(Global Torelli Theorem for K3 surface)

Two complex K3 surfaces S_1, S_2 are isomorphic if and only if $\exists$ a Hodge isometry $H^2(S_1, \mathbb{Z}) \to H^2(S_2, \mathbb{Z})$.

Moreover, for any Hodge isometry $\psi: H^2(S_1, \mathbb{Z}) \to H^2(S_2, \mathbb{Z})$ with $\psi(\mathcal{K}_{S_1}) \cap \mathcal{K}_{S_2} \neq \phi$, then $\exists!$ isomorphism $g: S_2 \to S_1$, with $\psi = g^*$.

Pf: $S_1 \overset{\sim}{\to} S_2 \Rightarrow \psi: H^2(S_1, \mathbb{Z}) \overset{\sim}{\to} H^2(S_2, \mathbb{Z})$ Hodge isometry.

Conversely, if $\exists$ Hodge isometry $H^2(S_1, \mathbb{Z}) \overset{\sim}{\to} H^2(S_2, \mathbb{Z})$, then we can find markings f_1, f_2 for S_1, S_2 respectively, such that

$$\begin{tikzcd} H^2(S_1, \mathbb{Z}) \arrow[rr, "\psi"] \arrow[dr, "f_1"'] & \arrow[d, phantom, "\circlearrowright"] & H^2(S_2, \mathbb{Z}) \arrow[dl, "f_2"] \\ & \Lambda_{K3} & \end{tikzcd}$$

commutes.

$(S_1, f_1), (S_2, f_2) \in N$

ψ Hodge isometry $\Rightarrow \mathcal{P}(S_1, f_1) = \mathcal{P}(S_2, f_2)$ in D_{K3},

take $f_2' = f_2$ or $-f_2$, s.t.

$(S_1, f_1), (S_2, f_2')$ lie in the same connected component of N,

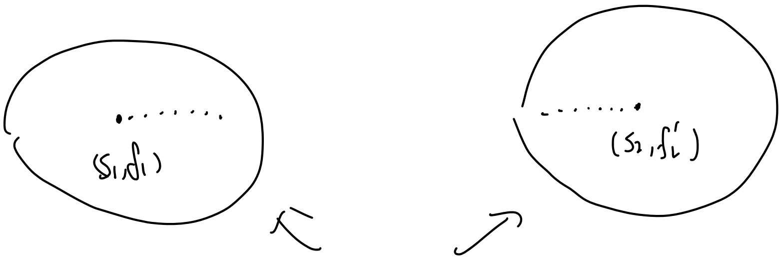
then $(S_1, f_1), (S_2, f_2')$ are inseparable points in N,

$\Rightarrow S_1 \cong S_2$.

Now we assume $\psi(K_{S_1}) \cap K_{S_2} \neq \phi$,

i.e. $\exists\ \alpha \in K_{S_1}$, s.t. $\psi(\alpha) \in K_{S_2}$.

(S_1, f_1), (S_2, f_2') are inseparable points.



intersect.

$(S_1^{(j)}, f_1^{(j)}) \to (S_1, f_1)$ $\qquad$ $(S_2^{(j)}, f_2^{(j)}) \to (S_2, f_2')$.

$(S_1^{(j)}, f_1^{(j)})$, $(S_2^{(j)}, f_2^{(j)})$ represent the same point in N,

$\Rightarrow\ \exists$ isomorphism $S_1^{(j)} \xrightarrow{\simeq} S_2^{(j)}$ compatible with $f_1^{(j)}, f_2^{(j)}$

$$\begin{tikzcd} H^2(S_1^{(j)}, \mathbb{Z}) \arrow[dr, "f_1^{(j)}"'] & & H^2(S_2^{(j)}, \mathbb{Z}) \arrow[ll] \arrow[dl, "f_2^{(j)}"] \\ & \Lambda_{K3} & \end{tikzcd}$$

graph $\Gamma_j \subset S_1^{(j)} \times S_2^{(j)}$

$j \to \infty$, $(\Gamma_j)_j$ has a limit Γ_∞ is an analytic cycle in $S_1 \times S_2$,

$$\Gamma_\infty = Z + \sum_{i=1}^{k} (C_i \times C_i'),$$

Z is the graph induced by an isomorphism $S_2 \underset{\simeq}{\xrightarrow{f}} S_1$

$C_i \subset S_1$, $C_i' \subset S_2$ are curves.

