Proposition: Let $\mathcal{C}$ be a rigid monoidal category, and let $\mathcal{M}$ be a $\mathcal{C}$-module category. There is a canonical isomorphism

$$\hom_{\mathcal{M}}(X^* \otimes M, N) \xrightarrow{\sim} \hom_{\mathcal{M}}(M, X \otimes N)$$

natural in $X \in \mathcal{C}$, $M, N \in \mathcal{M}$

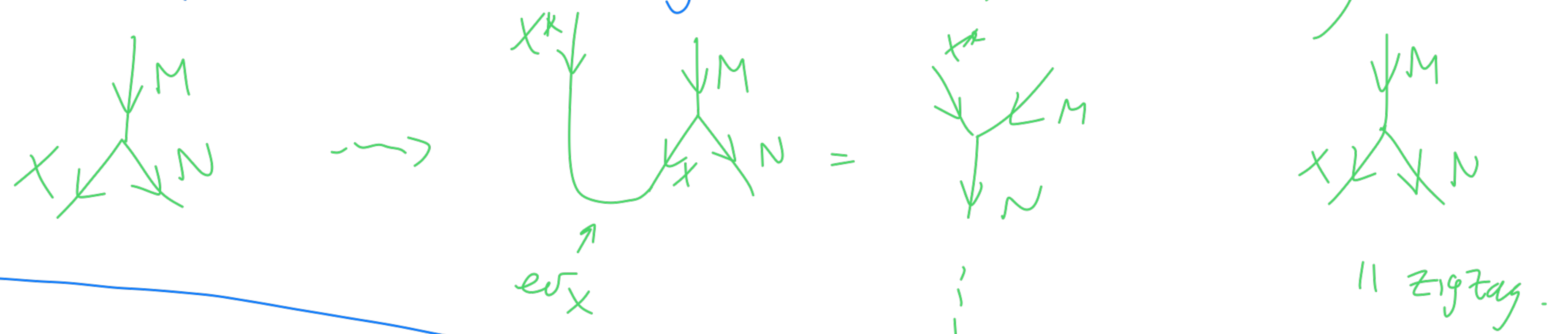
i.e. natural transformation of functors $\phi: F \to G$

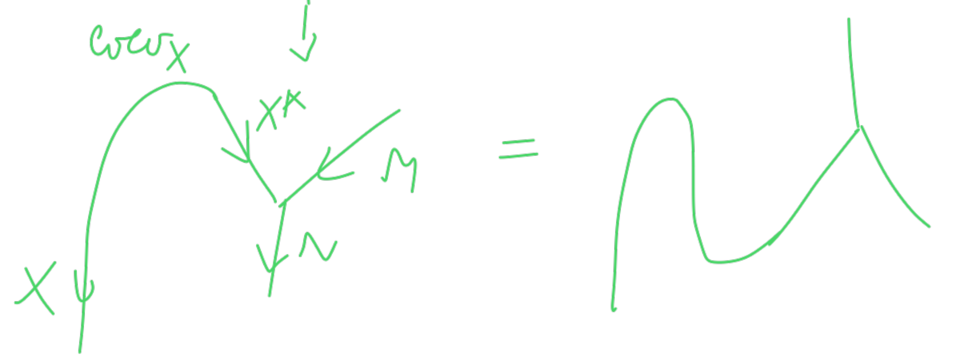
where $F: \mathcal{C} \times \mathcal{M} \times \mathcal{M} \to \mathrm{Set}$
$(X, M, N) \mapsto \hom_{\mathcal{M}}(X^* \otimes M, N)$

$G: \mathcal{C} \times \mathcal{M} \times \mathcal{M} \to \mathrm{Set}$
$(X, M, N) \mapsto \hom_{M}(M, X \otimes N)$

s.t. $\phi_{X,M,N}$ iso. $\forall X \in \mathcal{C}$, $N, M \in \mathcal{M}$.

Proof: As for the natural adjunction iso, i.e. pictorially:





$\square$

In other words, for $\mathcal{C}$ rigid, the endofunctor $\mathcal{M} \to \mathcal{M}$

$$M \mapsto X^* \otimes M.$$

is a left adjoint to

$$M \mapsto X \otimes M$$

and

$$M \mapsto {}^*X \otimes M$$ is a right adjoint to. $M \mapsto X \otimes M$.

