The difference between fiber functor and quasi-fiber functor (F, J)

$(F : \mathcal{C} \to \mathrm{Vec})$

is whether the natural iso. $J : F(X) \otimes F(Y) \to F(X \otimes Y)$

satisfies the "monoidal structure axiom" or not.

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[rr, "\Phi_{F(X), F(Y), F(Z)}"] \arrow[d, "J_{X,Y} \otimes \mathrm{id}_{F(Z)}"'] & & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes J_{Y,Z}"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & & F(X) \otimes F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[rr, "F(a_{X,Y,Z})"'] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

where $\Phi_{F(X),F(Y),F(Z)}$ is defined by above diagram (ok because all arrows are iso.)

then $\Phi \in H^{\otimes 3}$ where $H = \mathrm{End}(F(\cdot))$, called the <u>associator</u>.

(<u>invertible</u> as $\Phi_{X,Y,Z}$ iso)

<u>Rk</u>: For a fiber functor, we need $\Phi = a$ (associativity constraint)
$= 1$ (because a is trivial on Vec)

<u>In</u> general the invertible $\Phi \neq 1$. (quasi-fiber functor).

As before, we can still define algebra homomorphism for $H = \mathrm{End}(F)$.

comultiplication Δ

counit ε

If $\Phi = 1$ then Δ is coassociative.

If $\Phi \neq 1$ then Δ may be non-coassociative

(but $\exists$ examples where Δ is coassociative with $\Phi \neq 1$)

<u>Def:</u> Let $\mathcal{C}$ be a finite tensor category. The quasi-fiber functor (F, J) is <u>normalized</u> if $J_{1,X} = J_{X,1} = \mathrm{id}_{F(X)}$.

Two quasi-fiber functor (F, J_1), (F, J_2) are called <u>twist equivalent</u> by twist $J_1^{-1} J_2$.

same

<u>Exo:</u> Show that (F, J) is equivalent to a normalized one.

(twist?)

Without loss of generality, we will assume that (F, J) is normalized.

Prop: The following holds:

(1) $(id\otimes\Delta)(\Delta(h)) = \Phi(\Delta\otimes id)(\Delta(h))\Phi^{-1}$, $h\in H$.

(2) $(id\otimes id\otimes\Delta)(\Phi)(\Delta\otimes id\otimes id)(\Phi) = (1\otimes\Phi)(id\otimes\Delta\otimes id)(\Phi)(\Phi\otimes 1)$

(3) $(\varepsilon\otimes id)(\Delta(h)) = h = (id\otimes\varepsilon)(\Delta(h))$

(4) $(id\otimes\varepsilon\otimes id)(\Phi) = 1\otimes 1$.

proof: hints: (1): def of Φ

(2): pentagon axiom of the finite tensor cat. $\mathcal{C}$.

(3): quasi-fiber functor (F, J) is normalized

(4): ________________ + triangle axiom of $\mathcal{C}$.

Def: An associative unital $\Bbbk$-algebra H equipped with unital algebra homomorphisms

$\Delta : H \to H \otimes H$. (comultiplication)

$\varepsilon : H \to \Bbbk$ (counit)

and invertible $\Phi \in H^{\otimes 3}$ (associator)

satisfying above four equalities (1), (2), (3), (4), is called quasi-bialgebra

Rk: A bialgebra is a quasi-bialgebra with $\Phi = 1$.

Rk: As for bialgebras, for any quasi-bialgebra H, then $\mathrm{Rep}(H)$ is a monoidal category. (in fact, a finite ring cat. if $\dim(H) < \infty$)

Def: A twist for a quasi-bialgebra H is an invertible $J \in H \otimes H$, s.t.

$$(\varepsilon \otimes id)(J) = (id \otimes \varepsilon)(J) = 1.$$

It permits to define a new quasi-bialgebra denoted H^J, with same counit ε, but comultiplication $\Delta^J(x) = J^{-1}\Delta(x)J$

associator $\Phi^J = (id\otimes J)^{-1}(id\otimes\Delta)(J^{-1})\Phi(\Delta\otimes id)(J)(J\otimes id)$

H^J is called twist-equivalent to H, by twist J.

Rk: twist equivalent quasi-fiber functors $\longleftrightarrow$ twist equivalent quasi-bialgebras

Reconstruction Theorem: the assignments $(\mathcal{C}, F) \mapsto H = End(F)$
are mutually inverse bijection between $H \mapsto (Rep(H), forget)$

(1) the set of monoidal equiv. classes of finite ring cat. $\mathcal{C}$ over k with a quasi-fiber functor F

(2) the set of equiv. classes of finite dim. quasi-bialgebras H over k up to twist equivalence.

