Prop 9.7 Let $\{U_i\}$ be an open covering of $X\in Sm/k$ and f be a morphism in $\mathrm{DM}^{\mathrm{eff},-}(X)$. If $f|_{U_i}$ is an isomorphism for every i, then f is an isomorphism.

Proof: We may assume that the index i is finite. By the condition, $C(f)|_{U_i}=0$ in $\mathrm{DM}^{\mathrm{eff},-}(U_i)$ for every i. It suffices to prove that $C(f)=0 \in \mathrm{DM}^{\mathrm{eff},-}(X)$. For any $Y\in Sm/U_i$ and $n\in\mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(U_i)}(\mathbb{Z}_{U_i}(Y)[n], C(f)|_{U_i}) = \underline{\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(X)}(\mathbb{Z}_X(Y)[n], C(f))} = 0$$

by adjunction. So for any $Y\in Sm/X$, $Y=\bigcup_i Y_i$ where $Y_i\in Sm/U_i$ and $Y_i\subseteq Y$ is open. Then by MV-sequence we have

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(X)}(\underline{\mathbb{Z}_X(Y)[n]}, C(f)) = 0 \quad \forall n\in\mathbb{Z}$$

so $C(f)=0$ by Lem 5.22. □

Now for any two maps $f_i:\mathbb{Z}(X)\to C_i$ $(i=1,2)$, we define $f_1\otimes f_2$ as the composite $\mathbb{Z}(X)\xrightarrow{\Delta}\mathbb{Z}(X)\otimes\mathbb{Z}(X)\xrightarrow{f_1\otimes f_2} C_1\otimes C_2$.

Thm 9.8 (Projective bundle theorem) Let E be a vector bundle of rank n on $X\in Sm/S$. The map

$$\mathbb{Z}_S(\mathbb{P}(E))\xrightarrow{p\boxtimes c_1(\mathcal{O}_E(1))^i}\bigoplus_{i=0}^{n-1}\mathbb{Z}_S(X)(i)[2i]$$

is an isomorphism in $\mathrm{DM}^{\mathrm{eff},-}(S)$, where $p:\mathbb{P}(E)\to X$ is the structure map.

Proof: By Cor 9.6, we have $\mathbb{Z}_S(\mathbb{P}^n\times S)\xrightarrow{c_1(\mathcal{O}(1))^i}\bigoplus_{i=0}^{n}\mathbb{Z}_S(i)[2i]$ after pullback along the structure map $S\to\mathrm{Spec}\,k$. Now choose a trivialization $\{U_i\}$ of E i.e. $E|_{U_i}=\mathcal{O}_{U_i}^{\oplus n}$. Then the map

$$\mathbb{Z}_X(\mathbb{P}(E))\xrightarrow{c_1(\mathcal{O}_E(1))^i}\bigoplus_{i=0}^{n-1}\mathbb{Z}_X(i)[2i]$$

is an isomorphism when restricting to each U_i. Hence it is an isomorphism by Prop 9.7. Finally apply $f_\#$ for the structure map $f:X\to S$ to conclude. □

Cor 9.9 $H^{*,*}(\mathbb{P}(E),\mathbb{Z})=\bigoplus_{i=0}^{n-1}H^{*-2i,*-i}(X,\mathbb{Z})\cdot c_1(\mathcal{O}_E(1))^i$

Proof: Follow from cancellation theorem. □

Then the $c_1(\mathcal{O}_E(1))^n$ can be uniquely written an $\sum_{i=0}^{n-1}a_{n-i}\cdot c_1(\mathcal{O}_E(1))^i$. The $c_i(E)=(-1)^{i-1}a_i\in H^{2i,i}(X,\mathbb{Z})=CH^i(X)$, $i=1,\cdots,n$ is defined to be the i-th Chern class of E.

From Thm 9.8 we obtain split injections

$$c_r(E)\cdot\mathbb{Z}_S(X)(r)[2r]\longrightarrow\bigoplus_{i=0}^{n-1}\mathbb{Z}_S(X)(i)[2i]=\mathbb{Z}(\mathbb{P}(E))$$

for $r=0,\cdots,n-1$.

For every vector bundle E, we can consider its projective completion $\underline{\mathbb{P}(E^\vee\oplus\mathcal{O}_X)}$ {quotient line bdl}. We have a diagram

$$\begin{tikzcd} \mathbb{P}(E^\vee) \arrow[r] \arrow[dr] & \mathbb{P}(E^\vee\oplus\mathcal{O}_X) & (0:\cdots:0:1) \\ & \mathbb{P}(E^\vee\oplus\mathcal{O}_X)\setminus X \arrow[u] & \end{tikzcd}$$

The $\mathbb{P}(E^\vee\oplus\mathcal{O}_X)\setminus\mathbb{P}(E^\vee)$ is identified with $E\xrightarrow{p}X$ i by the quotient $p^*E^\vee\oplus\mathcal{O}_E\to p^*E^\vee\to\mathcal{O}_E$ where the second map is induced by id_E. There is an $\mathbb{A}^1$-bundle $\mathbb{P}(E^\vee\oplus\mathcal{O}_X)\setminus X\to\mathbb{P}(E^\vee)$, $(s,t)\longmapsto s$ with zero section i so $\mathbb{Z}_S(i)$ is an isomorphism in $\mathrm{DM}^{\mathrm{eff},-}(S)$.

