Reconstruction Theorem: the assignment $(\mathcal{C}, F) \mapsto H = \mathrm{End}(F)$
are mutually inverse bijection between $H \mapsto (\mathrm{Rep}(H), \mathrm{forget})$

(1) tensor equiv. classes of finite tensor cat. $\mathcal{C}$ (over $\Bbbk$) with a quasi-fiber functor F,

(2) Equiv. classes of f.d. quasi-Hopf algebra, up to twist equiv and iso.

Finally, let us mention (from Chap 6) the following important application of the notion of quasi-Hopf algebra:

Prop: A finite tensor category is integral iff it is equivalent to the rep. cat. of a finite dim. quasi-Hopf algebra.

i.e. its Grothendieck ring is integral,
i.e. $\mathrm{FPdim}(X) \in \mathbb{Z}$ $\forall X$ simple

⚠ In non-semisimple case, $\mathrm{FPdim}(\mathcal{C}) \neq \mathrm{FPdim}(Gr(\mathcal{C})) = \sum_i \mathrm{FPdim}(X_i)^2$ (see Section 6.1 in the book)

Above Prop. leads to the following improved reconstruction theorem (w/o mention of quasi-fiber functor)

Reconstruction Theorem: the assignment $H \mapsto \mathrm{Rep}(H)$ defines a bijection between:

(1) finite dim. quasi-Hopf algebras H over $\Bbbk$ (up to twist equiv. and iso.)

(2) integral finite tensor categories over $\Bbbk$ (up to tensor equiv.)

Remark: Let $V \in \mathrm{Rep}(H)$, then $\mathrm{FPdim}(V) = \dim_{\Bbbk}(V)$.

for non-integral case ($\Leftrightarrow \nexists$ quasi-fiber functor), there still exist an underlying algebraic structure called weak Hopf algebra, but ⚠ $\mathrm{FPdim}(V) \neq \dim_{\Bbbk}(V)$ (in general)

Remark: Above reconstruction theorem exists also for ∞-setting as for Hopf algebras...

Module categories

The notion of tensor category "categorifies" the notion of ring. Similarly:

———— module category ———————— module over a ring.

Definition of module category over a general monoidal category

Let $\mathcal{C} = (\mathcal{C}, \otimes, 1, a, l, r)$ be a monoidal category

Def: A left module category over $\mathcal{C}$ (also called left $\mathcal{C}$-module)
is a category $\mathcal{M}$ equipped with a action (or module product), i.e. a bifunctor
$\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, and a natural isomorphism m given by

$$m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M), \quad X, Y \in \mathcal{C},\ M \in \mathcal{M}.$$

called <u>module associativity constraint</u>, s.t. the functor

$$\begin{aligned} \mathcal{M} &\to \mathcal{M} \\ M &\mapsto 1 \otimes M \end{aligned}$$ is an autoequivalence (unit axiom)

and the following pentagon diagram commutes $\forall X, Y, Z \in \mathcal{C}, \forall M \in \mathcal{M}$.

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes M \arrow[dl, "a_{X,Y,Z} \otimes \mathrm{id}_M"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \otimes M \arrow[d, "m_{X, Y \otimes Z, M}"] & \circlearrowright & (X \otimes Y) \otimes (Z \otimes M) \arrow[d, "m_{X, Y, Z \otimes M}"] \\
X \otimes ((Y \otimes Z) \otimes M) \arrow[rr, "\mathrm{id}_X \otimes m_{Y,Z,M}"] & & X \otimes (Y \otimes (Z \otimes M))
\end{tikzcd}$$

(pentagon axiom)

Rk: the notion right $\mathcal{C}$-module cat. $\mathcal{M}$, can be defined similarly,

or equiv. as a left $\mathcal{C}^{op}$-module cat. $\left(\begin{array}{l} X \otimes^{op} Y := Y \otimes X \\ a^{op}_{X,Y,Z} := a^{-1}_{Z,Y,X} \end{array} \right)$

⚠ $\mathcal{C}^{op} \neq \mathcal{C}^{\vee}$ (arrow reversed)

or also s.t. $\mathcal{M}^{\vee}$ = left $\mathcal{C}$-module (when $\mathcal{C}$ is rigid), ... see later...

Notation: $\mathcal{C}$-module cat. will always mean left $\mathcal{C}$-module cat. (unless specified...)

As for monoidal cat., the definition of (left) $\mathcal{C}$-module admits the alternative equivalent definition, replacing unit axiom by unit natural isomorphism ℓ

$$\ell_M : 1 \otimes M \xrightarrow{\sim} M$$

called the unit constraint, satisfying the triangle axiom i.e.

$$\begin{tikzcd}
(X\otimes 1)\otimes M \arrow[rr, "m_{X,1,M}"] \arrow[dr, "r_X\otimes \mathrm{id}_M"'] & \circlearrowleft & X\otimes(1\otimes M) \arrow[dl, "\mathrm{id}_X\otimes l_M"] \\
& X\otimes M &
\end{tikzcd}$$

the one from $\mathcal{C}$ ($\nexists$ such r for $\mathcal{M}$ as left $\mathcal{C}$-module)

the one above

commute $\forall X\in\mathcal{C},\ M\in\mathcal{M}$.

<u>Exercice</u>: Show the equivalence of the two above definition (hint: it works as for monoidal cat.)

