$\{M_n\}$ - homotopy invariant Nisnevich sheaves with transfers $(M_{n+1})_{-1} = M_n$

$$\updownarrow$$

$M \to$ cycle module

D4: E/k $v \in DV(E/k)$ $M(E) \xrightarrow{\partial} M(k(v))$

Suppose $v \in DV(E/k)$, by [Déglise, Lemme 2.1.32], $O_v = \varinjlim A_i$ where A_i is smooth/k, $A_i \subseteq O_v$ and O_v is a localization of A_i at some prime ideal. For every A_i, we have a closed immersion $Z_i \subseteq \operatorname{Spec} A_i$ where $Z_i \in (\operatorname{Spec} A_i)^{(1)}$ and give the valuation v. We may also suppose Z_i is smooth/k by generic smoothness and $N_{Z_i/\operatorname{Spec} A_i}$ is trivial.

Then we define a map

$$M_{n+1}(\operatorname{Spec} A_i \setminus Z_i) \xrightarrow{\partial} H^1_{Z_i}(\operatorname{Spec} A_i, M_{n+1}) = \underbrace{H^1_{Z_i}(Z_i \times \mathbb{A}^1, M_{n+1})}$$

where the last equality comes from Thm 8.20. We have an exact sequence

$$M_{n+1}(Z_i \times \mathbb{A}^1) \to M_{n+1}(Z_i \times \mathbb{G}_m) \to H^1_{Z_i}(Z_i \times \mathbb{A}^1, M_{n+1}) \to \overbrace{H^1(Z_i \times \mathbb{A}^1, M_{n+1})}^{H^1(Z_i, M_{n+1})} \to H^1(Z_i \times \mathbb{G}_m, M_{n+1})$$

where the last arrow admits a section, hence injective. So we find that

$$H^1_{Z_i}(Z_i \times \mathbb{A}^1, M_{n+1}) = (M_{n+1})_{-1}(Z_i) = M_n(Z_i).$$

Hence we get a map $\partial: M_{n+1}(\operatorname{Spec} A_i \setminus Z_i) \to M_n(Z_i)$. Taking limit we obtain a map $\partial_v: M_{n+1}(E) \to M_n(k(v))$.

Thm 8.21 (k is perfect) The two procedures described above defines an equivalence between the category of cycle modules and $\underline{\{M_n\} \subseteq Sh(k)}$ with $\underline{M_n \text{ homotopy invariant and } (M_{n+1})_{-1} = M_n}$.

Proof: See [Déglise, Théorème 6.1.1]. □

homotopy modules $\cong$ heart of DM
if no transfers, M_n strictly homotopy invariant
$\{M_n\} \cong$ heart SH
$\cong$
heart $\widetilde{DM} \to$ Milnor-Witt motives

Cor 8.22 (k is perfect) Let $\{M_n\}$ as above. We have $H^p(X, M_n) = A^p(X, M_n)$ for every $X \in Sm/k$.

Thm 8.23 For any $X \in Sm/k$, we have

$$\underline{H^{p,q}(X, \mathbb{Z})} = H^{p-q}(X, K^M_q)$$

if $p \geq 2q-1$.

Proof. We may suppose that k is perfect by Rem 6.10. By the discussion in Prop 5.37, the two presheaves

$$X \mapsto \operatorname{Hom}_{D^-(k)}(\mathbb{Z}(X), (*\mathbb{Z}(q))[p]), \quad X \mapsto \overbrace{H^p((*\mathbb{Z}(q))(X))}^{=H^{p,q}(X,\mathbb{Z})}$$

have the same sheafification, denoted by $H^{p,q}_M$. Since $H^p((*\mathbb{Z}(q))) = H_{q-p}(C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))$ so $H^{p,q}_M = 0$ if $p > q$. If $p = q$, $H^{p,q}_M(E) = K^M_q(E)$ for every field E by Thm 4.7 hence $H^{q,q}_M = K^M_q$ by Thm 6.7. For any $F \in PSh(k)$ homotopy invariant, F is a Zariski sheaf on $\mathbb{A}^1$ by [MVW, Lemma 22.4]. So F is also a Nisnevich sheaf on $\mathbb{A}^1$ since regular birational morphisms between smooth curves are open immersions. Hence $F((\mathbb{G}_m)_E) = F^+((\mathbb{G}_m)_E)$ for every field E. So the map $(F_{-1})^+ \to (F^+)_{-1}$ induces an isomorphism on every field E, so it is an isomorphism by Thm 6.7. From this, together with the cancellation theorem Thm 6.9, we see that

$$(H^{p,q}_M)_{-1} = H^{p-1,q-1}_M$$

$$\operatorname{Hom}_{DM}(\mathbb{Z}(X)(1)[1], \mathbb{Z}(q)[p]) \xrightarrow{\text{cancellation}} \operatorname{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q-1)[p-1]).$$