Thm 9.10 Suppose $Z\subseteq X$ is a closed immersion in Sm/S and $n=d_X-d_Z$. There exists a unique family of isomorphisms of the form

$$P_{(X,Z)}: M_Z(X):=\mathbb{Z}(X)/\mathbb{Z}(X\setminus Z)\xrightarrow{\simeq}\mathbb{Z}(Z)(n)[2n]$$

s.t. the following holds:

(1) For every Cartesian diagram
$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$
between closed pair (X,Z) and (Y,T) of rel. dim. $=n$, the following diagram commutes

$$\begin{tikzcd} M_T(Y) \arrow[r, "(f{,}g)"] \arrow[d, "P_{(Y,T)}"'] \arrow[dr, phantom, "2"] & M_Z(X) \arrow[d, "P_{(X,Z)}"] \\ \mathbb{Z}(T)(n)[2n] \arrow[r, "g(n)[2n]"] & \mathbb{Z}(Z)(n)[2n]. \end{tikzcd}$$

(2) Let $X\in Sm/k$ and E be a v.b. of rk $=n$ on X. Consider the pair $(\mathbb{P}(E^\vee\oplus\mathcal{O}_X), X)$, where $\mathbb{P}(E^\vee\oplus\mathcal{O}_X)=:P$. Then $P_{(P,X)}$ is the inverse the morphism

$$\mathbb{Z}(X)(n)[2n]\xrightarrow{c_n}\mathbb{Z}(P)\longrightarrow M_X(P).$$

Proof: We have an isomorphism between split distinguished triangles for every E/X of rk $=n$:

$$\begin{tikzcd} \underline{\mathbb{Z}(\mathbb{P}(E^\vee))}=\mathbb{Z}(P\setminus X) \arrow[r] & \mathbb{Z}(P) \arrow[r] & M_X(P) \arrow[r] & {} \\ & & & \mathbb{Z}(P) \arrow[ul] \\ \bigoplus_{i=0}^{n-1}\mathbb{Z}(X)(i)[2i] \arrow[r] \arrow[uu] \arrow[dr] & \bigoplus_{i=0}^{n}\mathbb{Z}(X)(i)[2i] \arrow[r] \arrow[uu, "c_n"] \arrow[d] & \mathbb{Z}(X)(n)[2n] \arrow[r] \arrow[uu] \arrow[ur, "c_n"] & {}, \\ & \bigoplus_{i=0}^{n-1}\mathbb{Z}(X)(i)[2i] & & \end{tikzcd}$$

chern classes

which establishes the existence part of (2). For general closed pair (X,Z), we define $P_{(X,Z)}$ to be composite

$$M_Z(X)=\mathrm{Th}_S(N_{Z/X})=M_Z(\underbrace{\mathbb{P}(N_{Z/X}^\vee\oplus\mathcal{O}_Z)}_{P})\xrightarrow[\simeq]{P_{(P,Z)}}\mathbb{Z}(Z)(n)[2n]$$

where the first map comes from homotopy purity and second map comes from étale excision.

For the uniqueness, we adopt the following diagram

$$\begin{tikzcd} M_Z(X) \arrow[r] \arrow[d] & M_{Z\times\mathbb{A}^1}(D(X,Z)) \arrow[d] & \mathrm{Th}_S(N_{Z/X}) \arrow[l] \arrow[r] \arrow[d] & M_Z(P) \arrow[dl] \\ \mathbb{Z}(Z)(n)[2n] \arrow[r] & \mathbb{Z}(Z\times\mathbb{A}^1)(n)[2n] & \mathbb{Z}(Z)(n)[2n] \arrow[l] & \end{tikzcd}$$

where all maps are isomorphisms. □

The theorem above in particular shows that $\mathrm{Th}_S(E)=\mathbb{Z}_S(X)(n)[2n]$ for any v.b. E/X of rk $=n$.

Cor 9.11 (Gysin triangle) In the context above, we have distinguished triangle

$$\mathbb{Z}(X\setminus Z)\to\mathbb{Z}(X)\xrightarrow[P_{(X,Z)}]{}\mathbb{Z}(Z)(n)[2n]\to\mathbb{Z}(X\setminus Z)[1]$$

in $\mathrm{DM}^{\mathrm{eff},-}(S)$.

