Hyper Kähler Manifold.

Deformation. Beauville - Bogomolov - Fujiki form.

Defn: A hyperKähler manifold is a simply connected compact Kähler manifold X with an everywhere non-degenerate holomorphic 2-form σ_X such that $H^0(X, \Omega_X^2) = \mathbb{C}\sigma_X$.

(If X is not required to be Kähler, but satisfies other conditions, then it is called an irreducible holomorphic symplectic manifold).

Holonomy group of a hyper Kähler mfd with respect to a Ricci flat metric is $Sp_n(\mathbb{C})$.

$\Rightarrow$ tangent space has natural action of $\mathbb{H} = \mathbb{R}\{1, I, J, K\}$

$aI + bJ + cK$ for $(a,b,c) \in S^2$ is a complex structure that is a constant tensor w.r.t. the metric.

σ nondegenerate, $\Rightarrow \dim_{\mathbb{C}} X$ is even.

Denote $\dim_{\mathbb{C}} X = 2m$.

$\underline{m=1}$: X K3 surface.

$$H^0(X, \Omega_X^p) = \begin{cases} 0 & p \text{ odd} \\ \mathbb{C}\,\sigma_X^{\wedge(\frac{p}{2})} & p \text{ even}, \end{cases}$$

Deforming complex structure on a hyperkähler manifold still gives rise to hyperkähler manifold,

4 series of HK mflds :

(Beauville, Fujiki) : ① S a K3, the Douady space $S^{[m]}$ is a hyperkähler manifold of dim. $2m$.

② (Beauville). generalized Kummer manifold

T complex torus, $T^{[m+1]}$: $2m+2$ dimensional.

$$\mathrm{Ker}\left(T^{[m+1]} \xrightarrow{\text{summation}} T\right) = K_m(T).$$

$K_m(T)$: a HK mfd of dim $2m$.

③ ④ : (O'Grady) : OG_{10} OG_6.

These are all known examples up to deformation

Deformation theory for HK Mfd

X : a HK mfd, $\dim = 2m$.

$\sigma \in H^0(X, \Omega_X^2)$ $\sigma^{\wedge m}$ nowhere vanishing.

σ induces an isomorphism $T_X \simeq \Omega_X$ as holo. vector bundles.

$$v \mapsto v.\sigma$$

$$H^0(X, T_X) \simeq H^0(X, \Omega_X) = H^{1,0} = 0$$

X simply connected

Kuranishi: as a complex manifold, X admits a universal deformation, the tangent space of the base $\mathrm{Def}(X)$ at $[X]$ is identified with $H^1(X, T_X) = H^1(X, \Omega_X) = H^{1,1}$

$$(\dim \mathrm{Def}(X) = h^{1,1} = b_2 - 2)$$

$H^2(X, T_X) = H^2(X, \Omega_X) = H^{1,2}$ is possibly non vanishing.

But by (Tian-Todorov), deformation of Calabi-Yau manifolds are unobstructed.

We know deformation of X is unobstructed,

i.e. $\mathrm{Def}(X)$ is a smooth complex manifold of dim $b_2 - 2$.

Examples: (1) for X deformation type $S^{[m]}$,

$$H^2(S^{[m]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta. \qquad m \geq 2.$$

$$b_2(S^{[m]}) = b_2(S) + 1 = 23.$$

(2) for X of type $K_m(T)$,

$$H^2(X, \mathbb{Z}) \cong H^2(T, \mathbb{Z}) \oplus \mathbb{Z}\delta.$$

$\mathrm{rk}\, H^1(T, \mathbb{Z}) = 4$. $\qquad \mathrm{rk}\, H^2(T, \mathbb{Z}) = \mathrm{rk}\, \Lambda^2 H^1(T, \mathbb{Z}) = \binom{4}{2} = 6.$

$b_2(X) = 1.$

③ $b_2(OG_{10}) = 24 \qquad b_2(OG_6) = 8.$

X HK mfd, $\begin{matrix} \mathcal{X} & X \\ \downarrow & \downarrow \\ B & \ni 0 \end{matrix}$ unobstructed universal deformation.

B: simply connected, smooth,

can identify canonically $H^2(\mathcal{X}_b, \mathbb{Z})$ with $H^2(X, \mathbb{Z})$

local period map:

$B \ni b \xmapsto{\mathcal{P}}$ image of $H^{2,0}(\mathcal{X}_b)$ in $H^2(X, \mathbb{C})$.

$=$ a point in $\mathbb{P}\, H^2(X, \mathbb{C})$.

