Examples of module categories

<u>Example:</u> A (multi) tensor category $\mathcal{C}$ is itself a $\mathcal{C}$-module cat:

- module product = tensor product
- module associativity constraint $m = a$ (associativity constraint of $\mathcal{C}$)

$\longrightarrow$ it is a categorification of the notion of regular representation

More generally: if $\mathcal{D} \subset \mathcal{C}$ is a (multi) tensor subcat. then $\mathcal{C}$ is a $\mathcal{D}$-module cat.

<u>Example:</u> $\mathcal{C}$ is a $(\mathcal{C} \boxtimes \mathcal{C}^{op})$-module cat. with:

$(X \otimes^{op} Y := Y \otimes X)$

$(X \boxtimes Y) \otimes Z := X \otimes Z \otimes Y$

Exo: Write details

Religne tensor product

__More generally__: Let $\mathcal{C}, \mathcal{D}$ be multitensor categories, and $\mathcal{M}$ be a $(\mathcal{C}, \mathcal{D})$-bimodule cat. Then it is a $\mathcal{C} \boxtimes \mathcal{D}^{op}$-module cat. (and reciprocally)

__Exercice__: Let $\mathcal{M}$ be a $(\mathcal{C}, \mathcal{D})$ bimodule cat.

Show that $\mathcal{M}^\vee$ is a $(\mathcal{D}, \mathcal{C})$ bimodule cat.

__Example__: Let $\mathcal{C}$ be a multitensor cat., let $\mathcal{C} = \bigoplus_{i,j} \mathcal{C}_{ij}$ be its usual decomposition

(Recall that:
- every indecomposable obj belongs to some $\mathcal{C}_{ij}$
- $\mathcal{C}_{ii}$ is a tensor cat)

__Then__: $\mathcal{C}_{ij}$ is a $(\mathcal{C}_{ii}, \mathcal{C}_{jj})$-bimodule cat.

__Example__: When is $\mathcal{M} = Vec$ a $\mathcal{C}$-module cat?

__Exo__: Show that $End(\mathcal{M}) \cong Vec$ (as tensor cat)

So by previous (representation) Prop, Vec is a $\mathcal{C}$-module cat iff
$\exists F: \mathcal{C} \to \mathrm{End}(M)$ tensor functor, i.e. $F: \mathcal{C} \to \mathrm{Vec}$ fiber functor

Conclusion: Vec has a structure of $\mathcal{C}$-module cat
iff $\mathcal{C}$ has a fiber functor.
In that sense, the theory of module cat. is an extension of
the theory of fiber functors.

Exercice: Let $F: \mathcal{C} \to \mathcal{D}$ be a tensor functor.
Show that $\mathcal{M} = \mathcal{D}$ has a structure of $\mathcal{C}$-module cat with
module product $X \otimes Y := F(X) \otimes Y$

Let G be a finite group. What about $\mathrm{Rep}(G)$-module categories?

Let $L \subset G$ be a subgroup. The restriction $\mathrm{Res}: \mathrm{Rep}(G) \to \mathrm{Rep}(L)$ is a tensor functor, so by previous exercise:

$\mathrm{Rep}(L)$ is a $\mathrm{Rep}(G)$-module cat.

More generally, we may prove later that every semisimple indecomposable $\mathrm{Rep}(G)$-module cat is of the form $\mathrm{Rep}_\psi(L)$, where:

- $L \subset G$ is a subgroup
- $\psi \in Z^2(L, k^*)$ is a 2-cocycle called Schur multiplier

required to define a projective rep of L on $V \in \mathrm{Vec}$:

$\rho: L \to GL(V)$, $\rho(g) \circ \rho(h) = \psi(g,h)\rho(gh)$

$\mathrm{Rep}_\psi(L)$ is the abelian cat. of projective rep. of L with Schur multiplier ψ.

<u>Rk:</u> $\mathrm{Rep}_\psi(L) = \mathrm{Rep}(\mathbb{k}L_\psi)$ where $\mathbb{k}L_\psi$ is given by
$g.h = \psi(g,h)\, gh$.

Now, what about Vec_G-module cats: there are abelian cat $\mathcal{M}$ with:

(again by previous rep. Prop)

- collection of autoequivalence: $F_g : \mathcal{M} \to \mathcal{M}$
  $M \mapsto \delta_g \otimes M$
- collection of tensor functor iso: $\eta_{gh} : f_g \circ f_h \to f_{gh}$.
- 2-cocycle relations: $\eta_{gh,k} \circ \eta_{g,h} = \eta_{g,hk} \circ \eta_{h,k}$ for nat. iso. $f_g \circ f_h \circ f_k \xrightarrow{\sim} f_{ghk}$

(for associativity)

in other words (according to some previous session) a Vec_G-module cat is the same thing as an ab. cat with an action of G.