If $q \leq 0$, then $H^{p,q}_M = 0$ if $p \neq 0$ $\left(H^{p,0}_M = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{else} \end{cases} \right)$. We have

$$H^n(X, H^{p,q}_M) = H^n\Big(\bigoplus_{x \in X^{(0)}} H^{p,q}_M(k(x)) \to \bigoplus_{x \in X^{(1)}} H^{p-1,q-1}_M(k(x)) \to \cdots\Big)$$

by Cor 8.22. Hence $H^n(X, H^{p,q}_M) = 0$ if $\begin{cases} n > q \text{ and } p \neq q. \\ p > q \end{cases}$

We have the hypercohomology spectral sequence

$$H^n(X, H^{p,q}_M) \Rightarrow H^{n+p,q}(X, \mathbb{Z}).$$

So if $p \geq 2q-1$, the $H^a(X, H^{p-a,q}_M) = 0$ if $a \neq p-q$. Then we conclude. □

Cor 8.24 We have $H^{p,q}(X, \mathbb{Z}) = 0$ if $p > 2q$.

proof: $H^n(X, K^M_m) = 0$ if $n > m$ by $(K^M_m)_{-n} = 0$. □ $(\mathbb{Z}_{-1} = 0)$

Cor 8.25 We have $H^{2n,n}(X, \mathbb{Z}) = CH^n(X) = H^n(X, K^M_n) = A^n(X, K^M_n)$. □

§9 Orientation and Decomposition.

Def 9.1 Let $X \in Sm/S$ and E be a vector bundle over X, we define the Thom space of E

$$Th_S(E) = \mathbb{Z}_S(E)/\mathbb{Z}_S(E^\times).$$

Prop 9.2 1) Suppose that E_1 and E_2 are vector bundles over $X \in Sm/k$. Then

$$\underline{Th_X(E_1) \otimes_X Th_X(E_2)} = Th_X(E_1 \oplus E_2)$$

in $DM^{eff,-}(X)$.

2) Suppose that $f: S \to T$, $\begin{matrix} E \\ \downarrow \\ X \in Sm/T \end{matrix}$, then $f^* Th_T(E) = Th_S(f^*E)$.

3) f smooth, then $f_\# Th_S(E) = Th_T(E)$.

proof: 1) The total space of $E_1 \oplus E_2$ is just $E_1 \times_X E_2$. By definition, $Th_X(E)$ is quasi-isomorphic to the complex

$$\mathbb{Z}_X(E^\times) \to \mathbb{Z}_X(E).$$

Hence the left hand side is equal to the total complex

$$\mathbb{Z}_X(E_1^\times \times_X E_2^\times) \to \mathbb{Z}_X(E_1^\times \times_X E_2) \oplus \mathbb{Z}_X(E_1 \times_X E_2^\times) \to \mathbb{Z}_X(E_1 \times_X E_2).$$

By Thm 2.27, the complex

$$\mathbb{Z}_X(E_1^\times \times_X E_2^\times) \to \mathbb{Z}_X(E_1^\times \times_X E_2) \oplus \mathbb{Z}_X(E_1 \times_X E_2^\times)$$

is quasi-isomorphic to

$$0 \to \mathbb{Z}_X((E_1 \oplus E_2)^\times)$$

since $(E_1 \oplus E_2)^\times = (E_1^\times \times_X E_2) \cup (E_1 \times_X E_2^\times)$. Hence we have a quasi-isomorphism

$$\begin{tikzcd}
\mathbb{Z}_X(E_1^\times \times_X E_2^\times) \arrow[r] \arrow[d] & \mathbb{Z}_X(E_1^\times \times_X E_2) \oplus \mathbb{Z}_X(E_1 \times_X E_2^\times) \arrow[r] \arrow[d] & \mathbb{Z}_X(E_1 \oplus E_2) \arrow[d, equal] \\
0 \arrow[r] & \mathbb{Z}_X((E_1 \oplus E_2)^\times) \arrow[r] & \mathbb{Z}_X(E_1 \oplus E_2).
\end{tikzcd}$$

2), 3) HW. □

Prop 9.3 If E is a trivial vector bundle of rank n on $X \in Sm/S$. Then

$$Th_S(E) = \mathbb{Z}_S(X)(n)[2n]$$

in $DM^{eff,-}(S)$.

Proof. If $n = 1$, $Th_X(E) = \mathbb{Z}_X(\mathbb{A}^1_X)/\mathbb{Z}_X(\mathbb{G}_{m,X}) = \mathbb{Z}_X(1)[2]$. So for general n,

$Th_X(E) = Th_X(O_X^{\oplus n}) = Th_X(O_X)^{\otimes n} = (\mathbb{Z}_X(1)[2])^{\otimes n} = \mathbb{Z}_X(n)[2n]$ by Prop 9.2.