$[\Gamma_\infty]$: $H^2(S_1, \mathbb{Z}) \xrightarrow{\simeq} H^2(S_2, \mathbb{Z})$ is ψ or $-\psi$

$(f_2' = f_2) \quad (f_2 = -f_2)$

$\alpha \in K_{S_1}, \quad \psi(\alpha) \in K_{S_2},$

$$[\Gamma_\infty]\alpha = [Z]\alpha + \left[\sum_{i=1}^{k} C_i \times C_i'\right]\alpha$$

$$= f^*\alpha + \sum (C_i, \alpha)\cdot [C_i'] = f^*\alpha + [D].$$

D: effective divisor on S_2

take norm:

$$\underbrace{([\Gamma_\infty]\alpha)^2}_{\alpha^2} = (f^*\alpha + [D])^2$$

$$= \underbrace{(f^*\alpha)^2}_{\alpha^2} + 2(f^*\alpha, D) + (D, D),$$

So $\quad 2(f^*\alpha, D) + (D, D) = 0.$ ——— $(*)$

Assume $f_2' = f_2$, then $[\Gamma_\infty] = \psi$, then

$$\psi(\alpha) = [\Gamma_\infty]\alpha = f^*\alpha + [D].$$

$\psi(\alpha) \in K_{S_2} \Rightarrow (\psi(\alpha), D) \geq 0$, equality holds iff $D = 0$

$f^*\alpha \in K_{S_2} \Rightarrow (f^*\alpha, D) \geq 0$, equality holds iff $D = 0$

$(*) \Rightarrow$

$0 = 2(f^*\alpha, D) + (D, D) = (f^*\alpha, D) + (\psi(\alpha), D) \geq 0.$

$\Rightarrow$ We must have $D = 0$,

$\Rightarrow \Gamma_\infty = Z$ the graph of $f: S_2 \xrightarrow{\cong} S_1$,

$\Rightarrow \psi = [\Gamma_\infty] = f^*.$

Uniqueness of f is because: the action of the automorphism group of a complex K3 surface on its second cohomology is faithful. $\square$

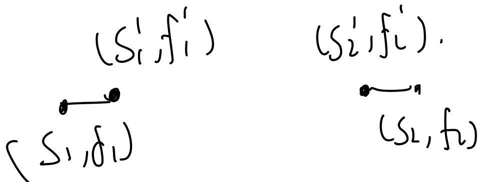
Lemma: $(S_1, f_1), (S_2, f_2) \in N$ inseparable points,

then $f_1^{-1} f_2 : H^2(S_2, \mathbb{Z}) \to H^2(S_1, \mathbb{Z})$ preserves positive cone

i.e. $f_1^{-1} f_2 (C_{S_1}) = C_{S_2}$.

$C_{S_1} :=$ component of the set $\{ x \in \underbrace{H^{1,1}(S_1, \mathbb{R})}_{(1,19)} \mid x^2 > 0 \}$

containing Kähler classes.



$(S_1', f_1') \cong (S_2', f_2') \Rightarrow (f_1')^{-1} f_2'$ preserves positive cone

$\Rightarrow$ so is $f_1^{-1} f_2$.

$\psi(K_{S_1}) \cap K_{S_2} \neq \phi \Rightarrow \psi(C_{S_1}) = C_{S_2}$.

(S_1, f_1) (S_2, f_2') inseparable $\Rightarrow (f_2')^{-1} f_1 = \psi$.

$\Rightarrow f_2' = f_2$.

Moduli of polarized K3 surface. fix d even positive integer.

A polarized K3 surface of degree d is a pair

(S, L) consisting of a complex K3 surface S and an ample line bundle L, s.t. $[L] \in \mathrm{Pic}(S)$ primitive

and $L^2 = d$.

$\Lambda_{K3} \ni l$, $l^2 = d$, l primitive,

A marking for (S, L) is an isometry

$f: H^2(S, \mathbb{Z}) \to \Lambda_{K3}$, $L \mapsto l$,

(S, L, f) : marked polarized K3,

$(S, L, f) \sim (S', L', f')$ equivalent, if $\exists$ isom

$g: S \xrightarrow{\cong} S'$, $g^* L' = L$, and:

$$\begin{tikzcd} H^2(S, \mathbb{Z}) \arrow[dr, "f"'] & & H^2(S', \mathbb{Z}) \arrow[ll, "g^*"'] \arrow[dl, "f'"] \arrow[dl, phantom, shift right=6, "\circlearrowright"] \\ & \Lambda_{K3} & \end{tikzcd}$$

Denote by N_d the set of equivalence classes of marked polarized K3 surface of degree d.