Let us state w/o proof the following classification result:

Prop: Every indecomposable semisimple Vec_G-module cat $\mathcal{M}$ is determined by:

- a subgroup $L \subset G$ s.t. the simple objects of $\mathcal{M} \leftrightarrow G/L$.
- an element $\psi \in H^2(L, k^*)$ encoding the Vec_G-module associativity constraint.

Such an $\mathcal{M}$ is denoted $\mathcal{M}(L, \psi)$.

Rk: Indecomposable semisimple module cat over $\boxed{\text{this similarity should be explained by the fact that } \mathrm{Vec}_G \text{ and } \mathrm{Rep}(G) \text{ are Morita equivalent, or equiv. have same Drinfeld center, ... (see later...)}}$

- $\mathrm{Rep}(G) \rightsquigarrow \mathrm{Rep}_\psi(L)$
- $\mathrm{Vec}_G \rightsquigarrow \mathcal{M}(L, \psi)$

with $L \subset G$, and $\psi \in Z^2(L, k^*)$ or $H^2(L, k^*) = Z^2/B^2$.

Exact module categories over finite tensor categories

The understanding of all the module cat. over a finite tensor cat $\mathcal{C}$ may be too complicated (even for $\mathcal{C} = \mathrm{Vec}$, where it requires the understanding of all the $\mathbb{k}$-linear locally finite ab. cat).

The following def makes classification much more manageable:

Def: Let $\mathcal{C}$ be a multitensor cat with enough projective objects (i.e. every simple object has a projective cover).
A $\mathcal{C}$-module cat. is called exact if $\forall P \in \mathcal{C}$ projective, $\forall M \in \mathcal{M}$ then $P \otimes M$ is also proj. in $\mathcal{M}$.

(Recall that $X \in \mathcal{D}$ is projective if (def) $\mathrm{hom}_{\mathcal{D}}(P, -)$ is exact)

__Exercice:__ Show that if $\mathcal{M}$ is a $\mathcal{C}$-module cat (w/o assuming exact) $X \in \mathcal{C}$, $Q \in \mathcal{M}$ proj, then $X \otimes Q$ is proj in $\mathcal{M}$.

__Example:__ Any semisimple $\mathcal{C}$ module cat. is exact (because every object is projective)

__Rk:__ A Vec-module cat is exact iff it is semisimple (because $1 \in \mathrm{Vec}$ is proj, so is $M = 1 \otimes M$ $\forall M \in \mathcal{M}$)

More generally, if $\mathcal{C}$ is semisimple, a $\mathcal{C}$-module cat is exact iff it is semisimple.

[So the introduction of this new notion "exact" is relevant in the non-semisimple case (of $\mathcal{C}$) otherwise it is the same as semisimple (for $\mathcal{M}$).

Example: Let $\mathcal{C}$ be a multitensor cat. $P, X \in \mathcal{C}$ with P proj. Then (recall) $P \otimes X$ is proj. So $\mathcal{C}$ is an exact $\mathcal{C}$-module cat

Idem: $\mathcal{C}$ is an exact $\mathcal{C} \boxtimes \mathcal{C}^{op}_{-}$ module cat.

Now, recall that a functor $F: \mathcal{C} \to \mathcal{D}$ is surjective if any simple object in $\mathcal{D}$ is a subquotient of some $F(X)$, with $X \in \mathcal{C}$.

Example: Let $\mathcal{C}, \mathcal{D}$ be finite multitensor categories and let $F: \mathcal{C} \to \mathcal{D}$ be a surjective tensor functor

Then, F induces a structure of exact $\mathcal{C}$-module cat on $\mathcal{D}$. (with $F(X) \otimes \_$)

Exo: find example with F non-surj. and $\mathcal{D}$ non-exact (as $\mathcal{C}$-module cat, induced by F)

(i.e. "surjective" is not superfluous to get exact).

First properties of exact module categories

We will state the main results of this section w/o proof because they are trivial the semisimple case (and for now, we are mainly interested in the semisimple case, and we do not want to take too much time for the non-semisimple case).

Let $\mathcal{C}$ be a multitensor cat.

→ Assume that an exact $\mathcal{C}$-module cat. $\mathcal{M}$ has finitely many simple objects (up to iso). Then $\mathcal{M}$ is finite

④

① $\dim_k(\hom_{\mathcal{M}}(A,B))<\infty$
② $\forall M\in\mathcal{M}$, M is finite length
③ enough projective
④ ...

→ In an exact $\mathcal{C}$-module cat, any proj. obj is injective, and vice-versa (i.e. quasi-Frobenius cat).

See you later...