Then the statement follows by applying $f_\#$ for $f: X \to S$. □

$$\boxed{\begin{aligned} f_\#(\mathbb{Z}_X(n)[2n]) &= (f_\# \mathbb{Z}_X)(n)[2n] \\ &= \mathbb{Z}_S(X)(n)[2n]. \end{aligned}}$$

Prop 9.4 We have a decomposition

$$\mathbb{Z}(\mathbb{P}^n) = \bigoplus_{i=0}^n \mathbb{Z}(i)[2i] \quad \text{in } DM^{eff,-}(k).$$

Proof: We do by induction on n. If $n = 0$, the statement is trivial. We have a distinguished triangle

$$\underline{\mathbb{Z}(\mathbb{P}^n \setminus (0:\cdots 0:1))} \to \mathbb{Z}(\mathbb{P}^n) \to \underline{\mathbb{Z}(\mathbb{P}^n)/\mathbb{Z}(\mathbb{P}^n \setminus (0:\cdots 0:1))} \to \cdots [1].$$

There is a Cartesian square

$$\begin{tikzcd}
\mathbb{A}^n \setminus 0 \arrow[r] \arrow[d] & \mathbb{P}^n \setminus (0:\cdots:0:1) \arrow[d] \\
\mathbb{A}^n \arrow[r] & \mathbb{P}^n
\end{tikzcd}$$

Hence

$$\mathbb{Z}(\mathbb{P}^n)/\mathbb{Z}(\mathbb{P}^n \setminus (0:\cdots:0:1)) = \underline{\mathbb{Z}(\mathbb{A}^n)/\mathbb{Z}(\mathbb{A}^n \setminus 0)}$$

by étale excision. The latter term is $Th(k^{\oplus})$. By Prop 9.3, it is isomorphic to $\mathbb{Z}(n)[2n]$. On the other hand, we have an identification

$$\begin{matrix} \mathbb{P}^n \setminus (0:\cdots:0:1) = O_{\mathbb{P}^{n-1}}(1) & (x_0:\cdots:x_n) \mapsto f:(x_0,\cdots,x_{n-1}) \mapsto x_n \\ \downarrow & \updownarrow \\ \mathbb{P}^{n-1} & (x_0:\cdots:x_{n-1}) \end{matrix}$$

hence we obtain a distinguished triangle

$$\underline{\mathbb{Z}(\mathbb{P}^{n-1})} \to \mathbb{Z}(\mathbb{P}^n) \to \mathbb{Z}(n)[2n] \to \mathbb{Z}(\mathbb{P}^{n-1})[1].$$

Now $\mathbb{Z}(\mathbb{P}^{n-1}) = \bigoplus_{i=0}^{n-1} \mathbb{Z}(i)[2i]$ by induction. So we are reduced to compute $\operatorname{Hom}_{DM^{eff,-}(k)}(\mathbb{Z}(n)[2n], \mathbb{Z}(i)[2i+1])$. But it vanishes since $\mathbb{Z}(n)[2n]$ is a direct summand of $(\mathbb{P}^1)^{\times n}$ and $H^{2i+1,i}((\mathbb{P}^1)^{\times n}, \mathbb{Z}) = 0$ by Cor 8.24. Hence the triangle splits. □

$\mathbb{P}^1 = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$

Prop 9.5 Suppose $\mathbb{Z}(X)$, $X \in Sm/k$ can be written as

$$\bigoplus_i \mathbb{Z}(n_i)[2n_i]$$

in $DM^{eff,-}(k)$. Given a map $\varphi = (\varphi_i): \mathbb{Z}(X) \to \mathbb{Z}(X) = \bigoplus_i \mathbb{Z}(n_i)[2n_i]$, the followings are equivalent:

a) For every $k \in \mathbb{N}$, we have $CH^k(X) = \bigoplus_{n_i = k} \mathbb{Z} \cdot \varphi_i$.

b) φ is an isomorphism in $DM^{eff,-}(k)$.

Proof: a) ⇒ b) It suffice to show that $\operatorname{Hom}_{DM}(\varphi, \mathbb{Z}(k)[2k])$ is an isomorphism. Let us compute $\operatorname{Hom}_{DM}(\underline{\mathbb{Z}(i)[2i], \mathbb{Z}(j)[2j]})$ for $i, j \in \mathbb{N}$. If $i = j$, it is $\mathbb{Z}$ by cancellation. If $i < j$, it is zero by Prop 2.35. If $i > j$, $\underline{\mathbb{Z}(i-j)[2i-2j]}$ is a direct summand of $((\mathbb{P}^1)^{\times(i-j)}, pt)$, so the group vanishes by $CH^0((\mathbb{P}^1)^{\times(i-j)}, pt) = 0$ and cancellation.

So $\underline{\operatorname{Hom}_{DM}(\varphi, \mathbb{Z}(k)[2k])}$ is the same as the map $\underline{\begin{matrix} \bigoplus_{n_i = k} \mathbb{Z} \to CH^k(X) \\ e_i \mapsto \varphi_i \end{matrix}}$. Hence we are done.

b) ⇒ a) Easy from discussion above. □

Cor 9.6 The map $\mathbb{Z}(\mathbb{P}^n) \xrightarrow{(c_1(O(1)))^n} \bigoplus_{i=0}^n \mathbb{Z}(i)[2i]$ is an isomorphism in DM.