$\mathcal{P}$: local period map

$$\begin{tikzcd}
T_0 B \arrow[r, "T_b \mathcal{P}"] \arrow[d, "\cong"', "\text{Kodaira-Spensor}"] & T_{\mathcal{P}(0)} \mathbb{P}\, H^2(X, \mathbb{C}) = \mathrm{Hom}(H^{2,0}(X), H^2(X, \mathbb{C})/H^{2,0}) \arrow[d, phantom, "\circlearrowright"] \\
H^1(X, T_X) \arrow[r, "\cong"] & \mathrm{Hom}(\underbrace{H^{2,0}(X)}_{= H^0(X, \Omega_X^2)}, H^1(X, \Omega_X)) \arrow[u, hook]
\end{tikzcd}$$

$T_b \mathcal{P}$ is injectivity.

$\Rightarrow$ Local Torelli Thm: the local period map of a HK mfd is injective and its image is a smooth analysic hypersurface in $\mathbb{P}(H^2(X, \mathbb{C}))$.

BBF form:

Thm (Beauville, Bogomolov, Fujiki): Let X HK mfd of dim $2m$, there exists a unique integral quadratic form q_X on $H^2(X, \mathbb{C})$, a unique constant $C_X \in \mathbb{Q}_{>0}$, s.t.

(a). $\forall \alpha \in H^2(X, \mathbb{C})$, one has $\int_X \alpha^{2m} = C_X\, q_X(\alpha)^m$.

(b). q_X is nondivisible on $H^2(X, \mathbb{Z})$ and takes positive values on Kähler classes in $H^2(X, \mathbb{C})$

(q_X is called the BBF form, C_X is called the Fujiki constant)

Pf:
$$\begin{array}{ccc} \mathcal{X} & & X \\ \downarrow & & \downarrow \\ B & \ni & 0 \end{array}$$
unobstructed universal deformation of X.

b_2-2 $\quad$ b_2-1.

local period map $\wp: B \longrightarrow \mathbb{P} H^2(X, \mathbb{C})$.
$$b \longmapsto [H^{2,0}(\mathcal{X}_b)]$$

B: simply connected, open, $\dim = b_2-2$, small enough such that $\wp$ is injectivity.

Let $D = \wp(B)$ be the image,

D is a smooth analytic hypersurface of $\mathbb{P} H^2(X, \mathbb{C})$,

denote $f(\alpha) = \int_X \alpha^{2m}$, this is a homogeneous polynomial of degree $2m$ if we fix a basis of $H^2(X, \mathbb{C})$.

$\forall\, b \in B,\quad H^{2,0}(\mathcal{X}_b) \rightsquigarrow$ a point in D.

$$\Lambda^{m+1} H^{2,0}(\mathcal{X}_b) = H^{2m+2,0}(\mathcal{X}_b) = 0.$$

f vanishes on D.

A partial derivative of f still vanishes on D.
$(\deg \leq m-1)$

$\bar{D} = Z(f) \subset \mathbb{P} H^2(X, \mathbb{C})$ is the Zariski closure of D.

f vanishes on $\bar{D}$ with multiplicity at least m.

$\bar{D}$ is an irreducible hypersurface of $\mathbb{P} H^2(X, \mathbb{C})$.

$\exists\ q : H^2(X, \mathbb{C}) \to \mathbb{C}$ irreducible polynomial,

such that $\bar{D} = Z(q)$, and $f = q^m \cdot h$.

Notice that $\deg q \neq 1$,

otherwise, $\bar{D}$ is a hyperplane in $\mathbb{P} H^2(X, \mathbb{C})$,

$\Longrightarrow$ in $H^2(X, \mathbb{C})$, $H^{2,0}(\mathcal{X}_b) \oplus H^{1,1}(X_b)$ does not depend on b,

this is not true.

So $\deg q \geq 2$, so h is a nonzero constant.

f is rational polynomial,

so there exists q_X which is integral, non divisible on

$H^2(X, \mathbb{Z})$, and proportional to q, such that

f is proportional to q_X^m.

$$q_X = a_1 x_1^2 + \cdots \qquad a_1 \in \mathbb{Q} - 0.$$

$$f \sim q_X^m = a_1^m x_1^{2m} + \underbrace{\cdots}_{\text{other coeff. are all rational.}}$$

$$q_X = a_1 x_1^2 + a_2 x_1 x_2 + \cdots$$

$$q_X^m = a_1^m x_1^{2m} + \underbrace{m \cdot a_1 \cdot a_2}_{\in \mathbb{Q}} \cdot x_1^{2m-1} x_2 + \cdots \quad \Rightarrow \quad a_2 \in \mathbb{Q}.$$

$q_X \to \pm q_X$ make q_X positive on Kähler cone of X.

$$f = c_X q_X^m. \qquad c_X \in \mathbb{Q}^+.$$

α Kähler. $f(\alpha) > 0$, $q_X(\alpha) > 0 \Rightarrow c_X > 0$.