We introduce a common situation where $\mathbb{Z}(X)$ can be written as a direct sum of the form $\mathbb{Z}(i)[2i]$ in $\mathrm{DM}^{\mathrm{eff},-}(k)$.

Thm 9.12 (Białynicki-Birula) Let X be a smooth projective variety over a field k equipped with an action of $\mathbb{G}_m$. Then:

1) The fixed point locus $X^{\mathbb{G}_m}$ is a smooth, closed subscheme of X.

2) There is a numbering $X^{\mathbb{G}_m}=\coprod_{i=1}^{n}Z_i$ of the connected components of the fixed points locus, a filtration of closed subsets

$$X=X_n\supseteq X_{n-1}\supseteq\cdots\supseteq X_0\supseteq X_{-1}=\emptyset$$

and affine bundles $\varphi_i: X_i\setminus X_{i-1}\to Z_i$.

3). The relative dimension a_i of the affine bundle φ_i is the dimension of the positive eigenspace of the action of $\mathbb{G}_m$ on the tangent space $T_{X,z}$, z is any closed point in Z_i. The dimension of Z_i is the dimension of $(T_{X,z})^{\mathbb{G}_m}$.

Proof: See [P. Brosnan, On the motivic decompositions arising from the method of Białynicki-Birula, Theorem 3.2]. □

Thm 9.13 In the context above, there is a decomposition

$$\mathbb{Z}(X)=\bigoplus_{i=0}^{n}\mathbb{Z}(Z_i)(b_i)[2b_i]$$

in $\mathrm{DM}^{\mathrm{eff},-}(k)$, where $b_i=\dim(T_{X,z}^-)$ for any closed $z\in Z_i$.

Proof: By Cor 9.11, we have a distinguished triangle

$$\mathbb{Z}(X\setminus X_i)\longrightarrow\mathbb{Z}(X\setminus X_{i-1})\longrightarrow\underset{\mathbb{Z}(Z_i)(c_i)[2c_i]}{\mathbb{Z}(X_i\setminus X_{i-1})(c_i)[2c_i]}\longrightarrow\underline{\mathbb{Z}(X\setminus X_i)[1]}$$

where $c_i=d_X-d_{X_i}=\dim(T_{X,z}^+)+\dim(T_{X,z}^-)+\dim(T_{X,z}^0)-(\dim(T_{X,z}^+)+\dim(T_{X,z}^0))$
$=\dim(T_{X,z}^-)$ for $z\in Z_i$.

We prove by induction that $\mathbb{Z}(X\setminus X_i)=\bigoplus_{j=i+1}^{n}\mathbb{Z}(Z_j)(b_j)[2b_j]$.

If $i=n$ this is trivial. So we see that it suffices to prove

$$\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(U)(n)[2n],\mathbb{Z}(V)(m)[2m+1])=0$$

for any $m,n\in\mathbb{N}$, $U,V\in Sm/k$ and U projective. We will prove in the last section that the group above is isomorphic to

$$\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(U\times V)(n-\dim U)[2n-2\dim U],\underline{\mathbb{Z}(m)[2m+1]})$$

by duality, which vanishes by Cor 8.24. □

In particular, if $\dim X^{\mathbb{G}_m}=0$, then $\mathbb{Z}(X)=\bigoplus_{i=0}^{n}\mathbb{Z}(b_i)[2b_i]$.

Prop 9.14 Let G be a connected reductive linear algebraic group split/k. Suppose P is a parabolic subgroup and T is a maximal torus contain in a Borel subgroup contained in P. Then G/P is smooth projective with a $\mathbb{G}_m$-action s.t. $(G/P)^{\mathbb{G}_m}$ is of dimension zero.

Proof: The G/P is smooth since G acts transitively. It is projective by [GTM 21, 21.3, Cor B]. The G/P has a T-action and there is a T-equivariant map $\pi: G/B\to G/P$. The $(G/B)^T$ is isomorphic to the Weyl group $N_G(T)/Z_G(T)$, which is a finite set. For every $xP\in(G/P)^T$, $\pi^{-1}(xP)=P/B$ which is a projective T-variety, so it admits a T-fixed point by [GTM 21, 21.2]. So map $(G/B)^T\to(G/P)^T$ is surjective. Hence $|(G/P)^T|<\infty$.

The tangent space $T_1(G/B)$ is identified with the space spanned by positive roots Φ^+. Choose a cocharacter $f:\mathbb{G}_m\to T$ s.t. $\langle f,\chi\rangle>0$ $\forall\chi\in\Phi^+$, then the $\mathbb{G}_m$-action of G/B induced by f satisfies $\underline{(G/B)^{\mathbb{G}_m}}=(G/B)^T$, which is a finite set. Hence $(G/P)^{\mathbb{G}_m}$ is also finite by the same argument as before. □