Definition (bimodule category): Let $\mathcal{C}$, $\mathcal{D}$ be monoidal cat.

A $(\mathcal{C},\mathcal{D})$-bimodule category is a category $\mathcal{M}$ that has:

- left $\mathcal{C}$-module structure with module associativity constraint

$$m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M)$$

- right $\mathcal{D}$-module

$$n_{M,W,Z} : M \otimes (W \otimes Z) \xrightarrow{\sim} (M \otimes W) \otimes Z$$

compatible by a natural isomorphism b defined by the collection

$$b_{X,M,Z} : (X \otimes M) \otimes Z \xrightarrow{\sim} X \otimes (M \otimes Z)$$

called <u>middle associativity constraint</u>, st the following two diagrams commute $\forall X,Y\in\mathcal{C}$, $Z,W\in\mathcal{D}$, $M\in\mathcal{O}b$.

$$\begin{tikzcd}
 & ((X\otimes Y)\otimes M)\otimes Z \arrow[dl, "m_{X,Y,M}\otimes id_Z"'] \arrow[dr, "b_{X\otimes Y,M,Z}"] & \\
(X\otimes(Y\otimes M))\otimes Z \arrow[d, "b_{X,Y\otimes M,Z}"] & \circlearrowright & (X\otimes Y)\otimes(M\otimes Z) \arrow[d, "m_{X,Y,M\otimes Z}"] \\
X\otimes((Y\otimes M)\otimes Z) \arrow[rr, "id_X\otimes b_{Y,M,Z}"] & & X\otimes(Y\otimes(M\otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
 & X\otimes(M\otimes(W\otimes Z)) \arrow[dl, "id_X\otimes n_{M,W,Z}"'] & \\
X\otimes((M\otimes W)\otimes Z) & \circlearrowright & (X\otimes M)\otimes(W\otimes Z) \arrow[ul, "b_{X,M,W\otimes Z}"'] \\
(X\otimes(M\otimes W))\otimes Z \arrow[u, "b_{X,M\otimes W,Z}"] & & ((X\otimes M)\otimes W)\otimes Z \arrow[ll, "b_{X,M,W}\otimes id_Z"] \arrow[u, "n_{X\otimes M,W,Z}"']
\end{tikzcd}$$

Module functors

monoidal cat.

Def: Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{C}$-module categories, with module associativity constraints m, n resp.

A $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ is given by a functor $F: \mathcal{M} \to \mathcal{N}$ and a natural iso. s defined as

$$s_{X,M}: F(X \otimes M) \to X \otimes F(M) \qquad X \in \mathcal{C},\ M \in \mathcal{M}.$$

s.t. the following diagrams commute $\forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$.

$$\begin{tikzcd}
 & F((X\otimes Y)\otimes M) \arrow[dl, "F(m_{X,Y,M})"'] \arrow[dr, "s_{X\otimes Y, M}"] & \\
F(X\otimes(Y\otimes M)) \arrow[d, "s_{X, Y\otimes M}"'] & & (X\otimes Y)\otimes F(M) \arrow[d, "n_{X,Y,F(M)}"] \\
X\otimes F(Y\otimes M) \arrow[rr, "id_X \otimes s_{Y,M}"] & & X\otimes(Y\otimes F(M))
\end{tikzcd}$$

$$\begin{tikzcd}
F(1\otimes M) \arrow[rr, "s_{1,M}"] \arrow[dr, "F(l_M)"'] & & 1\otimes F(M) \arrow[dl, "l_{F(M)}"] \\
 & F(M) &
\end{tikzcd}$$

Rk: The notion of $\mathcal{C}$-module functor categorifies the notion of homomorphism of modules over a ring (if ring = field, that corresponds to linear map between vector spaces)

Def: A $\mathcal{C}$-module equivalence is a $\mathcal{C}$-module functor (F, s) s.t. $F: \mathcal{M} \to \mathcal{N}$ is an equivalence of cat. (i.e. $\exists G: \mathcal{N} \to \mathcal{M}$ and nat. iso $\phi: G \circ F \to Id_{\mathcal{M}}$, $\phi': F \circ G \to Id_{\mathcal{N}}$)

