_0(\Gamma')$ even if we know $f \in D_0(\Gamma)$.

Idea of proof: Want to find sequence $\{\tilde{f}_n\}$ finite support and inside $\Psi^*(D(\Gamma'))$ that converge to f.

Approach: Project $\{f_n\}$ to $\psi^*(D(\Gamma'))$ to get $\{f_n\}$.

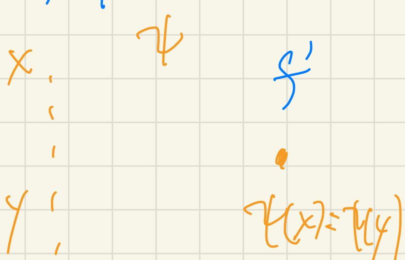
Let $x \in V$, define $(x) := \{ y \in V \mid \psi(x) = \psi(y) \}$.

$w(x) := \#(x)$ number of elements.

Given $g : V \to \mathbb{R}$, define $(Mg) : V \to \mathbb{R}$ by

$f := f' \circ \psi$

$$Mg(x) := \frac{1}{w(x)} \sum_{s \in (x)} g(s)$$



$$\Rightarrow \quad Mg(x) = Mg(y) \quad \text{if } y \in (x).$$

$\Rightarrow$ Mg is induced by some function g' on Γ'.

E.g. Suppose $z = \Psi(x) \in V'$.

$$g'(z) := Mg(x) \text{ is}$$

independent of the preimage in $\Psi^{-1}(z) \ni (x)$.

$$\Rightarrow \quad Mg = g' \circ \Psi.$$

Ex: If $g \in D(\Gamma) \Rightarrow Mg \in D(\Gamma)$

$$\Rightarrow \quad M : \begin{array}{ccc} D(\Gamma) & \to & \Psi^*(D(\Gamma')) \subset D(\Gamma). \\ \cup & & \\ D_0(\Gamma) & \to & D_0(\Gamma) \cap \Psi^*(D(\Gamma')). \end{array}$$

Take $\xi_n := M f_n$ finite support and $\psi^*(D(\Gamma''))$,

$\Rightarrow \xi_n'$ finit support $\subset D_0(\Gamma')$.

$\Rightarrow \xi_n' \xrightarrow{\text{as } n\to\infty} f' \in D_0(\Gamma')$.

$\Rightarrow \psi^* : D_0(\Gamma') \longrightarrow D_0(\Gamma) \cap \psi^*(D(\Gamma'))$

surjective

and <u>isomorphism</u>

Claim: $\Gamma \in O_{HD} \Rightarrow \Gamma' \in O_{HD}$.

Pf: Assume Γ, Γ' transient. and $\Gamma \in O_{HD}$,

Take $f' \in HD(\Gamma')$,

$$f := f' \circ \psi \in D(\Gamma)$$

$$f = g + \kappa \quad \text{where } g \in D_0(\Gamma), \ \kappa \text{ is const.}$$

Note: $\kappa = \kappa' \circ \psi \Rightarrow f, \kappa \in \psi^*(D(\Gamma'))$

$$\Rightarrow g \in \psi^*(D(\Gamma')),$$

$$\Rightarrow \underset{\substack{\uparrow \\ D_0(\Gamma')}}{0} + \underset{\substack{\uparrow \\ \text{harmonic}}}{f'} = \underset{\substack{\uparrow \\ D_0(\Gamma')}}{g'} + \underset{\substack{\uparrow \\ \text{const.}}}{\kappa'}$$

Uniqueness of Royden decomposition $\Rightarrow$ $g' \equiv 0$ and $f' = x'$

$\Rightarrow$ $\Gamma' \in O_{HD}$.

Corollary: Γ recurrent $\Rightarrow$ Γ' recurrent

Pf. Γ recurrent $\Rightarrow$ $\exists f_n$ finite support s.t.

$\|e - f_n\| \to 0$ as $n \to \infty$.

Define $\hat{f}_n := M f_n$ $\Rightarrow$ $\exists \hat{f}_n'$ s.t. $\hat{f}_n = \hat{f}_n' \circ \varphi$

has finite support on Γ'.

Note: $Me = e$.

$\|e' - \hat{f}_n'\| \le C \|e - \hat{f}_n\| = C\|M(e - f_n)\| \le \hat{C} \|e - f_n\| \to 0.$

$\Rightarrow e' \in D_0(\Gamma') \Rightarrow \Gamma'$ recurrent.

Recall: $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ not surjective

Take $\Gamma' := \varphi(\Gamma_1) \subset \Gamma_2$

We proved: Γ_1 transient $\Rightarrow \Gamma'$ transient $\Rightarrow \Gamma_2$ transient

Γ_1 recurrent $\Rightarrow \Gamma'$ recurrent $\not\Rightarrow \Gamma_2$ recurrent / transient

(e.g. $\mathbb{Z}^2 \Rightarrow \mathbb{Z}^2 \Rightarrow \mathbb{Z}^3$)

Thm Γ_1, Γ_2 roughly isometric graphs of bounded degree. Then Γ_1 transient $\Leftrightarrow$ Γ_2 transient.

Pf: $\phi: \Gamma_1 \to \Gamma_2$ roughly isometric, i.e. $\exists\, a, b > 0$

$$(*) \qquad a\, d_{\Gamma_1}(x,y) - b \le d_{\Gamma_2}(\phi(x), \phi(y)) \le a\, d_{\Gamma_1}(x,y) + b.$$

! ϕ might not be a morphism.

Claim: $\phi: \Gamma_1 \to \Gamma_2^k$ is morphism for $k > a+b$.

Moreover, ϕ is roughly isometric.

Pf: Suppose $x \sim y \in Y(\Gamma_1)$

$\Rightarrow d_{\Gamma_1}(x,y) \leq 1$

$\Rightarrow d_{\Gamma_2}(\phi(x), \phi(y)) \leq a+b < K$

$\Rightarrow \phi(x) \sim \phi(y) \in Y(\Gamma_2^K)$.

Recall: $\Gamma_1 \sim \Gamma_2 \sim \Gamma_2^K$ roughly isometric.

Claim: $\exists m>0$, $\phi(x) = \phi(y) \Rightarrow d_{\Gamma_1}(x,y) < m$.

Pf: $\underset{\substack{\| \\ 0}}{d_{\Gamma_2^K}(\phi(x), \phi(y))} \Rightarrow d_{\Gamma_2}(\phi(x), \phi(y)) = 0 \overset{(*)}{\Rightarrow} a\, d_{\Gamma_1}(x,y) - b \leq 0$

$\Rightarrow d_{\Gamma_1}(x,y) \leq ab$.

$$\Gamma_1 \text{ transient} \implies \phi(\Gamma_1) \text{ transient}$$

$$\subset \Gamma_2^K$$

$$\implies \Gamma_2^K \text{ transient}$$

$$\implies \Gamma_2 \text{ transient.}$$