Here is another alternative def. for $\mathcal{C}$-module cat. $\mathcal{M}$ (stated in the following Prop), using the fact that for a category $\mathcal{M}$, the category $\mathrm{End}(\mathcal{M})$ of endofunctors (i.e. functors $G:\mathcal{M}\to\mathcal{M}$) is a monoidal cat. (with $\otimes$ given by composition $\circ$).

<u>Prop</u>: There is a bijective correspondence between structures of $\mathcal{C}$-module category on $\mathcal{M}$, and monoidal functors $F:\mathcal{C}\to\mathrm{End}(\mathcal{M})$.

<u>proof:</u> Let $F : \mathcal{C} \to \mathrm{End}(\mathcal{M})$ be a monoidal functor with monoidal structure J, given by $J_{X,Y} : F(X) \circ F(Y) \xrightarrow{\sim} F(X \otimes Y)$

satisfying the monoidal structure axiom

We define a $\mathcal{C}$-module cat. structure on $\mathcal{M}$ as follows:

Set $X \otimes M := F(X)(M)$, $X \in \mathcal{C}$, $M \in \mathcal{M}$, and define the module associativity constraint m of $\mathcal{M}$ using J:

$$m_{X,Y,M} : (X \otimes Y) \otimes M = F(X \otimes Y)(M) \xrightarrow{J_{X,Y}^{-1}} (F(X) \circ F(Y))(M) = F(X)(F(Y)(M)) = X \otimes (Y \otimes M)$$

$\forall\, X, Y \in \mathcal{C}$, $M \in \mathcal{M}$.

<u>Conversely:</u> let $\mathcal{M}$ be a $\mathcal{C}$-module category. We define a monoidal functor $F : \mathcal{C} \to \mathrm{End}(\mathcal{M})$ as follows:

for any $X \in \mathcal{C}$, we have the functor of left tensor product by X: $M \mapsto X \otimes M$

Thus, we have a functor $f: \mathcal{C} \to \mathrm{End}(\mathcal{M})$. Now, what about the monoidal structure J: it can defined using the module associativity constraint m:

$$\left(J_{X,Y}\right)_M : \left(f(X) \circ f(Y)\right)(M) = f(X)\left(f(Y)(M)\right) = X \otimes (Y \otimes M) \xrightarrow{m^{-1}_{X,Y,M}} \underset{\displaystyle (X \otimes Y) \otimes M}{\overset{f(X \otimes Y)(M)}{\shortparallel}}$$

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<u>Finally</u>: the Pentagon axiom of $\mathcal{C}$-module is exactly the monoidal structure axiom of J (hexagon.., but one arrow disappear because $\mathrm{End}(\mathcal{M})$ is strict, i.e. trivial associativity constraint..) ∎

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<u>Rk</u>: This proposition categorifies the fact that (in elementary alg.) module over ring is the same as representation.

Def: A module subcategory $\mathcal{N}$ of a $\mathcal{C}$-module category $\mathcal{M}$ is a full subcategory $\mathcal{N} \subset \mathcal{M}$ which is closed the action of $\mathcal{C}$.

Full: $\forall N_1, N_2 \in \mathcal{N}$, $\hom_{\mathcal{N}}(N_1, N_2) = \hom_{\mathcal{M}}(N_1, N_2)$

Closed: $\forall N \in \mathcal{N}$, $\forall X \in \mathcal{C}$, then $X \otimes N \in \mathcal{N}$.

Remark (left/right $\mathcal{C}$-module and dual cat.): let $\mathcal{C}$ rigid monoidal cat. and let $\mathcal{M}$ be a right $\mathcal{C}$-module cat.

Then: $\mathcal{M}^\vee$ (dual cat, i.e. arrow reversed) is a left $\mathcal{C}$-module cat. with $\mathcal{C}$-action

$$\odot : \mathcal{C} \times \mathcal{M}^\vee \to \mathcal{M}^\vee, \quad X \odot M := M \otimes {}^*X,$$

and module associativity constraint:

$$(X \otimes Y) \odot M = M \otimes {}^*(X \otimes Y) \simeq M \otimes ({}^*Y \otimes {}^*X) \xrightarrow[\text{mod. asso. const. of } \mathcal{M}]{\sim} (M \otimes {}^*Y) \otimes {}^*X = X \odot (Y \odot M)$$

$\forall X, Y \in \mathcal{C}$, $M \in \mathcal{M}^\vee$.

Idea: ... let $\mathcal{N}$ be a left $\mathcal{C}$ module cat.

Then $\mathcal{N}^\vee$ is a right $\mathcal{C}$-module cat. / with $\mathcal{C}$-action: $N \odot X := X^* \otimes N$

But we can also define left $\mathcal{C}$-module ${}^\vee\mathcal{M}$: $X \odot M := M \otimes X^*$

right $\mathcal{C}$-module ${}^\vee\mathcal{N}$: $N \odot X := {}^*X \otimes N$.

Thm: ${}^\vee(\mathcal{M}^\vee) \simeq ({}^\vee\mathcal{M})^\vee \simeq \mathcal{M}$.

But: ⚠ in general $(\mathcal{M}^\vee)^\vee \not\simeq \mathcal{M}$ as $\mathcal{C}$-module, it is (just) the twist of $\mathcal{M}$ by the tensor autoequivalence $X \mapsto X^{**}$ of $\mathcal{C}$.

See you later ...