We can define a category of $\mathcal{C}$-module functors between two $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$, where the morphisms between (F, s), (G, t) are the natural transformations $\nu: F \to G$ s.t. the following commutes $\forall X \in \mathcal{C}, M \in \mathcal{M}$

$$
\begin{tikzcd}
F(X \otimes M) \arrow[r, "s_{X,M}"] \arrow[d, "\nu_{X \otimes M}"'] & X \otimes F(M) \arrow[d, "id_X \otimes \nu_M"] \\
G(X \otimes M) \arrow[r, "t_{X,M}"] & X \otimes G(M)
\end{tikzcd}
$$

Exo:

$\mathcal{C}$-module cat

$$\begin{tikzcd} \mathcal{M} \arrow[r, "(F_1, s_1)"'] & \mathcal{N} \arrow[r, "(F_2, s_2)"'] & \mathcal{L} \end{tikzcd}$$

$\mathcal{C}$-module functor

Show that (F, s) is a $\mathcal{C}$-module functor, where $F = F_2 \circ F_1$ and s:

$$\begin{tikzcd} F(X \otimes M) \arrow[d, equal] \arrow[rr, "s_{X,M}"] & & X \otimes F(M) \arrow[d, equal] \\ F_2 \circ F_1(X \otimes M) \arrow[r, "s_{1\,X,M}"] & F_2(X \otimes F_1(M)) \arrow[r, "s_{2\,X, F_1(M)}"] & X \otimes F_2(F_1(M)) \end{tikzcd}$$

Rk: We can prove a Maclane strictness thm for module categories (allowing to remove brackets...), and we can assume w/o loss of generality that: $1 \otimes M = M$, $l_M = id_M$ $\forall M \in \mathcal{M}$.

So that, the second diagram for $\mathcal{C}$-module functor means just $s_{1,M} = id_{F(M)}$

Module categories over multitensor categories

Recall: $\forall X, Y \in \mathcal{M}$
- $\dim_{\Bbbk} \hom_{\mathcal{M}}(X,Y) < \infty$
- X has finite length

$\exists$ Jordan–Hölder series.

Let $\mathcal{C}$ be a multitensor category over a field $\Bbbk$.

Def: A $\mathcal{C}$-module cat. is a locally finite abelian category $\mathcal{M}$ (over $\Bbbk$) which is a $\mathcal{C}$-module cat (with $\mathcal{C}$ as monoidal cat.) such that the module product bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is bilinear on the morphisms and exact in the first variable (so in both variable, using left/right adjoint... EMO)

Recall that we proved the following Prop for $\mathcal{C}$ a monoidal cat:

<u>Prop</u>: There is a bijective correspondence between structures of $\mathcal{C}$-module cat. on $\mathcal{M}$, and monoidal functors $F: \mathcal{C} \to \mathrm{End}(\mathcal{M})$.

Now, with same proof, for $\mathcal{C}$ a multitensor cat:

<u>Prop</u>: ....

.... exact monoidal functors $F: \mathcal{C} \to \mathrm{End}_\ell(\mathcal{M})$

where $\mathrm{End}_\ell(\mathcal{M})$ is the abelian cat. of left exact functors from $\mathcal{M}$ to $\mathcal{M}$.

<u>Rk</u>: According to Ulrich Thiel, ! requires $\mathcal{M}$ to be an essentially small locally finite abelian cat over a field ($\Bbbk$)

(ie equiv. to a small cat.)

The module functors between module categories over multitensor cat. will be assumed to be left exact, and just called "module functors".

There is an obvious construction of <u>direct sum</u> $\mathcal{M}_1 \oplus \mathcal{M}_2$ of $\mathcal{C}$-module categories $\mathcal{M}_1, \mathcal{M}_2$ (using sum of: - module product - associ. constraint - unit constraint.)

Def: A $\mathcal{C}$-module cat. $\mathcal{M}$ is called <u>indecomposable</u> if $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \Rightarrow \mathcal{M}_1$ or $\mathcal{M}_2 = 0$.

Next session will start by providing examples of (bi)module categories... See You later...