Rmk: N non-Hausdorff, we will see N_d is Hausdorff.

N_d is naturally a complex manifold of dimension 19.

$\Lambda_d = l^{\perp} := \{ x \in \Lambda_{K3} \mid (x, l) = 0 \}$.

lattice of signature $(2, 19)$.

$\hat{D}_d = \mathbb{P}\{ x \in (\Lambda_d)_{\mathbb{C}} \mid \varphi(x, x) = 0,\ \varphi(x, \bar{x}) > 0 \}$,

$=$ set of oriented positive two planes in $(\Lambda_d)_{\mathbb{R}}$.

Rmk: D_{K3} is connected, $\hat{D}_d$ has two connected components, (corresponding to different orientations)

Let D_d be a connected component of $\hat{D}_d$.

$N_d \ni (S, L, f)$, $\quad f(H^{2,0}(S)) \perp f(L) = \ell \in \Lambda_{K3}$.

$\Rightarrow f(H^{2,0}(S))$ is a complex line in $(\Lambda_d)_{\mathbb{C}}$

$\varphi(\cdot,\cdot) = 0, \quad \varphi(\cdot, \bar{\cdot}) > 0$,

$\Rightarrow f(H^{2,0}(S)) \in \hat{D}_d$.

We have local period map for polarized K3 surface of degree d;

$$\mathcal{P}_d : N_d \longrightarrow \hat{D}_d$$

Deformation theory $\Rightarrow$ $\mathcal{P}_d$ is locally biholomorphic.

Prop: $\mathcal{P}_d$ is injective.

Pf: Assume $\mathcal{P}_d(S, L, f) = \mathcal{P}_d(S', L', f')$.

$\psi = (f')^{-1} \circ f : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ is a Hodge isometry,

$$L \longmapsto L'$$

L, L' ample, $\psi(K_S) \cap K_{S'} \neq \phi$.

So by global Torelli thm, $\exists !$ $g : S' \xrightarrow{\sim} S$, s.t. $g^* = \psi$.

So (S, L, f), (S', L', f') represent the same point in N_d.

So Φ_d is injective. $\square$

So N_d is also a Hausdorff complex manifold.

Prop: N_d also has two connected components.

Pf: take $(S, L, f) \in N_d$.

there exists $\delta \in \Lambda_d$ with norm 2 (Lattice theory)

S_δ reflection in δ, Spinor norm of S_δ -1.

S_δ exchanges two components of $\hat{D}_d$.

$\Phi_d(S, L, S_\delta \circ f)$ and $\Phi_d(S, L, f)$ lie in different conn. components,

$\Rightarrow$ N_d has two connected components.

$\Gamma = \mathrm{Aut}(\Lambda_{K3}, \ell)$. $\Gamma^+ < \Gamma$ index two.

$\Gamma^+ = \mathrm{Aut}^+(\Lambda_{K3}, \ell)$

$$\begin{tikzcd} \Phi_d : N_d \arrow[r, hook] \arrow[loop below, "\Gamma"] & \hat{D}_d . \arrow[loop below, "\Gamma"] \end{tikzcd}$$

$$\Phi_d : N_d/\Gamma \longleftrightarrow \hat{D}_d/\Gamma \cong D_d/\Gamma^+ .$$

$$N_d/\Gamma = \{\text{equivalences classes of } (S, L, f)\}/\Gamma$$

$= \{$ equivalence classes of (S, L) $\}$

$=$ certain quasi-proj. variety constructed by GIT.

take $d = 4$ as example,

Smooth quartic surfaces are polarized K3 of degree 4

$V_0 = \{$ Smooth complex homogeneous polynomials of degree 4 $\}$

every point is stable w.r.t. to this action.

$SL(4, \mathbb{C})$.

$M_4^\circ = SL(4, \mathbb{C}) /\!/ V_0$: quasi-proj. var.

GIT quotient

M_4° is a Zariski-open subset of N_4 / Γ.

open embedding of analytic spaces

$$\mathcal{P}_4 : \underbrace{M_4^\circ}_{q\text{-proj}} \hookrightarrow \underbrace{D_4 / \Gamma}_{}.$$

is also quasi-proj, due to existence of the Baily-Borel compactification

Thm: $\mathcal{P}_4 : M_4^\circ \hookrightarrow D_4/\Gamma$.

is an algebraic open embedding.