Prop: If in addition $\mathcal{C}$ (finite ring cat.) has left duals (e.g. tensor cat) a quasi-fiber functor (F, J) (if exists) is unique up to iso and twisting (i.e. changing J)

Rk: above Prop is false for ∞-setting. Take $\mathcal{C} = \mathrm{Rep}(SL_2(\mathbb{C}))$ tensor cat., take V to be a 2-dim rep. then $\forall n \in \mathbb{N}_{\geq 2}$ $\exists$ fiber functor on $\mathcal{C}$ with $\dim_{\mathbb{C}}(F(V)) = n$ (in particular, non-unique up to changing J)

Now, what about antipode?

Assume that a finite ring cat (with a quasi-fiber functor (F, J)) has left duals. Then on one hand:

$F(V^*)^*$ provides another quasi-fiber functor, isomorphic to F (by previous Prop). Let $\xi = (\xi_V)$ (natural iso), $\xi_V : F(V) \to F(V^*)^*$

<u>On the other hand:</u> $F(V^*)$ is a left dual of $F(V)$ and the (co)ev maps:

$$F(V^*) \otimes F(V) \to \Bbbk \qquad \in \hom_{\mathcal{C}}(F(V^*)\otimes F(V), \Bbbk)$$

$$\Bbbk \longrightarrow F(V) \otimes F(V^*) \qquad \in \hom_{\mathcal{C}}(\Bbbk, F(V)\otimes F(V^*))$$

which (by natural adjunction iso) provides elements

$$\begin{tikzcd}
F(V) \arrow[r, "\alpha_V"] \arrow[dr, "\hat{\alpha}_V"'] & F(V) \arrow[r, "\beta_V"] \arrow[d, "\xi_V"] & F(V) \\
& F(V^*)^* \arrow[ur, "\hat{\beta}_V"'] &
\end{tikzcd}$$

$$\hat{\alpha}_V \in \hom_{\mathcal{C}}(F(V), F(V^*)^*)$$

$$\hat{\beta}_V \in \hom_{\mathcal{C}}(F(V^*)^*, F(V))$$

Thus, the quasi-bialgebra $H = \mathrm{End}(F)$ has the following additional structure:

(1) element $\alpha, \beta \in H$ given by $\alpha_V := \xi_V^{-1} \circ \hat{\alpha}_V$, $\beta_V := \hat{\beta}_V \circ \xi_V$

(2) Antipode $S: H \to H$, $S(a)_{F(V)} = \xi_V^{-1} \circ a^*_{F(V^*)} \circ \xi_V$, unital alg. antihomo. s.t.

If $\Delta(a) = \sum_i a_i^1 \otimes a_i^2$, $a \in H$, then

$$(*) \quad \boxed{\sum_i S(a_i^1)\,\alpha\, a_i^2 = \varepsilon(a)\alpha, \quad \sum_i a_i^1 \beta\, S(a_i^2) = \varepsilon(a)\beta}$$

Let us write associator $\Phi = \sum_i \underset{\in H}{\Phi_i^1} \otimes \underset{\in H}{\Phi_i^2} \otimes \underset{\in H}{\Phi_i^3} \in H^{\otimes 3}$

$$\Phi^{-1} = \sum_i \overline{\Phi_i^1} \otimes \overline{\Phi_i^2} \otimes \overline{\Phi_i^3}$$

then:

$$(**) \quad \boxed{\begin{cases} \sum_i \Phi_i^1 \beta\, S(\Phi_i^2)\,\alpha\, \Phi_i^3 = 1 \\ \sum_i S(\overline{\Phi_i^1})\,\alpha\, \overline{\Phi_i^2}\, \beta\, S(\overline{\Phi_i^3}) = 1. \end{cases}}$$

the proof follows from the def. of left dual.

Def: An antipode on a quasi-bialgebra H is a triple (S, α, β) where $S: H \to H$ unital antihomomorphism, and $\alpha, \beta \in H$ satisfying (*), (**)

Def: A quasi-Hopf algebra is a quasi-bialgebra $(H, \Delta, \varepsilon, \Phi)$ with an antipode (S, α, β) which is bijective.

Rk: A Hopf algebra is a quasi-Hopf algebra with $\Phi = 1$, $\alpha = \beta = 1$.

Rk: Let H be a quasi-Hopf alg. Then $Rep(H)$ is a tensor cat.

As above (for "finite ring cat with left duals is tensor cat."), a finite dim. quasi-bialgebra with antipode S is a quasi-Hopf alg. (i.e. S is automatically invertible).

See You later...